Mathematics II – Examples

IV. Line integral

IV.1. Parametrization of curves

Let $P(t) = [x(t),\, y(t),\, z(t)]$ be a morphism of interval $\langle a,\, b\rangle$ into $\mathbb{E}_3$. If

(1) $P(t)$ is continuous and simple on $\langle a, b\rangle$

(2) derivative $\dot{\mathbf{P}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ is bounded and continuous mapping of (a, b),

(3) $\dot{\mathbf{P}}(t) \neq \vec{0}$ for all $t \in (a, b)$,

then we will call the set $c = \{X \in \mathbb{E}_3 :\ X = \mathbf{P}(t),\ t \in \langle a, b\rangle\,\}$ **simple smooth curve** in $\mathbb{E}_3$ with **parametrization $\mathbf{P}(t)$**.

Analogously we define parametrization of a curve in $\mathbb{E}_2$.

The orientation of a curve determines a direction of move along the curve when value of parameter t increases. The orientation can be done by a unit tangent vector $\vec{\tau}$.

We say that a curve is oriented in accordance with its parametrization $P(t)$, if $P(a)$ is its initial point or if $\vec{\tau} = \dfrac{\dot{\mathbf{P}}(t)}{||\dot{\mathbf{P}}(t)||}$. If $\vec{\tau} = -\dfrac{\dot{\mathbf{P}}(t)}{||\dot{\mathbf{P}}(t)||}$ or if the initial point of a given curve is $P(b)$, then we say that the orienation of the given curve is in contrary with the parametrization $P(t)$.

A simple smooth closed curve c in $\mathbb{E}_2$ is positively oriented (with respect to its interior) if "*moving along the curve in its orientation, the interior is on the left hand*". This is in accordance with "*counterclockwise*" orientation. The opposite orientation is called negative.

Example 424: Consider the curve $c = \{[x, y] \in \mathbb{E}_2 :\ y = x^2,\ x \in \langle -4, 4\rangle\,\}$ with starting point $A = [-4,\, 16\,]$. Verify, that $P(t) = [x(t), y(t)]$ is a parametrization of a simple smooth curve c, if

a) $P(t) = [t, t^2],\quad t \in \langle -4, 4\rangle$,

b) $P(t) = [t^2, t^4],\quad t \in \langle -2, 2\rangle$,

c) $P(t) = [\sqrt{t}, t],\quad t \in \langle 0, 16\rangle$.

Example 425: Consider a half circle $c = \{[x, y] \in \mathbb{E}_2 :\ x^2 + y^2 = a^2,\ y \geq 0\,\}$ with starting point $A = [-a,\, 0\,]$. Verify, if $P(t)$ is its parametrization, where

a) $P(t) = [a\cos t, a\sin t],\quad t \in \langle 0, \pi\rangle$

b) $P(t) = [t, \sqrt{a^2 - t^2}],\quad t \in \langle -a, a\rangle$

c) $P(t) = \left[\dfrac{at}{\sqrt{1+t^2}}, \dfrac{a}{\sqrt{1+t^2}}\right],\quad t \in \mathbb{R}$

- Find a parametrization of a curve with initial point A and determine if its orientation is in accordance with the parametrization.

Example 426: The curve c is a line segment with initial point $A = [4,\, -1,\, 3]$ and terminal point $B = [3,\, 1,\, 5]$.

Example 427: $c = \{\, [x,\, y] \in \mathbb{E}_2 : (x+3)^2 + (y-2)^2 = 9,\ x \le -3 \,\},\quad A = [-3, -1]$

Example 428: $c = \{\, [x,\, y] \in \mathbb{E}_2 : \dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1,\ y \ge 0 \,\},\quad A = [3, 0]$

Example 429: $c = \{\, [x,\, y,\, z] \in \mathbb{E}_3 : x^2 + y^2 = 4x,\ y + z = 0,\ z \ge 0 \,\},\quad A = [0, 0, 0]$

Example 430: $c = \{\, [x,\, y,\, z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2,\ x = y,\ x \ge 0 \,\},\quad A = [0, 0, -a]$

- A curve is given $\mathbb{E}_2$ by its parametrization. Find an implicit formula and name this curve.

Example 431: $c = \{\, [x,\, y] \in \mathbb{E}_2 : x = 2t + 1,\ y = 3 - t,\ t \in \langle 1, 4 \rangle \,\}$, the orientation is in accordance with parametrization. [*the line segment with initial point* $A = [3, 2] = P(1)$.]

Example 432: $c = \{\, [x,\, y] \in \mathbb{E}_2 : x = t^2 - 2t + 3,\ y = t^2 - 2t + 1,\ t \in \langle 0, 3 \rangle \,\}$, the orientation is in contrary with parametrization. [*the line segment with initial point* $A = [6, 4] = P(3)$.]

Example 433: $c = \{\, [x,\, y] \in \mathbb{E}_2 : x = 2\sin^2 t,\ y = 4\cos^2 t,\ t \in \langle 0, \frac{\pi}{2} \rangle \,\}$, the orientation is in accordance with parametrization. [*the line segment with initial point* $A = [0, 4] = P(0)$.]

Example 434*: The curve is given in polar coordinates by equation $r(\varphi) = 4\sin\varphi,\ \varphi \in \langle \frac{\pi}{2},\, \pi \rangle \,\}$, the orientation is in contrary with parametrization. [*the circle.*]

- Verify that $c = c_1 \cup c_2$ is a simple, closed and smooth by parts curve. Find parametrization of curves c_1 and c_2. Sketch them and determine their orientation in a case when a point A is both an initial point of c_1 and a terminal point of c_2:

Example 435: $c_1,\, c_2 \in \mathbb{E}_2,\ A = [0, 0];\ c_1 = \{\, [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4x,\ y \ge 0 \,\}$,
$c_2 = \{\, [x, y] \in \mathbb{E}_2 : y = 0,\ x \in \langle 0, 4 \rangle \,\}$.

Example 436: $c_1,\, c_2 \in \mathbb{E}_2,\ A = [1, 8];\ c_1 = \{\, [x, y] \in \mathbb{E}_2 : xy = 8,\ x \in \langle 1, 4 \rangle \,\}$,
$c_2 = \{\, [x, y] \in \mathbb{E}_2 : y + 2x = 10,\ x \in \langle 1, 4 \rangle \,\}$.

Example 437: $c_1,\, c_2 \in \mathbb{E}_2,\ A = [1, 1];\ c_1 = \{\, [x, y] \in \mathbb{E}_2 : y = \sqrt{x},\ x \in \langle 0, 1 \rangle \,\}$,
$c_2 = \{\, [x, y] \in \mathbb{E}_2 : y = x^2,\ x \in \langle 0, 1 \rangle \,\}$.

Example 435: $c_1,\, c_2 \in \mathbb{E}_3,\ A = [3, 0, 2];\ c_1 = \{\, [x, y, z] \in \mathbb{E}_3 : x^2 + y^2 = 9,\ x - z = 1,\ y \ge 0 \,\}$,
$c_2 = \{\, [x, y, z] \in \mathbb{E}_3 : x - z = 1,\ y = 0 \,\}$.

- Suggest a parametrization of a curve c with initial point A: