Mathematics II – Examples
II. Differential Calculus of Several Variables

II.4. Total differential and tangent plane

Notation: let us have a function $z = f(x, y)$. Then
the (total) differential of the function f in the point $A = [x_0, y_0]$:
$$\mathrm{d}f(A) = \frac{\partial f}{\partial x}(A) \cdot (x - x_0) + \frac{\partial f}{\partial y}(A) \cdot (y - y_0)$$
Denote $\mathrm{d}x = x - x_0, \mathrm{d}y = y - y_0$. Then
$$\mathrm{d}f(A) = \frac{\partial f}{\partial x}(A) \cdot \mathrm{d}x + \frac{\partial f}{\partial y}(A) \cdot \mathrm{d}y$$

Example 91: Let $f(x, y) = \frac{y}{x} - \frac{x}{y}$.
a) Determine and sketch domains, where the function f is differentiable.
b) Write the differential of f in the point $A = [x_0, y_0]$.

Example 92: Determine total differential and an approximate increment of the function $z = \frac{y}{x}$ in in the point $A = [2, 1]$ for $\triangle x = 0.1$ and $\triangle y = 0.2$. Compare them.

Example 93: Using total differential compute approximate increment of the function $z = \operatorname{arctg}\frac{y}{x}$ in if x changes from $x_0 = 1$ to $x_1 = 1.2$ and y changes from $y_0 = -3$ to $y_1 = -3.1$.

Example 94: Approximate a value of the expression $\ln\left(\sqrt{9.03} - \sqrt{0.99} - 1\right)$ using total differential of an appropriate function.

Example 95: Compute approximate value of the expression $0.98^{3.04}$ using total differential of an inappropriate function.

Example 96: Find an equation of both tangent plane τ and normal line n to the graph of the function $z = 2x^2 - 4y^2$ in the point $T = [2, 1, ?]$. Compute approximate value of this function in the point $[2.2, 1.3]$.

Example 97: Find an equation of tangent plane τ to the surface $z = x^2 + xy - y^2 + x + 3$ and parallel to the given plane $\rho:\ 5x - 3y - z = 0$.

- **Compute approximate values of a given expressions using total differential:**

Example 98: $\sqrt[3]{7.95} \cdot \sqrt{8.96}$

Example 99: $\dfrac{\sqrt[4]{0.97}}{1.02^3 \cdot \sqrt[3]{0.99}}$

Example 100: $\sqrt{4.04} \cdot \ln 1.02 \cdot \operatorname{arctg} 0.9$

- **Find an equation of both tangent plane τ and normal line n to the surface $z = f(x,y)$ in the point T:**

Example 101: $z = 4\sqrt{x^2+y^2}, \quad T = [3,4,?]$

Example 102: $z = xy, \quad T = [0,0,?]$

Example 103: $z = x^2 \cdot \cos\frac{1}{y}, \quad T = [1, \frac{2}{\pi}, ?]$

Example 104: $z = \frac{1}{x} \cdot \arcsin y, \quad T = [\frac{1}{2}, \frac{\sqrt{2}}{2}, ?]$

- **Find an equation of tangent plane τ to the surface $z = f(x,y)$ and parallel to the plane ρ:**

Example 105: $z = 2x^2 - y^2, \quad \rho: \; 8x - 6y - z - 15 = 0$

Example 106: $z = \ln(x^2 + 2y^2), \quad \rho: \; 2x - z + 5 = 0$

Example 107: $z = x^2 - y^2 + 6xy + 2x, \quad \rho: \; 4x + 6y - z = 0$

II.5. Derivatives and differentials of high order

Example 108: Find all partial derivatives of second order of the function $f(x,y) = xy^3 - y \cdot \mathrm{e}^{x+y^2}$.

Example 109: Prove that the function $u = u(x,t) = \operatorname{arctg}(2x - t)$ satisfies the partial differential equation $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial t \partial x} = 0$ in $\mathbb{E}_2$.

