

# Mathematics II – Examples

## IV. Line integral

### IV.1. Parametrization of curves

Let  $P(t) = [x(t), y(t), z(t)]$  be a morphism of interval  $\langle a, b \rangle$  into  $\mathbb{E}_3$ . If

- (1)  $P(t)$  is continuous and simple on  $\langle a, b \rangle$
- (2) derivative  $\dot{\mathbf{P}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$  is bounded and continuous mapping of  $(a, b)$ ,
- (3)  $\dot{\mathbf{P}}(t) \neq \vec{0}$  for all  $t \in (a, b)$ ,

then we will call the set  $c = \{X \in \mathbb{E}_3 : X = \mathbf{P}(t), t \in \langle a, b \rangle\}$  **simple smooth curve** in  $\mathbb{E}_3$  with **parametrization**  $\mathbf{P}(t)$ .

Analogously we define parametrization of a curve in  $\mathbb{E}_2$ .

The orientation of a curve determines a direction of move along the curve when value of parameter  $t$  increases. The orientation can be done by a unit tangent vector  $\vec{\tau}$ .

We say that a curve is oriented in accordance with its parametrization  $P(t)$ , if  $P(a)$  is its initial

point or if  $\vec{\tau} = \frac{\dot{\mathbf{P}}(t)}{\|\dot{\mathbf{P}}(t)\|}$ . If  $\vec{\tau} = -\frac{\dot{\mathbf{P}}(t)}{\|\dot{\mathbf{P}}(t)\|}$  or if the initial point of a given curve is  $P(b)$ , then we say that the orientation of the given curve is in contrary with the parametrization  $P(t)$ .

A simple smooth closed curve  $c$  in  $\mathbb{E}_2$  is positively oriented (with respect to its interior) if "moving along the curve in its orientation, the interior is on the left hand". This is in accordance with "counterclockwise" orientation. The opposite orientation is called negative.

**Example 424:** Consider the curve  $c = \{[x, y] \in \mathbb{E}_2 : y = x^2, x \in \langle -4, 4 \rangle\}$  with starting point  $A = [-4, 16]$ . Verify, that  $P(t) = [x(t), y(t)]$  is a parametrization of a simple smooth curve  $c$ , if

- a)  $P(t) = [t, t^2], t \in \langle -4, 4 \rangle,$
- b)  $P(t) = [t^2, t^4], t \in \langle -2, 2 \rangle,$
- c)  $P(t) = [\sqrt{t}, t], t \in \langle 0, 16 \rangle.$

**Example 425:** Consider a half circle  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2, y \geq 0\}$  with starting point  $A = [-a, 0]$ . Verify, if  $P(t)$  is its parametrization, where

- a)  $P(t) = [a \cos t, a \sin t], t \in \langle 0, \pi \rangle$
- b)  $P(t) = [t, \sqrt{a^2 - t^2}], t \in \langle -a, a \rangle$
- c)  $P(t) = \left[ \frac{at}{\sqrt{1+t^2}}, \frac{a}{\sqrt{1+t^2}} \right], t \in \mathbb{R}$

- Find a parametrization of a curve with initial point  $A$  and determine if its orientation is in accordance with the parametrization.

**Example 426:** The curve  $c$  is a line segment with initial point  $A = [4, -1, 3]$  and terminal point  $B = [3, 1, 5]$ .

**Example 427:**  $c = \{ [x, y] \in \mathbb{E}_2 : (x + 3)^2 + (y - 2)^2 = 9, x \leq -3 \}, A = [-3, -1]$

**Example 428:**  $c = \{ [x, y] \in \mathbb{E}_2 : \frac{(x - 1)^2}{4} + \frac{y^2}{9} = 1, y \geq 0 \}, A = [3, 0]$

**Example 429:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x^2 + y^2 = 4x, y + z = 0, z \geq 0 \}, A = [0, 0, 0]$

**Example 430:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2, x = y, x \geq 0 \}, A = [0, 0, -a]$

- A curve is given  $\mathbb{E}_2$  by its parametrization. Find an implicit formula and name this curve.

**Example 431:**  $c = \{ [x, y] \in \mathbb{E}_2 : x = 2t + 1, y = 3 - t, t \in \langle 1, 4 \rangle \}$ , the orientation is in accordance with parametrization. [*the line segment with initial point  $A = [3, 2] = P(1)$ .*]

**Example 432:**  $c = \{ [x, y] \in \mathbb{E}_2 : x = t^2 - 2t + 3, y = t^2 - 2t + 1, t \in \langle 0, 3 \rangle \}$ , the orientation is in contrary with parametrization. [*the line segment with initial point  $A = [6, 4] = P(3)$ .*]

**Example 433:**  $c = \{ [x, y] \in \mathbb{E}_2 : x = 2 \sin^2 t, y = 4 \cos^2 t, t \in \langle 0, \frac{\pi}{2} \rangle \}$ , the orientation is in accordance with parametrization. [*the line segment with initial point  $A = [0, 4] = P(0)$ .*]

**Example 434\*:** The curve is given in polar coordinates by equation  $r(\varphi) = 4 \sin \varphi, \varphi \in \langle \frac{\pi}{2}, \pi \rangle$ , the orientation is in contrary with parametrization. [*the circle.*]

- Verify that  $c = c_1 \cup c_2$  is a simple, closed and smooth by parts curve. Find parametrization of curves  $c_1$  and  $c_2$ . Sketch them and determine their orientation in a case when a point  $A$  is both an initial point of  $c_1$  and a terminal point of  $c_2$ :

**Example 435:**  $c_1, c_2 \in \mathbb{E}_2, A = [0, 0]; c_1 = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4x, y \geq 0 \}, c_2 = \{ [x, y] \in \mathbb{E}_2 : y = 0, x \in \langle 0, 4 \rangle \}$ .

**Example 436:**  $c_1, c_2 \in \mathbb{E}_2, A = [1, 8]; c_1 = \{ [x, y] \in \mathbb{E}_2 : xy = 8, x \in \langle 1, 4 \rangle \}, c_2 = \{ [x, y] \in \mathbb{E}_2 : y + 2x = 10, x \in \langle 1, 4 \rangle \}$ .

**Example 437:**  $c_1, c_2 \in \mathbb{E}_2, A = [1, 1]; c_1 = \{ [x, y] \in \mathbb{E}_2 : y = \sqrt{x}, x \in \langle 0, 1 \rangle \}, c_2 = \{ [x, y] \in \mathbb{E}_2 : y = x^2, x \in \langle 0, 1 \rangle \}$ .

**Example 438:**  $c_1, c_2 \in \mathbb{E}_3, A = [3, 0, 2]; c_1 = \{ [x, y, z] \in \mathbb{E}_3 : x^2 + y^2 = 9, x - z = 1, y \geq 0 \}, c_2 = \{ [x, y, z] \in \mathbb{E}_3 : x - z = 1, y = 0 \}$ .

- Suggest a parametrization of a curve  $c$  with initial point  $A$ :

**Example 439:**  $c = \{ [x, y] \in \mathbb{E}_2 : 3x + y = 1, x \in \langle -1, 2 \rangle \}$ ,  $A = [-1, 4]$

**Example 440:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : 2x - y = 2, x + z = 3, y \in \langle 0, 2 \rangle \}$ ,  $A = [2, 2, 1]$

**Example 441:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : 4x^2 + z^2 = 4, y + z = 0, y \leq 0 \}$ ,  $A = [-1, 0, 0]$

## IV.2. Line integral of a scalar function

- Check the existence of line integral  $\int_c f \, ds$ . If this integral exists, compute it.

**Example 442:**  $\int_c \frac{1}{x-2y} \, ds$ ;  $c$  is a line segment between points  $A, B$ , where  
 a)  $A = [1, -2], B = [3, 0]$ ,      b)  $A = [1, -2], B = [3, 4]$ .

**Example 443:**  $\int_c \frac{x+2}{\sqrt{x^2+y^2}} \, ds$ ;    a)  $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4x \}$   
 b)  $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4 \}$

**Example 444:**  $\int_c x^2 \, ds$ ;     $c = \{ [x, y] \in \mathbb{E}_2 : y = \ln x, x \in \langle 1, 3 \rangle \}$

**Example 445:**  $\int_c \frac{x^2}{y} \, ds$ ;     $c = \{ [x, y] \in \mathbb{E}_2 : y^2 = 2x, y \in \langle \sqrt{2}, 2 \rangle \}$

**Example 446:**  $\int_c (x^2 + y^2 + z^2) \, ds$ ;  $c$  is the first thread of the helix  $x = a \cos t, y = a \sin t, z = bt$ .

- Justify on which curve  $c$  the integral  $\int_c f \, ds$  exists. Compute the integral.

**Example 447:**  $\int_c \frac{3-y}{y-x+2} \, ds$ ,    a)  $c$  is the circle  $x^2 - 2x + y^2 = 0$ ,  
 b)  $c$  is the line segment  $AB$ , where  $A = [2, 3], B = [0, 1]$ .  
 [a) doesn't exist, b) exists]

**Example 448:**  $\int_c \frac{1}{x^2+y^2} \, ds$ ,    a)  $c = \{ [x, y] \in \mathbb{E}_2 : x = t-3, y = 3-t, t \in \langle 1, 4 \rangle \}$ ,  
 b)  $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2 \}$ .  
 [a) doesn't exist, b) exists]

**Example 449:**  $\int_c \frac{1}{x^2-y} \, ds$ ,    a)  $c = \{ [x, y] \in \mathbb{E}_2 : y = 2x, x \in \langle 1, 3 \rangle \}$ ,  
 b)  $c = \{ [x, y] \in \mathbb{E}_2 : y = 9, x \in \langle 0, 2 \rangle \}$ .  
 [a) doesn't exist, b) exists]

**Example 450:**  $\int_c xy \, ds$ ,  $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2, x \leq 0, y \geq 0 \}$ ,

**Example 451:**  $\int_c \sqrt{2y} \, ds$ ,  $c = \{ [x, y] \in \mathbb{E}_2 : x = a(t - \sin t), y = a(1 - \cos t), t \in \langle 0, 2\pi \rangle \}$

**Example 452:**  $\int_c \sqrt{x} \, ds$ ,  $c = \{ [x, y] \in \mathbb{E}_2 : y = \sqrt{x}, x \in \langle 1, 2 \rangle \}$

**Example 453:**  $\int_c (xy + 2) \, ds$ ,  $c = \{ [x, y] \in \mathbb{E}_2 : x = \cos t, y = 2 \sin t, t \in \langle 0, 2\pi \rangle \}$

**Example 454:**  $\int_c z \, ds$ ,  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = t \cos t, y = t \sin t, z = t, t \in \langle 0, \pi \rangle \}$

**Example 455:**  $\int_c (x + y) \, ds$ ,  $c = \{ [x, y, z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2, y = x, x \geq 0, y \geq 0, z \geq 0 \}$

**Example 456:**  $\int_c xyz \, ds$ ,  $c = \{ [x, y, z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2, x^2 + y^2 = \frac{a^2}{4}, \text{ in the first octant} \}$

- Consider a scalar function  $f$  and a curve  $c$ , which is intersection of two given surfaces.
  - a) Suggest a parametrization of the curve  $c$ .
  - b) Write the vector  $\dot{\mathbf{P}}(t)$  and compute its length  $\|\dot{\mathbf{P}}(t)\|$ .
  - c) Compute line integral of a given scalar function  $f$ .

**Example 457:**  $f(x, y, z) = y^2 + 2z^2$ , curve  $c$  is intersection of the planes  $x + y + 2z = 5$ ,  $2x + 5y - 2z = 4$  in the first octant.

**Example 458:**  $f(x, y, z) = z^2$ , curve  $c$  is cut cylindrical surface  $\frac{x^2}{9} + \frac{y^2}{25} = 1$  and the plane  $4x - 3z = 0$ .

### IV.3. Application of line integral of a scalar function

- Compute length  $l$  of curve  $c$ .

**Example 459:**  $c = \{ [x, y] \in \mathbb{E}_2 : y = 2 - \ln(\cos x), x \in \langle 0, \frac{\pi}{4} \rangle \}$

**Example 460:**  $c = \{ [x, y] \in \mathbb{E}_2 : x = t^2, y = t - \frac{t^3}{3}, t \in \langle -\sqrt{3}, \sqrt{3} \rangle \}$

**Example 461\*:**  $c = \{ [x, y] \in \mathbb{E}_2 : x = a \cos^3 t, y = a \sin^3 t, t \in \langle 0, 2\pi \rangle \}$

**Example 462\*:**  $c$  is a part of logarithmic spiral  $r = ae^{k\varphi}$  inside a circle  $r = a$ ,  $k \geq 0$ ,  $a \geq 0$ .

**Example 463\*:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = \int_1^t \frac{\cos z}{z} dz, y = \int_1^t \frac{\sin z}{z} dz, t \in \mathbb{R} \}$ . Compute the distance between origin and the closest point, in which a tangent is parallel with  $y$ -axis.

- Compute area of given cylindrical surfaces between  $xy$ -plane and given surfaces.

**Example 464\*:**  $y^2 = 4x$ ,  $z = 2\sqrt{x - x^2}$

**Example 465\*:**  $x^2 + y^2 = \frac{1}{4}$ ,  $z = xy$ ,  $x \geq 0$ ,  $y \geq 0$

- Compute mass of a curve  $c$  with longitudinal mass density  $\rho = \rho(x, y)$  resp.  $\rho(x, y, z)$ :

**Example 466:**  $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2, x \geq 0, y \geq 0 \}$ ,  $\rho(x, y) = x$

**Example 467:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = at, y = \frac{a}{\sqrt{2}}t^2, x = \frac{a}{3}t^3, t \in \langle 0, 1 \rangle \}$ ,  $\rho = \sqrt{\frac{2y}{a}}$

**Example 468:** The curve  $c$  is the first thread of the helix  $x = a \cos t$ ,  $y = a \sin t$ ,  $z = at$  and the longitudinal density equals to square distance from  $z$ -axis.

**Example 469:**  $c = c_1 \cup c_2$ ;  $c_1 = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 2ax, y \geq 0 \}$ ,  
 $c_2 = \{ [x, y] \in \mathbb{E}_2 : y = 0, x \in \langle 0, 2a \rangle, a > 0 \}$ ,  $\rho(x, y) = x^2 + y^2$

- Find a center of mass of a curve  $c$  with mass density  $\rho(x, y)$  or  $\rho(x, y, z)$  respectively:

**Example 470:**  $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 1, y \geq 0 \}$ ,  $\rho(x, y) = a(1 - y)$ ,  $a > 0$

**Example 471:**  $c = \{ [x, y] \in \mathbb{E}_2 : x = a(t - \sin t), y = a(1 - \cos t), a > 0, t \in \langle 0, 2\pi \rangle \}$ ,  $\rho(x, y) = 1$

**Example 472:**  $c = c_1 \cup c_2 \cup c_3$ ;  $c_1 = \{ [x, y, z] \in \mathbb{E}_3 : x^2 + y^2 = a^2, z = 0, x \geq 0, y \geq 0 \}$ ,  
 $c_2 = \{ [x, y, z] \in \mathbb{E}_3 : x^2 + z^2 = a^2, y = 0, x \geq 0, z \geq 0 \}$ ,  
 $c_3 = \{ [x, y, z] \in \mathbb{E}_3 : y^2 + z^2 = a^2, x = 0, y \geq 0, z \geq 0 \}$ ,  $\rho(x, y) = 1$

**Example 473:** Compute moment of inertia relative to  $yz$ -plane of the curve

$$c = \{ [x, y, z] \in \mathbb{E}_3 : x = a \cos t, y = a \sin t, z = bt, t \in \langle 0, 2\pi \rangle \}, \text{ if } \rho(x, y, z) = x^2 + y^2.$$

**Example 474:** Compute moment of inertia relative to  $z$ -axis of the curve  $c \subset \mathbb{E}_3$ :

$$c = \{ [x, y, z] \in \mathbb{E}_3 : 2x^2 + y^2 = 2, x + z = 1 \}, \text{ if } \rho(x, y, z) = z.$$

- Consider a curve  $c$ , which has a longitudinal mass density  $\rho$ .
  - a) Suggest a parametrization  $X = P(t)$ ,  $t \in \langle a, b \rangle$  of the given curve  $c$  and compute length of vector  $\dot{P}(t)$ .
  - b) Compute mass of the curve  $c$  if a longitudinal mass density is  $\rho$ .
  - c) Indicate, what can be a meaning of the given integral, e.g., static moment or moment of inertia, under which density and relative to which object (point, line or plane).

**Example 475:** The curve is line segment  $AB$ , where  $A = [1, 0]$ ,  $B = [2, 3]$ ,  $\rho(x, y) = x^2 + y^2$ .

**Example 476:** The curve is line segment  $AB$ , where  $A = [1, 0]$ ,  $B = [1, 2]$ ,  $\rho(x, y) = x^2 y$ .

**Example 477:**  $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 9, x \geq 0 \}$ ,  $\rho(x, y) = x$

**Example 478:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = 3 \cos t, y = 3 \sin t, z = t/3, t \in \langle 0, 3 \rangle \}$ ,  
 $\rho(x, y, z) = x^2 + y^2 + z^2$

**Example 479:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = 2 \cos t, y = 2 \sin t, z = t/4, t \in \langle 0, 2\pi \rangle \}$ ,  
 $\rho(x, y, z) = z^2 / (x^2 + y^2)$

**Example 480:**  $c = \{ [x, y] \in \mathbb{E}_2 : y = \frac{x^2}{2} + 2 \text{ between points } A = [0, 2], B = [2, 4] \}$ ,  $\rho(x, y) = x$

**Example 481:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = t \cos t, y = t \sin t, z = t, t \in \langle 0, 1 \rangle \}$ ,  $\rho(x, y, z) = z$

- Compute length  $l$  of a given curve  $c$ :

**Example 482:**  $c = \{ [x, y] \in \mathbb{E}_2 : y = \frac{1}{3}x\sqrt{x}, x \in \langle 0, 5 \rangle \}$

**Example 483:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = 3t, y = 3t^2, z = 2t^3, t \in \langle 0, 1 \rangle \}$

**Example 484:**  $c = \{ [\varphi, r] \in \mathbb{E}_2 : r(\varphi) = a(1 + \cos \varphi), \varphi \in \langle 0, \pi \rangle, a > 0 \}$   
*[upper half of cardioide]*

**Example 485:**  $c = \{ [\varphi, r] \in \mathbb{E}_2 : r(\varphi) = \sin^3 \frac{\varphi}{3}, \varphi \in \langle 0, 3\pi \rangle \}$

**Example 486:**  $c = \left\{ [x, y] \in \mathbb{E}_2 : y = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x} \, dx \right\}$  *[use  $\cos x \geq 0 \Rightarrow x \in (-\pi/2, \pi/2)$ ]*

**Example 487:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = 2 \cos t, y = 2 \sin t, z = t/2, t \in \langle 0, 2\pi \rangle \}$

**Example 488:**  $c = \{ [x, y] \in \mathbb{E}_2 : y = \frac{1}{3}x\sqrt{x}, x \in \langle 0, 5 \rangle \}$

**Example 489:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = R \cos t, y = R \sin t, z = at, t \in \langle 0, 2\pi \rangle \}$

- Compute mass  $m$  of a curve  $c$  with longitudinal mass density  $\rho(x, y)$  or  $\rho(x, y, z)$  resp.

**Example 490:**  $\rho = x(x^2 + z^2)$ ,  $c = \{ [x, y, z] \in \mathbb{E}_3 : y^2 + 2z^2 = 4, x = z, x \geq 0 \}$

**Example 491:**  $\rho = x(y + 2)$ ,  $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4, x \geq 0 \}$

**Example 492:**  $\rho = x^{4/3} + y^{4/3}$ ,  $c = \{ [x, y] \in \mathbb{E}_2 : x = a \cos^3 t, y = a \sin^3 t, t \in \langle 0, \pi/2 \rangle \}$

**Example 493:**  $\rho = e^{x^2+y^2}$ ,  $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2, x \geq 0, y \geq 0 \}$

**Example 494:** Compute moment of inertia of homogeneous half circle with radius  $a$  relative to axis of symmetry.

**Example 495:** Compute moment of inertia of the curve  $c = \{ [x, y] \in \mathbb{E}_2 : x = a \cos^3 t, y = a \sin^3 t, t \in \langle 0, \pi/2 \rangle \}$  relative to  $x$ -axis, if  $\rho = 1$ .

- Find center of mass  $T$  of a curve  $c$  with longitudinal mass density  $\rho$ .

**Example 496:**  $c = \{ [x, y, z] \in \mathbb{E}_3 : x = a \cos t, y = a \sin t, z = at, a > 0, t \in \langle 0, 2\pi \rangle \}$ ,

$$\rho = \frac{z^2}{x^2 + y^2}$$

**Example 497:**  $c = c_1 \cup c_2$ ;  $c_1 = \{ [x, y] \in \mathbb{E}_2 : y = 6\sqrt{x}, x \in \langle 1, 6 \rangle \}$ ,  
 $c_2 = \{ [x, y] \in \mathbb{E}_2 : y = -6\sqrt{x}, x \in \langle 1, 6 \rangle \}$ ,  $\rho = 1$

## IV.4. Line integral of a vector function

Consider a smooth single curve  $c$  with parametrization  $P(t)$ ,  $t \in \langle a, b \rangle$ . Then

$$\int_c \vec{f}(X) d\vec{s} = \pm \int_a^b \vec{f}(P(t)) \cdot \dot{P}(t) dt$$

The sign plus or minus depends on accordance of curve's orientation and orientation induced by parametrization  $P(t)$ .

Comment: Line integral of vector function  $\vec{f} = (U, V, W)$  can be written using differentials in the form

$$\int_c \vec{f} d\vec{s} = \int_c (U, V, W) \cdot (dx, dy, dz) = \int_c (U dx + V dy + W dz).$$

Analogously for  $c \subset \mathbb{E}_2$ .

- Compute given line integrals on oriented curve  $c$  with initial point  $A$ .

**Example 498:**  $\int_c (x, -y^2) d\vec{s}$ ,  $c$  is the line segment from  $A = [1, -2]$  into  $B = [3, 2]$ .

**Example 499:**  $\int_c (x^2 - y^2, 1) d\vec{s}$ ,  $c = \{[x, y] \in \mathbb{E}_2 : y = x^3\}$ , from  $A = [0, 0]$  into  $B = [3, 27]$ .

**Example 500:**  $\int_c (-y, x) d\vec{s}$ ,  $c = \{[x, y] \in \mathbb{E}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x \geq 0, y \geq 0\}$ ,  $A = [a, 0]$ .

**Example 501:**  $\int_c (y, -x, z) d\vec{s}$ ,  $c = \{[x, y, z] \in \mathbb{E}_3 : x = R \cos t, y = R \sin t, z = \frac{at}{2\pi} R > 0\}$ ,  
from intersection with the plane  $z = 0$  into intersection with the plane  $z = a$ ,  $a > 0$ .

**Example 502:**  $\int_c -x \cos y dx + y \sin x dy$ ,  $c$  is the line segment from  $A = [0, 0]$  into  $B = [\pi, 2\pi]$ .

**Example 503:**  $\oint_c x dy$ ,  $c$  is a perimeter of the triangle between lines  $x = 0$ ,  $y = 0$  and  $2x + 7y = 14$ . The curve  $c$  is oriented positively.

**Example 504:**  $\int_c (x \cos y, 0) d\vec{s}$ ,  $c$  is oriented line segment from  $A = [0, 1]$  into  $B = [1, 2]$ .

**Example 505:**  $\int_c (x^2 + y^2, x^2 - y^2) d\vec{s}$ ,  $c$  is oriented curve  $y = 1 - |1 - x|$ ,  $x \in \langle 0, 2 \rangle$ , initial point is  $A = [0, 0]$ .

**Example 506:**  $\int_c (x^2 - 2xy) dx + (y^2 - 2xy) dy$ ,  $c$  is arc of parabola  $y = x^2$  from  $A = [-1, 1]$  into  $B = [1, 1]$ .

**Example 507:**  $\int_c (y, x) d\vec{s}$ ,  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2, x \geq 0, y \geq 0, a > 0\}$ , initial point is  $A = [a, 0]$ .

## IV.5. Work along a curve

- Compute a work  $W$  done by a given force  $\vec{f}$  along a curve  $c$ :

**Example 508:**  $\vec{f} = (x + y, 2x)$ ,  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = R^2, y \geq 0\}$ , initial point is  $B = [-R, 0]$ .

**Example 509:**  $\vec{f} = \left( \frac{2y}{x^2 + 4y^2}, \frac{-2x}{x^2 + 4y^2} \right)$ ,  $c = \left\{ [x, y] \in \mathbb{E}_2 : \frac{x^2}{4} + y^2 = 1 \right\}$ , the curve  $c$  is oriented positively.



**Example 510:**  $\vec{f} = -y\vec{i} + x\vec{j} + a\vec{k}$ ,  $c = \{[x, y, z] \in \mathbb{E}_3 : x^2 + y^2 - 4x + 3 = 0, z = 2\}$ , the orientation of  $c$  is given by tangent vector  $\vec{\tau}([2, 1, 2]) = -\vec{i}$ .

**Example 511:**  $\vec{f} = (xy, x + y)$ ,  $c = c_1 \cup c_2$ ,  $c$  is closed curve, where  $c_1$  is a part of parabola  $y = x^2$  and  $c_2$  is a part of straight line  $y = x$ ,  $c$  is positively oriented.

- Compute a work done by a given force  $\vec{f}$  along a curve  $c$ :

**Example 512:**  $\vec{f} = \frac{(x - y, x + y)}{x^2 + y^2}$ ,  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4\}$ , curve  $c$  is negatively oriented.

**Example 513:**  $\vec{f} = \frac{2}{x^2 + y^2}(y, -x)$ ,  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = 16\}$ , curve  $c$  is positively oriented.

**Example 514:**  $\vec{f} = \left(\frac{1}{y}, -\frac{1}{x}\right)$ ,  $c$  is a perimeter of  $\triangle ABC$ , where  $A = [1, 1]$ ,  $B = [2, 1]$ ,  $C = [2, 2]$ . Curve  $c$  is positively oriented.

**Example 515:**  $\vec{f} = \frac{(y^2, -x^2)}{x^2 + y^2}$ ,  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2, a > 0, y \geq 0\}$  from the point  $[a, 0]$  into the point  $[-a, 0]$ .

**Example 516:**  $\vec{f} = (y, 2)$ ,  $c$  is a closed curve formed by the positive parts of the half-axes and quarter of an ellipse  $x = 2 \cos t$ ,  $y = \sin t$  in the first quadrant. Orientation is negative.

**Example 517:**  $\vec{f} = (x + y, 2x)$ ,  $c = \{[x, y] \in \mathbb{E}_2 : x = a \cos t, y = a \sin t, t \in \langle 0, 2\pi \rangle\}$ , orientation is positive.

**Example 518:**  $\vec{f} = (y, z, x)$ ,  $c$  is a line segment with initial point  $[a, 0, 0]$  and terminal point  $[a, a, a]$ .

**Example 519:**  $\vec{f} = (y, z, x)$ ,  $c$  is an intersection of surfaces  $z = xy$ ,  $x^2 + y^2 = 1$  from the point  $[1, 0, 0]$  into the point  $[0, 1, 0]$ .

**Example 520:**  $\vec{f} = (yz, z\sqrt{R^2 - y^2}, xy)$ ,  $c = \{X \in \mathbb{E}_3 : X = P(t), P(t) = \left(R \cos t, R \sin t, \frac{at}{2\pi}\right), a > 0, R > 0, t \in \langle 0, 2\pi \rangle\}$ , curve's orientation is in accordance with parametrization.

**Example 521:**  $\vec{f} = (x, y, xz - y)$ ,  $c = \{X \in \mathbb{E}_3 : X = P(t), P(t) = (t^2, 2t, 4t^3), t \in \langle 0, 1 \rangle\}$ , curve's orientation is in accordance with parametrization.

**Example 522:**  $\vec{f} = (x, y, z)$ ,  $c$  is a quarter of ellipse  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4, x + z = 2\}$  from  $[2, 0, 0]$  into  $[0, 2, 2]$ .

**Example 523:**  $\vec{f} = (y^2, z^2, x^2)$ ,  $c = \{X \in \mathbb{E}_3 : X = P(t), P(t) = (5, 2 + 4 \sin t, -3 + 4 \cos t), t \in \langle 0, 2\pi \rangle\}$ , curve's orientation is in accordance with parametrization.

- Consider a vector field  $\vec{f}$  and an oriented curve  $c$ .
  - a) Sketch the given curve  $c \in \mathbb{E}_2$ .
  - b) Suggest its parametrization  $P(t)$  and justify, if the curve  $c$  is oriented in accordance with your parametrization.
  - c) Compute a work, which is done by the force  $\vec{f}$  along the oriented curve  $c$ .

**Example 524:**  $\vec{f} = (y, z, x)$ ,  $c$  is a line segment with initial point  $[a, 0, 0]$  and terminal point  $[a, a, a]$ .

**Example 525:**  $\vec{f} = (xy, y - 1)$ ,  $c$  is a segment of the curve  $y = x^2$  with initial point  $A = [0, 0]$  and terminal point  $B = [2, 4]$ .

**Example 526:**  $\vec{f} = (\sqrt{x} + y, x + \sqrt{y})$ ,  $c$  is a segment of the curve  $x = y^2$  from the point  $A = [8, 2]$  into the point  $B = [2, 1]$ .

**Example 527:**  $\vec{f} = \left(x^3, \frac{1}{y} \ln y\right)$ ,  $c$  is given by  $y = e^x$ , where  $|x| \leq 1$  and the initial point is at  $x = -1$ .

**Example 528:**  $\vec{f} = (2, xy)$ ,  $c = \{[x, y] \in \mathbb{E}_2 : y = \ln x + 1, x \in \langle 1, e \rangle\}$

**Example 529:**  $\vec{f} = (0, x)$ ,  $c = \{[x, y] \in \mathbb{E}_2 : xy = 1, x \in \langle 2, 1 \rangle\}$

**Example 530:**  $\vec{f} = \left(\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2}\right)$ ,  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4, x \geq 0\}$  is oriented from  $[0, 2]$  to  $[0, -2]$ .

**Example 531:**  $\vec{f} = \left(\frac{2y}{x^2 + 4y^2}, -\frac{2x}{x^2 + 4y^2}\right)$ ,  $c = \{[x, y] \in \mathbb{E}_2 : \frac{x^2}{4} + y^2 = 1\}$ , with negative orientation.

- Let us have a line segment  $AB$ , a vector function  $\vec{f}$  and a scalar function  $\rho$ .
  - a) Suggest a parametrization  $P(t)$  of the line segment and compute tangent vector  $\dot{P}(t)$ .
  - b) Using a line integral of the vector function compute a work, which is done by force  $\vec{f}$  along the line segment  $AB$  from  $A$  to  $B$ .
  - c) Compute a mass of the curve  $k$ , if a longitudinal mass density is  $\rho(x, y)$ .

**Example 532:**  $A = [1, 0]$ ,  $B = [2, 3]$ ,  $\vec{f}(x, y) = (x^2, xy)$ ,  $\rho(x, y) = x^2 + y^2$

**Example 533:**  $A = [0, 1]$ ,  $B = [1, 2]$ ,  $\vec{f}(x, y) = (x\sqrt{y^2 - 2x}, 0)$ ,  $\rho(x, y) = xy$

## IV.6. Green's theorem

The line integral of vector field along a closed curve we call *circulation of vector field  $\vec{f}$  around the circle  $c$*  and write  $\oint_c \vec{f} \cdot d\vec{s}$ .

Let 1) the vector function  $\vec{f} = (U(x, y), V(x, y))$  has continuous partial derivatives in a region  $G \subset \mathbb{E}_2$ ,

2) curve  $c \subset G$  is positively oriented, closed, simple and smooth by parts,

3)  $\text{int}c \subset G$ .

Then

$$\oint_c \vec{f} \cdot d\vec{s} = \iint_{\text{int}c} \left( \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) dx dy.$$

Comment: If the curve is oriented negatively, the integral has negative sign.

Comment: In differentials, the Green's theorem has the following form:

$$\oint_c U dx + V dy = \iint_{\text{int}c} \left( \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) dx dy.$$

**Example 534:** Using Green's theorem, compute circulation of the vector field

$\vec{f} = (2x + 3y, 5x - y - 4)$  along a perimeter of triangle  $ABC$  in direction  $A \rightarrow B \rightarrow C$ , where  $A = [1, 0]$ ,  $B = [1, -3]$ ,  $C = [-3, 0]$ .

**Example 535:** Prove the existence of integral  $\oint_c \left( \ln(x^2 + y^2), -2\text{arctg}\frac{y}{x} \right) \cdot d\vec{s}$  and justify the possibility of use the Green's theorem in this case, if  $c \subset \mathbb{E}_2$  is positively oriented curve defined by one of the following equations:

a)  $x^2 + y^2 = 1$ ,

b)  $(x + 1)^2 + y^2 = 1$ ,

c)  $(x - 2)^2 + y^2 = 1$ ,

d)  $c$  is a perimeter of square with vertices  $A = [1, 0]$ ,  $B = [0, 1]$ ,  $C = [-1, 0]$ ,  $D = [0, -1]$ .

If the integral exists, compute them using Green's theorem.

**Example 536:** Compute circulation of the vector field  $\vec{f} = (-y, x)$  along the negatively oriented curve  $c = c_1 \cup c_2$ , where  $c_1 = \{[x, y] \in \mathbb{E}_2 : x^2 - 2x + y^2 = 0, y \geq 0\}$  and

$c_2 = \{[x, y] \in \mathbb{E}_2 : y = 0, x \in \langle 0, 2 \rangle\}$

a) directly as line integral,

b) using Green's theorem.

**Example 537\*:** Using line integral, compute area of interior of the curve

$c = \{[x, y] \in \mathbb{E}_2 : x^{2/3} + y^{2/3} = a^{2/3}, a > 0\}$ .

- There is given a set  $D$  and vector field  $\vec{f}$ .
  - a) Write Green's theorem (both assumptions and statement).
  - b) Sketch the set  $D$  and highlight a curve  $c$ , which forms a positively oriented boundary of  $D$ .
  - c) Compute a circulation of vector field  $\vec{f}$  along curve  $c$ .

**Example 538:**  $D = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$ ,  $\vec{f} = (xy, x^2 + 2x)$

**Example 539:**  $D = \{[x, y] \in \mathbb{E}_2 : 0 \leq x \leq 2, 0 \leq y \leq 2x\}$ ,  $\vec{f} = (y^2 - 3y, xy)$

**Example 540:**  $D = \{[x, y] \in \mathbb{E}_2 : 0 \leq x \leq 3, 0 \leq y \leq 1\}$ ,  $\vec{f} = (\frac{1}{3}y^3, x^2 + y^2)$

**Example 541:**  $D = \{[x, y] \in \mathbb{E}_2 : 0 \leq x \leq 4, 4 - 4x \leq y \leq 4\}$ ,  $\vec{f} = (y^2, (x + y)^2)$

**Example 539:** Examine an existence of integral  $\oint_c \left(-\frac{1}{x^2}, 2x\right) d\vec{s}$  and decide if Green's theorem can be used, if  $c \subset \mathbb{E}_2$  is negatively oriented curve:

- a)  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = 1\}$ ,
- b)  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + (y - 2)^2 = 1\}$ ,
- c)  $c = \{[x, y] \in \mathbb{E}_2 : (x - 2)^2 + y^2 = 1\}$ .

If yes, compute the integral using Green's theorem.

- There are given a vector field  $\vec{f}$  and a curve  $c$ .
  - a) Write Green's theorem and verify if its assumptions needed for computation the integral  $\oint \vec{f} d\vec{s}$  are met.
  - b) Compute the integral using Green's theorem.
  - c) Compute the same integral directly (without Green's theorem).

**Example 543:**  $\vec{f} = (-y, x)$ ,  $c = \{[x, y] \in \mathbb{E}_2 : x^2 + y^2 = 16\}$ , orientation is negative.