Mathematics II – Examples
II. Differential Calculus of Several Variables

II.1. Domain of definition of $z = f(x, y)$

- **Determine and draw domains of definitoin for the following functions:**

Example 42: $z = \dfrac{\ln(x^2 y)}{\sqrt{y-x}}$

Example 43: $z = \arcsin\dfrac{y-1}{x}$

Example 44: $z = \dfrac{\sqrt{x-y^2}}{\ln(1-x^2-y^2}$

Example 45: $z = \sqrt{\ln\dfrac{16}{x^2+y^2}}$

Example 46: $z = 3 - 7\ln(x + \ln y)$

Example 47: $z = \sqrt{2x+y-4} + \sqrt{16-x^2-y^2}$

Example 48: $z = \dfrac{3}{\sqrt{1-|x|-|y|}}$

Example 49: $z = \sqrt{xy-4}$

- **Determine and draw domain of definitoin, write the set gr(f) (graph of the function) and sketch the surface in $\mathbb{E}_3$, which represents the graph of the given function:**

Example 50: $f(x, y) = 4 - \sqrt{x^2+y^2}$

Example 51: $f(x, y) = \sqrt{x-y^2+2}$

Example 52: $f(x, y) = \sqrt{x^2-9y^2-36}$

II.2. Limits and continuity

- **Compute the following limits:**

Example 53: $\displaystyle\lim_{[x,y]\to[0,0]} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Example 54*: $\displaystyle\lim_{[x,y]\to[1,1]} \frac{x^3-y^3}{x^4-y^4}$, $[x, y] \in M$, where $M = \{[x, y] \in \mathbb{E}_2;\ x - y \neq 0\}$

Example 55: $\displaystyle\lim_{[x,y]\to[0,0]} \frac{e^{2(x^2+y^2)}-1}{x^2+y^2}$

Example 56: $\displaystyle\lim_{[x,y]\to[0,0]} \frac{\operatorname{tg}(x^2+y^4)}{3(x^2+y^4)}$

Example 57: Determine continuity of the function $f(x, y) = \begin{cases} \frac{x^2y^2}{\sqrt{x^2y^2+1}-1}, & [x, y] \neq [0, 0] \\ 2, & [x, y] = [0, 0] \end{cases}$

in the point $[0, 0]$.

- **Find domains of definitoin for the following functions:**

Example 58: $f(x,y) = \dfrac{x^2+x-12}{x^2+y^4+1}$

Example 59: $f(x,y,z) = \mathrm{e}^{z^2+x} \cdot \sin(x+y)$

Example 60: $f(x,y) = \dfrac{1}{x^2-2y}$

Example 61: $f(x,y) = \dfrac{x^4-y^4}{x^2+y^2}$

Example 62: $f(x,y,z) = \dfrac{1}{\ln\sqrt{(x^2+y^2+z^2)}}$

Example 63: $f(x,y,z) = \dfrac{\sin(x^2+y^2+z^2)^2}{x^2+y^2+z^2}$

Example 64: $f(x,y,z) = \dfrac{1}{|xy|+|z|}$

Example 65: $f(x,y,z) = \dfrac{y+4}{x^2y-xy+4x^2-4x}$

II.3. Partial derivatives

Notation: let us have a function $z = f(x,y)$. Then

first partial derivatives: $$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = z_x = \frac{\partial z}{\partial x}$$
$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = z_y = \frac{\partial z}{\partial y}$$

first partial derivatives in the point $[a,b]$:
$$\left.\frac{\partial z}{\partial x}\right|_{[a,b]} = f_x(a,b); \qquad \left.\frac{\partial z}{\partial y}\right|_{[a,b]} = f_y(a,b)$$

second partial derivatives: $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}; \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{yx};$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_y)_x = f_{xy}$$

- **Find all partial derivatives of the first order of the following functions:**

Example 66: $f(x,y) = (2x-3y)^4$

Example 67: $f(x,y) = 5x^4y^2 + \dfrac{x}{y} + 2x^2 - 3y$

Example 68: $f(x,y) = y^{x^2+3}, \quad y > 0$

Example 69: $f(x,y) = \dfrac{y}{\sqrt{x^2-y^2}}$, for $|x| > |y|$

Example 70: $f(x,y,z) = (x^y)^z$

- **Compute all partial derivatives of the first order of the following functions:**

Example 71: $f(x,y) = \ln(3x - y + 2)$

Example 72: $f(x,y) = \dfrac{3x-2y}{y}$

Example 73: $f(x,y) = \ln(xy^2)$

Example 74: $f(x,y) = \cos x + \dfrac{1}{2}\sin y - 3$

Example 75: $f(x,y) = \dfrac{y}{\sqrt{x}} - x\sqrt{y}$

Example 76: $f(x,y) = (x^2 + y)\mathrm{e}^{-2x}$

Example 77: $f(x,y) = \dfrac{xy^2}{2} + \operatorname{arctg}\left(\dfrac{y}{x}\right)$

Example 78: $f(x,y) = \ln(xy) - \sqrt{x^2 + y^2 - 20}$

Example 79: $f(x,y) = x^3 + \dfrac{y^3}{3} - \dfrac{1}{6}x^2y^4 - 15x$

Example 80: $f(x,y) = \dfrac{y^2}{x^2 + y^2}$

Example 81: $f(\phi,\psi) = \sin\phi\cos\psi$

Example 82: $g(u,v) = v \cdot \operatorname{tg}(u^2v^3)$

Example 83: $f(t,u,v) = \ln(tu) - \mathrm{e}^{uv} + \cos(tv)$

Example 84: Find domain of differentiability of the function $f(x,y) = x\sqrt{x^2 - y^2}$.

Example 85: Consider the functio $f(x,y) = \ln(|x| + |y|) + \frac{1}{\sqrt{9-x^2-y^2}}$. Determine

a) inequalities, which describe domain of definition and sketch this domain,
b) the value of $f(A)$, where $A = [-1,2]$,
c) $\dfrac{\partial f}{\partial x}(A)$.

Example 86: Prove that the function $z = f(x,y) = y^2\sin(x^2 - y^2)$ satisfies the differential equation $y^2 z_x + xyz_y = 2xz$ for all $[x,y] \in \mathbb{E}_2$.
[Hint: It suffices to compute z_x and z_y and put them into the equation.]

- **Compute partial derivatives of the given function in the point A:**

Example 87: $z = \sqrt{x^2 - y^2}, \quad A = [2,0]$

Example 88: $z = \dfrac{y}{x}, \quad A = [3,2]$

Example 89: $f = x^2\mathrm{e}^y\sin z, \quad A = [1,0,\pi/6]$

Example 90: $f = \ln(x^2 - y + 3z), \quad A = [2,1,1]$