Example 110*: There is given the function $f(x,y)$

$$f(x,y) = \begin{cases} xy\frac{x^2-2y^2}{x^2+y^2} & \text{for } [x,y] \neq [0,0], \\ 0 & \text{for } [x,y] = [0,0]. \end{cases}$$

Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Example 111: Let us consider the scalar field $\phi(x,y,z) = xy^2 + z^3 - xyz + 3$. Compute $\operatorname{grad}\phi, \quad \operatorname{rot}\operatorname{grad}\phi$.

Example 112: Let us consider the vector field $\vec{f} = (U,V,W) = \left(xy, x^2 - z^2, \frac{y}{x+z}\right)$. Compute $\operatorname{div}\vec{f}, \quad \operatorname{rot}\vec{f}, \quad \operatorname{div}\operatorname{rot}\vec{f}$.

Example 113*: Let the scalar field $\phi(x,y,z)$ has continuous partial derivatives of second order in a domain $D \subset \mathbb{E}_3$. Prove that $\operatorname{rot}\operatorname{grad}\phi = \vec{0}$ in D.

Example 114*: Let the vector field $\vec{f} = (U,V,W)$ has continuous partial derivatives of second order in a domain $D \subset \mathbb{E}_3$. Prove that $\operatorname{div}\operatorname{rot}\vec{f} = 0$ in D.

- **Find differentials of the given order:**

Example 115*: $z = \sin(2x+y),\quad \mathrm{d}^2 z = ?$

Example 116*: $z = x^3 - y^3 - xy + y^2,\quad \mathrm{d}^3 z = ?$

Example 117*: $u = \mathrm{e}^{2x-3y},\quad \mathrm{d}^2 u(A) = ?,\quad \mathrm{d}^3 u(A) = ?,\quad \mathrm{d}^n u(A) = ?,\quad A = [0,0].$

Differentials can be used in the iportant **Taylor theorem**:
Let a function $f(x,y)$ is differentiable$(n+1)$ times in any interior point of a rectangle M with a center in the point $A = [x_0, y_0]$. Then for any point $[x,y] \in M$ there exists a point $[\xi, \eta] \in M$, such that

$$f(x,y) = f(A) + df(A) + \frac{d^2 f(A)}{2!} + \cdots + \frac{d^n f(A)}{n!} + R_{n+1},$$

where $df(A) = df(x_0,y_0) = \dfrac{\partial f}{\partial x}(A)\cdot(x-x_0) + \dfrac{\partial f}{\partial y}(A)\cdot(y-y_0),$

$$d^2 f(A) = \frac{\partial^2 f}{\partial x^2}(A)\cdot(x-x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(A)\cdot(x-x_0)(y-y_0) + \frac{\partial^2 f}{\partial y^2}(A)\cdot(y-y_0)^2,$$

$$\vdots$$

$$d^n f(A) = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^n f}{\partial x^k \partial y^{n-k}}(A)\cdot(x-x_0)^k (y-y_0)^{n-k},$$

$$R_{n+1} = \frac{1}{(n+1)!} d^{n+1} f(\xi, \eta).$$

Example 118*: Write Taylor expansion of the function $f(x,y) = x^3 - 3xy^2 + y^2 + 4x - 5y$ in a neighborhood of the point $A = [2,-1]$ and use the result for approximation of the value of the function f in the point $[2.1, -1.1]$.

Example 119*: Write Taylor expansion of the fourth order of the function $f(x,y) = \cos(x^2+y^2)$ in a neighborhood of the point $[0,0]$.

- **Find partial derivatives of second order of the given function:**

Example 120: $\phi(s,t) = \ln(s^3+t)$

Example 121: $\phi(x,t) = \dfrac{\cos x^2}{t}$

Example 122: $f(x,y) = \mathrm{e}^{ax+by}$

Example 123: Verify that the function $u(x,t) = \sin(x-ct)$ and the function $u(x,t) = \sin(\omega ct)\cdot\sin(\omega t)$ satisfy the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

Example 124: Verify that the function $u(x,y) = \mathrm{e}^x \cdot \sin y$ satisfies the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

- **Expand the function $f(x, y)$ using Taylor's theorem in a neighborhood of a point A:**

Example 125*: $f(x, y) = x^3 + 5x^2 - 6xy + 2y^2, \quad A = [1, -2]$

Example 126*: $f(x, y) = x^2 + 3xy - y^3, \quad A = [2, -1]$

II.6. Gradient. Directional derivative

- **Determine the angle between two gradients of given functions in a point A:**

Example 127: $f(x, y, z) = x^y + yz, \quad g(x, y, z) = \sin(xz) + x + y - \frac{z}{y} - 1, \quad A = [1, 1, 0]$

Example 128: $f(x, y) = \operatorname{arctg} \frac{x}{,} \quad g(x, y) = y\sqrt{x}, \quad A = [1, 1]$

Example 129: Find a set in $D(f) \equiv \mathbb{E}_3$, in which the function $f(x, y, z) = x^2 + y^2 + z^2 - 2xyz$ has gradient, which equals to zero-vector $\vec{0}$.

Example 130: Find a set in $D(f) \equiv \mathbb{E}_2$, in which the function $f(x, y) = (x^2 + y^2)^{3/2}$ has a gradient of size 9.

- **Compute directional derivative of function f in direction $\vec{s}$ in a point** A:

Example 131: $f(x, y) = 2x^4 + yz + y^3, \quad \vec{s} = (3, -4), \quad A = [1, 2]$

Example 132: $f(x, y, z) = x^2 + 2y^2 - z^2, \quad A = [-3, 2, 4]$, direction $\vec{s}$ is given by the vector $\overrightarrow{A, B}$, where $B = [-2, 4, 2]$

Example 133: Compute directional derivative of the function $z = x^2 + \ln(x + y^2)$ in the point $A = [3, 2\sqrt{3}]$ in direction given by tangent line to parabola $y^2 = x$. Consider the vector with sharp angle with vector $\vec{i}$.

Example 134: Determine a direction, in which derivative of the function $f(x, y) = x^3y + \frac{x}{y^2} + 2y$ in the point $A = [-1, 1]$ in maximal. Compute this derivative.

Example 135: Let us have the function $z = \sqrt{2x + y}$, the point $A = [1, 2]$, vector $\vec{s} = (-1, 1)$. Determine

a) in which points the function z is differentiable.
b) $\frac{\partial z}{\partial \vec{s}}(A)$,
c) tangent plane to the graph of function z in the point $T = [1, 2, ?]$.

Example 136: Determine a vector $\vec{s}$, in which direction the speed of change of the function values of $f(x, y, z) = x^2 + y^2 + z^2 - 2xyz$ in the point $A = [1, -1, 2]$ is maximal. Compute this maximal speed.

Example 137: Determine points, in which gradient of the function $f(x,y,z) = x^2+y^2+z^2-2xyz$ is orthogonal to the axis x.

Example 138: Determine angle of vectors $\operatorname{grad} f(A)$ and $\operatorname{grad} g(A)$, where $f(x,y,z) = x - 3y + \sqrt{3xy} + z^3$, $g(x,y,z) = z\sqrt{x^2+y^2} + xyz$, $A = [3,4,0]$.

- **Compute gradient of given function**:

Example 139: $f(x,y) = \dfrac{1}{\sqrt{(x^2+y^2)^3}}$

Example 140: $f(x,y) = \sin(x^2y) + \dfrac{x^2}{3}$

Example 141: $f(x,y) = \ln(x + \sqrt{x^2+y^2})$

Example 142: $f(x,y,z) = x^2yz + \ln y - 15$

Example 143: $f(u,v,t) = t\sqrt{u^2+v^2}$

- **Compute gradient of function in a given point** A: