Mathematics II – Examples
II. Differential Calculus of Several Variables

II.4. Total differential and tangent plane

Notation: let us have a function $z = f(x, y)$. Then

the (total) differential of the function f in the point $A = [x_0, y_0]$:

$$\mathrm{d}f(A) = \frac{\partial f}{\partial x}(A) \cdot (x - x_0) + \frac{\partial f}{\partial y}(A) \cdot (y - y_0)$$

Denote $\mathrm{d}x = x - x_0, \mathrm{d}y = y - y_0$. Then

$$\mathrm{d}f(A) = \frac{\partial f}{\partial x}(A) \cdot \mathrm{d}x + \frac{\partial f}{\partial y}(A) \cdot \mathrm{d}y$$

Example 91: Let $f(x, y) = \dfrac{y}{x} - \dfrac{x}{y}$.

a) Determine and sketch domains, where the function f is differentiable.

b) Write the differential of f in the point $A = [x_0, y_0]$.

Example 92: Determine total differential and an approximate increment of the function $z = \dfrac{y}{x}$ in in the point $A = [2, 1]$ for $\triangle x = 0.1$ and $\triangle y = 0.2$. Compare them.

Example 93: Using total differential compute approximate increment of the function $z = \operatorname{arctg}\dfrac{y}{x}$ in if x changes from $x_0 = 1$ to $x_1 = 1.2$ and y changes from $y_0 = -3$ to $y_1 = -3.1$.

Example 94: Approximate a value of the expression $\ln\left(\sqrt{9.03} - \sqrt{0.99} - 1\right)$ using total differential of an appropriate function.

Example 95: Compute approximate value of the expression $0.98^{3.04}$ using total differential of an inappropriate function.

Example 96: Find an equation of both tangent plane τ and normal line n to the graph of the function $z = 2x^2 - 4y^2$ in the point $T = [2, 1, ?]$. Compute approximate value of this function in the point $[2.2, 1.3]$.

Example 97: Find an equation of tangent plane τ to the surface $z = x^2 + xy - y^2 + x + 3$ and parallel to the given plane $\rho:\ 5x - 3y - z = 0$.

- **Compute approximate values of a given expressions using total differential:**

Example 98: $\sqrt[3]{7.95} \cdot \sqrt{8.96}$

Example 99: $\dfrac{\sqrt[4]{0.97}}{1.02^3 \cdot \sqrt[3]{0.99}}$

Example 100: $\sqrt{4.04} \cdot \ln 1.02 \cdot \operatorname{arctg} 0.9$

- **Find an equation of both tangent plane τ and normal line n to the surface $z = f(x,y)$ in the point T:**

Example 101: $z = 4\sqrt{x^2+y^2}$, $\quad T = [3, 4, ?]$ **Example 102**: $z = xy$, $\quad T = [0, 0, ?]$

Example 103: $z = x^2 \cdot \cos\frac{1}{y}$, $\quad T = [1, \frac{2}{\pi}, ?]$ **Example 104**: $z = \frac{1}{x} \cdot \arcsin y$, $\quad T = [\frac{1}{2}, \frac{\sqrt{2}}{2}, ?]$

- **Find an equation of tangent plane τ to the surface $z = f(x,y)$ and parallel to the plane ρ:**

Example 105: $z = 2x^2 - y^2$, $\quad \rho: \; 8x - 6y - z - 15 = 0$

Example 106: $z = \ln(x^2 + 2y^2)$, $\quad \rho: \; 2x - z + 5 = 0$

Example 107: $z = x^2 - y^2 + 6xy + 2x$, $\quad \rho: \; 4x + 6y - z = 0$

II.5. Derivatives and differentials of high order

Example 108: Find all partial derivatives of second order of the function $f(x,y) = xy^3 - y \cdot \mathrm{e}^{x+y^2}$.

Example 109: Prove that the function $u = u(x,t) = \operatorname{arctg}(2x - t)$ satisfies the partial differential equation $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial t \partial x} = 0$ in $\mathbb{E}_2$.

Example 110*: There is given the function $f(x,y)$

$$f(x,y) = \begin{cases} xy\frac{x^2-2y^2}{x^2+y^2} & \text{for } [x,y] \neq [0,0], \\ 0 & \text{for } [x,y] = [0,0]. \end{cases}$$

Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Example 111: Let us consider the scalar field $\phi(x,y,z) = xy^2 + z^3 - xyz + 3$. Compute $\operatorname{grad}\phi$, $\quad \operatorname{rot}\operatorname{grad}\phi$.

Example 112: Let us consider the vector field $\vec{f} = (U, V, W) = \left(xy, x^2 - z^2, \frac{y}{x+z}\right)$. Compute $\operatorname{div}\vec{f}$, $\quad \operatorname{rot}\vec{f}$, $\quad \operatorname{div}\operatorname{rot}\vec{f}$.

Example 113*: Let the scalar field $\phi(x,y,z)$ has continuous partial derivatives of second order in a domain $D \subset \mathbb{E}_3$. Prove that $\operatorname{rot}\operatorname{grad}\phi = \vec{0}$ in D.

Example 114*: Let the vector field $\vec{f} = (U, V, W)$ has continuous partial derivatives of second order in a domain $D \subset \mathbb{E}_3$. Prove that $\operatorname{div}\operatorname{rot}\vec{f} = 0$ in D.

- **Find differentials of the given order:**

Example 115*: $z = \sin(2x+y)$, $\mathrm{d}^2 z = ?$

Example 116*: $z = x^3 - y^3 - xy + y^2$, $\mathrm{d}^3 z = ?$

Example 117*: $u = \mathrm{e}^{2x-3y}$, $\mathrm{d}^2 u(A) = ?$, $\mathrm{d}^3 u(A) = ?$, $\mathrm{d}^n u(A) = ?$, $A = [0,0]$.

Differentials can be used in the iportant **Taylor theorem**:
Let a function $f(x,y)$ is differentiable$(n+1)$ times in any interior point of a rectangle M with a center in the point $A = [x_0, y_0]$. Then for any point $[x,y] \in M$ there exists a point $[\xi, \eta] \in M$, such that

$$f(x,y) = f(A) + d\,f(A) + \frac{d^2 f(A)}{2!} + \cdots + \frac{d^n f(A)}{n!} + R_{n+1},$$

where $d\,f(A) = d\,f(x_0,y_0) = \frac{\partial f}{\partial x}(A)\cdot(x-x_0) + \frac{\partial f}{\partial y}(A)\cdot(y-y_0)$,

$$d^2 f(A) = \frac{\partial^2 f}{\partial x^2}(A)\cdot(x-x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(A)\cdot(x-x_0)(y-y_0) + \frac{\partial^2 f}{\partial y^2}(A)\cdot(y-y_0)^2,$$

$$\vdots$$

$$d^n f(A) = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^n f}{\partial x^k \partial y^{n-k}}(A)\cdot(x-x_0)^k (y-y_0)^{n-k},$$

$$R_{n+1} = \frac{1}{(n+1)!} d^{n+1} f(\xi, \eta).$$

Example 118*: Write Taylor expansion of the function $f(x,y) = x^3 - 3xy^2 + y^2 + 4x - 5y$ in a neighborhood of the point $A = [2,-1]$ and use the result for approximation of the value of the function f in the point $[2.1, -1.1]$.

Example 119*: Write Taylor expansion of the fourth order of the function $f(x,y) = \cos(x^2+y^2)$ in a neighborhood of the point $[0,0]$.

- **Find partial derivatives of second order of the given function:**

Example 120: $\phi(s,t) = \ln(s^3 + t)$

Example 121: $\phi(x,t) = \dfrac{\cos x^2}{t}$

Example 122: $f(x,y) = \mathrm{e}^{ax+by}$

Example 123: Verify that the function $u(x,t) = \sin(x - ct)$ and the function $u(x,t) = \sin(\omega ct)\cdot\sin(\omega t)$ satisfy the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

Example 124: Verify that the function $u(x,y) = \mathrm{e}^x \cdot \sin y$ satisfies the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

- **Expand the function $f(x,y)$ using Taylor's theorem in a neighborhood of a point A:**

Example 125*: $f(x,y) = x^3 + 5x^2 - 6xy + 2y^2, \quad A = [1,-2]$

Example 126*: $f(x,y) = x^2 + 3xy - y^3, \quad A = [2,-1]$

II.6. Gradient. Directional derivative

- **Determine the angle between two gradients of given functions in a point A:**

Example 127: $f(x,y,z) = x^y + yz, \quad g(x,y,z) = \sin(xz) + x + y - \dfrac{z}{y} - 1, \quad A = [1,1,0]$

Example 128: $f(x,y) = \operatorname{arctg} \dfrac{x}{,} \quad g(x,y) = y\sqrt{x}, \quad A = [1,1]$

Example 129: Find a set in $D(f) \equiv \mathbb{E}_3$, in which the function $f(x,y,z) = x^2 + y^2 + z^2 - 2xyz$ has gradient, which equals to zero-vector $\vec{0}$.

Example 130: Find a set in $D(f) \equiv \mathbb{E}_2$, in which the function $f(x,y) = (x^2+y^2)^{3/2}$ has a gradient of size 9.

- **Compute directional derivative of function f in direction $\vec{s}$ in a point A:**

Example 131: $f(x,y) = 2x^4 + yz + y^3, \quad \vec{s} = (3,-4), \quad A = [1,2]$

Example 132: $f(x,y,z) = x^2 + 2y^2 - z^2, \quad A = [-3,2,4]$, direction $\vec{s}$ is given by the vector $\overrightarrow{A,B}$, where $B = [-2,4,2]$

Example 133: Compute directional derivative of the function $z = x^2 + \ln(x+y^2)$ in the point $A = [3, 2\sqrt{3}]$ in direction given by tangent line to parabola $y^2 = x$. Consider the vector with sharp angle with vector $\vec{i}$.

Example 134: Determine a direction, in which derivative of the function $f(x,y) = x^3y + \frac{x}{y^2} + 2y$ in the point $A = [-1,1]$ in maximal. Compute this derivative.

Example 135: Let us have the function $z = \sqrt{2x+y}$, the point $A = [1,2]$, vector $\vec{s} = (-1,1)$.
Determine
a) in which points the function z is differentiable.
b) $\frac{\partial z}{\partial \vec{s}}(A)$,
c) tangent plane to the graph of function z in the point $T = [1,2,?]$.

Example 136: Determine a vector $\vec{s}$, in which direction the speed of change of the function values of $f(x,y,z) = x^2 + y^2 + z^2 - 2xyz$ in the point $A = [1,-1,2]$ is maximal. Compute this maximal speed.

Example 137: Determine points, in which gradient of the function $f(x,y,z) = x^2 + y^2 + z^2 - 2xyz$ is orthogonal to the axis x.

Example 138: Determine angle between vectors $\operatorname{grad} f(A)$ and $\operatorname{grad} g(A)$, where $f(x,y,z) = x - 3y + \sqrt{3xy} + z^3, \quad g(x,y,z) = z\sqrt{x^2+y^2} + xyz, \; A = [3,4,0]$.

- **Compute gradient of given function**:

Example 139: $f(x,y)=\dfrac{1}{\sqrt{(x^2+y^2)^3}}$

Example 140: $f(x,y)=\sin(x^2y)+\dfrac{x^2}{3}$

Example 141: $f(x,y)=\ln(x+\sqrt{x^2+y^2})$

Example 142: $f(x,y,z)=x^2yz+\ln y-15$

Example 143: $f(u,v,t)=t\sqrt{u^2+v^2}$

- **Compute gradient of function in a given point** A:

Example 144: $f(x,y)=\operatorname{arccotg}(x-2y),\quad A=[4,1]$

Example 145: $f(x,y,z)=x\sqrt{yz},\quad A=[-2,1,4]$

Example 146: $f(x,y,z)=\dfrac{x^2}{z}+\dfrac{z^2}{2y}-\dfrac{4}{x},\quad A=[1,2,-3]$

Example 147: Find points in which gradient of the function $f(x,y,z)=\ln\left(x+\frac{1}{y}\right)$ equals to the vector $\left(1,-\frac{16}{9}\right)$.

Example 148: In which points the gradient of the function $f(x,y,z)=x^2+y^2-2z^2+xy+3y+8z$
a) is orthogonal to axis x, b) is parallel to axis z, c) equals to zero vector.

Example 149: Find angle φ of gradients of the function $f(x,y)=\arcsin\frac{x-1}{y},\quad y\neq 0$, in points $A=[1,1],\ B=[3,4]$.

- **Let us have a given function** $f(x,y)$**, point** A **and vector** $\vec{s}$**. Determine where the function is differentiable, compute** $\frac{\partial f}{\partial \vec{s}}(A)$ **and write equation of a tangent plane to the graph of the function** f **in the point** $T=[A,f(A)]$:

Example 150: $f(x,y)=|x|+y,\quad A=[1,0],\quad \vec{s}=(-1,1)$

Example 151: $f(x,y)=x^2+3xy+y^2,\quad A=[1,0],\quad \vec{s}=(1,2)$

Example 152: Determine the direction in which the directional derivative of the function $f(x,y)=\ln\dfrac{x+y}{x-y}$ in the point $A=[3,0]$ is maximal. Compute value of this derivative.

Example 153: Compute directional derivative of the function $f(x,y)=x^2-y^2$ in the point $A=[2,3]$ in the direction $\vec{s}$ in angle $\alpha=\frac{\pi}{3}$ with vector $\vec{i}$ (α is so called *directional angle*).

Example 154*: Compute directional derivative of the function $f(x,y,z)=x^2-3xy+y^2z-5z$ in the point $A=[1,-2,-1]$ in the direction $\vec{s}$, which is determine by its three directional angles $\alpha=\frac{\pi}{3},\ \beta=\frac{3\pi}{4},\ \gamma\in\left(0,\frac{\pi}{2}\right)$ (directional angles are the angles between vector $\vec{s}$ and axes $x,\ y,\ z$).

Example 155: Let us consider the function $z = f(x, y) = \sqrt{18 - x^2 - 2y^2}$

a) Write a sufficient condition for differentiability of function of n-variables in an open set $M \subset \mathbb{E}_n$. Determine and sketch domain D in $\mathbb{E}_2$, where the function $f(x, y)$ is differentiable. Justify this.

b) Determine and sketch a graph of the given function $f(x, y) = \sqrt{18 - x^2 - 2y^2}$.

c) Compute directional derivative of the function f in the point $A = [1, -2]$ in the direction $\vec{s} = \overrightarrow{AB}$, where $B = [0, 0]$. Describe a behaviour of the function f in the point A in a given direction (e.g., if the function is increasing or decreasing, how fast, etc.).

d) Write an equation and parametrization of the normal line to the graph of the function f in the touch point $T = [1, -2, ?]$.

e) Write equations of contours $f(x, y) = k$ for $k = 0$ and $k = 3$. Sketch these contours.

Example 156: Let us have the function $z = f(x, y) = -\sqrt{5y - x^2}$.

a) Determine and sketch the domain D in $\mathbb{E}_2$, in which the function $f(x, y)$ is differentiable. Justify this.

b) Compute partial derivatives of first order of the given function in the point $A = [4, 5]$. Describe a behaviour of the function f in the point A (e.g., if the function is increasing or decreasing, in which direction and how fast, etc.).

c) Determine a direction $\vec{s}$ of the fastest descent of the function f. Compute the directional derivative of f in this direction $\vec{s}$.

d) Write a differentinal of the function f in the point $A = [4, 5]$. Using this, compute an approximate value of f in $[4.3, 5.3]$.

e) Sketch the graph of the function $z = f(x, y)$.

Example 157: Let us have the function $z = f(x, y) = \ln(xy - 4)$.

a) Determine and sketch domains in $\mathbb{E}_2$, in which the function $f(x, y)$ is differentiable. Justify this.

b) Compute gradient of the given function in the point $A = [-2, -4]$. Explain what the computed vector says about a behaviour of f in A.

c) Compute a value of directional derivative of f in A in gradient's direction.

d) Find a direction $\vec{s}$ in which a directional derivative is null. Justify by calculation.

e) Write equations of contours $f(x, y) = k$ for $k = 0$ and $k = \ln 4$.

Example 158: Let us have the function $z = f(x, y) = x^2 - y^2 + 6xy + 2x$, the point $A = [-1, 2]$ and the direction $\vec{s} = (2, -2)$.

a) Compute $\dfrac{\partial f}{\partial \vec{s}}(A)$. Is the vector $\vec{s}$ the direction of maximum growth of f in A?

b) Determine a set of points in which $\operatorname{grad} f(x, y) = \vec{0}$.

c) Find a touch point T and tangent plane τ to a graph of the function f, which is parallel to the plane $\rho: \; 4x + 6y - z + 3 = 0$.

Example 159: Let us have the function $z = f(x, y) = y^2 \sin(x^2 - y^2)$.

a) Determine where is the function f differentiable and compute $f_x, \; f_y$.

b) Write an equation of a tangent plane to the graph of the function f in touch point $T = [1, 1, ?]$.

c) Using total differential compute approximation of a value $f(0.9, \; 1.1)$.