Mathematics II – Examples
III. Double and triple intgral

III.1. Existence

Existence Theorem: Let D be a measurable (in the sense of Jordan) set in $\mathbb{E}_2$ (resp. in $\mathbb{E}_3$) and let f be a bounded function on D. Let the set of discontinuities has measure 0. Then f is integrable on D, i.e., there exists the integral

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad \left(\iiint_D f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \text{ resp.}\right)$$

Note: More detailed information you find in the textbook J. Neustupa, Mathematics II.

Example 237: Decide if exists the integral $\displaystyle\iint_D \frac{1}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, where

a) $D = \{[x,y] \in \mathbb{E}_2 : x^2 + (y-1)^2 \le \frac{1}{4}\}$;
b) $D = \{[x,y] \in \mathbb{E}_2 : x^2 + y^2 \le 1\}$;
c) $D = \{[x,y] \in \mathbb{E}_2 : x^2 + (y-1)^2 \le 2\}$;

Example 238: Consider the set $D = \{[x,y] \in \mathbb{E}_2 : x^2 + y^2 \le 4\}$. Check existence of the following integrals:

a) $\displaystyle\iint_D \frac{\sin(x^2+y^2)}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, b) $\displaystyle\iint_D \frac{x\,\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$, c) $\displaystyle\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2+1}$,

d) $\displaystyle\iint_D \frac{1}{1+xy}\,\mathrm{d}x\,\mathrm{d}y$, e) $\displaystyle\iint_D \frac{1}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y$.

Example 237: Decide if exists the following integrals:

a) $\displaystyle\iiint_W \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(1+x+z)^3}$, $W = \{[x,y,z] \in \mathbb{E}_3 : 0 \le x \le 2,\ -1 \le y \le 0,\ -2 \le z \le 2\}$;

b) $\displaystyle\iiint_W (x+yz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W = \{[x,y,z] \in \mathbb{E}_3 : x^2 \le y \le 2,\ z \ge 3\}$;

c) $\displaystyle\iiint_W \frac{1}{x^2+y^2+z^2-9}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W = \{[x,y,z] \in \mathbb{E}_3 : 1 < x^2+y^2+z^2 < 4\}$;

III.2. Fubini's Theorem for double integral

The set $D = \{[x,y] \in \mathbb{E}_2;\ a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\,\}$, where $\phi_1(x)$ and $\phi_2(x)$ are continuous functions on $\langle a, b\rangle$ and $\phi_1(x) \le \phi_2(x)$, is called the elementary region relative to the x-axis.

Let D be an elementary region relative to the x-axis. Let the function $f(x,y)$ is continuous in D. Then

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Note: The same assertion is valid relative to the y-axis.

- **Compute the following double integrals od given rectangular domains:**

Example 240: $I = \displaystyle\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{x^2 - 2xy + y^2}, \quad D = \{[x,y] \in \mathbb{E}_2 :\ 3 \le x \le 4,\ 0 \le y \le 2\,\}$

Example 241: $I = \displaystyle\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{(x - 2y + 3)^2}, \quad D = \{[x,y] \in \mathbb{E}_2 :\ 0 \le x \le 2,\ 0 \le y \le 1\,\}$

Example 242: $I = \displaystyle\iint_D y^2 \sin^2 x\,\mathrm{d}x\,\mathrm{d}, \quad D = \{[x,y] \in \mathbb{E}_2 :\ 0 \le x \le \frac{\pi}{2},\ 1 \le y \le 2\,\}$

Hint: If f can be written in the form $f(x,y) = g(x)\cdot h(y)$ and the set D is a rectangle $D = \langle a,b\rangle \times \langle c,d\rangle$, then $\iint_D f(x,y)\ dx\ dy = \int_a^b g(x)\ dx \int_c^d h(y)\ dy$.

Example 243: $I = \displaystyle\iint_D \frac{xye^{x^2}}{y^2+3}\,\mathrm{d}x\,\mathrm{d}y, \quad D = \{[x,y] \in \mathbb{E}_2 :\ 0 \le x \le 2,\ 0 \le y \le 3\,\}$

- **Let the set M be bounded by given curves. Sketch this set and express them as an elementary region.**

Example 244: $2x - y = 1, \quad 2x - y = 5, \quad x = 0, \quad x = 2$

Example 245: $y = 0, \quad x = 2y, \quad x = 4$

Example 246: $y = 18 - x^2, \quad y = x^2$

Example 247: $xy = 4, \quad y = x, \quad y = 4x, \quad (x \ge 0)$

- **Change order of integration:**

Example 248: $I = \int_0^1 \left(\int_0^{1-y} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$

Example 249: $I = \int_0^1 \left(\int_{x^2}^{x} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$

Example 250: $I = \int_0^4 \left(\int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$

Example 251: $I = \int_0^4 \left(\int_0^{\frac{y}{2}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y + \int_4^6 \left(\int_0^{6-y} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$

- **Using Fubini's theorem transform the double integral $\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ to two-fold integrals in both directions of integration (relative to x-axis and/or relative to y-axis), if the set $D \subset \mathbb{E}_2$ is bounded by the given curves:**

Example 252: $x = 0, \quad y = x^2, \quad x + y = 2, \quad (x \geq 0)$

Example 253: $x = y^2 - 4, \quad x = -3y$

- **Sketch the set $D \subset \mathbb{E}_2$ bounded by given curves and compute given integral:**

Example 254: $\iint_D \frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; xy = 1, \quad y = 4x, \quad x = 3$

Example 255: $\iint_D x^3 y^2\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; x^2 + y^2 = a^2, \quad x = 0 \;\; (x \geq 0)$

Example 256: $\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+1}, \quad D: \; y = 2x - x^2, \quad y = -x$

Example 257: $\iint_D (x+y)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; y^2 - x^2 \leq 1, \quad y \geq 0, \quad x \in \langle -2,\, 2 \rangle$

Example 258: $\iint_D (1+x)y\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; y = x^2 - 4, \quad y = -3x, \quad x \leq 0$

Example 259: $\iint_D (x+1)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; y = 2x, \quad 2y = x, \quad y = 2$

Example 260: $\iint_D \frac{1}{y}\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; xy = 1, \quad y = x, \quad x = 4, \quad x \geq 0$

Example 261: $\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{(x+y)^2}, \quad D: \; x = 3, \quad x = 4, \quad y = 1, \quad y = 2$

Example 262: $\displaystyle\iint_D \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; x=0, \quad y=\pi, \quad y=x$

Example 263: $\displaystyle\iint_D (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; y=0, \quad y=1-x, \quad y=1+x$

Example 264: $\displaystyle\iint_D (x+2y)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; x=y^2-4, \quad x=5$

Example 265: $\displaystyle\iint_D xy\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; y=x-4, \quad y^2=2x$

Example 266: $\displaystyle\iint_D \frac{1}{y+1}\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; x=0, \quad y=2, \quad y=4, \quad y^2=x$

Example 267: $\displaystyle\iint_D \frac{x}{y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; y^2=x, \quad y^2=4x, \quad y=2$

Example 268: $\displaystyle\iint_D (xy+y)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; x=1, \quad x=2, \quad xy=4, \quad y=0$

Example 269: Transform double integral of the function $f(x,y)=1/x$ on set $D=\{[x,y]\in\mathbb{E}_2: y\geq \ln x,\ x\geq 1,\ y\leq 1\}$ into two-fold integrals (i.e., use both orders of integration relative to both x-axis and y-axis) and compute the integral.

- **The set $D\subset\mathbb{E}_2$ is bounded by given inequalities and/or border curves. There is given a function $f(x,y)$.**

  a) Sketch the set D (along with description of all important points, axes, etc.)

  b) Verify all conditions for use of Fubini's theorem.

  c) Express the set D as an elementary region relative to appropriately chosen axis.

  d) Compute $iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

Example 270: $D=\{[x,y]\in\mathbb{E}_2;\ 0\leq x\leq 1,\ 0\leq y\leq 2x+1\}, \quad f(x,y)=x^2y$

Example 271: $D=\{[x,y]\in\mathbb{E}_2;\ x+y\leq\pi,\ x-y\leq\pi,\ x\geq 0\}, \quad f(x,y)=\sin(x+y)$

Example 272: $D\subset\mathbb{E}_2$ is bounded by curves $y=x/2,\ y=3x,\ y=2.\ \ f(x,y)=x\sqrt{y}$

Example 273: $D=\{[x,y]\in\mathbb{E}_2;\ x\geq 0,\ x+y\leq 2,\ x\leq y^2\}, \quad f(x,y)=xy$

Example 274: $D=\{[x,y]\in\mathbb{E}_2;\ y\geq x^2,\ y\leq 12-x^2\}, \quad f(x,y)=|x|$

Example 275: $D\subset\mathbb{E}_2$ is bounded by curves $y=\sqrt{x},\ y=2\sqrt{x},\ x=1.\ \ f(x,y)=2xy$

Example 276: $D\subset\mathbb{E}_2$ is bounded by curves $y=x,\ y=1/x,\ y=2.\ \ f(x,y)=xy^2$

III.3. Substitution theorem for double integral

Let equations $x = \phi_1(u,v),\ y = \phi_2(u,v)$ define a mutually unique transformation of a region $B(u,v) \subset \mathbb{E}_2$ on a region $D \subset \mathbb{E}_2$. Then

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{B(u,v)} f\left(\phi_1(u,v), \phi_2(u,v)\right)\,|J|\,\mathrm{d}u\,\mathrm{d}v, \quad \text{where} \quad J = \begin{vmatrix} \frac{\partial\phi_1}{\partial u} & \frac{\partial\phi_1}{\partial v} \\ \frac{\partial\phi_2}{\partial u} & \frac{\partial\phi_2}{\partial v} \end{vmatrix}$$

Example 277: Decide if exists the integral $\displaystyle\iint_D \operatorname{arctg}\frac{y}{x}\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{[x,y] \in \mathbb{E}_2;\ x^2+y^2 \leq 1,\ y \geq 0,\ x \geq 0\}$. If yes, then compute this.

- **Compute the following integrals:**

Example 278: $\displaystyle\iint_D \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad D = \{[x,y] \in \mathbb{E}_2;\ x^2+y^2-bx \leq 0\},\ b > 0$

Example 279: $\displaystyle\iint_D \ln(1+x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y, \quad D = \{[x,y] \in \mathbb{E}_2;\ x^2+y^2 \leq a^2,\ x \leq 0\}$

Example 280*: $\displaystyle\iint_D x^3\,\mathrm{d}x\,\mathrm{d}y$, where $D \subset \mathbb{E}_2$ is a set bounded by curves $xy = 1,\ xy = 3,$ $y = \frac{x^2}{2},\ y = 2x^2.$

Example 281*: $\displaystyle\iint_D (2x-y)\,\mathrm{d}x\,\mathrm{d}y$, $D \subset \mathbb{E}_2$ is bounded by primes $x+y = 1,\ x+y = 4,$ $y = x,\ y = 5x.$

Example 282: $\displaystyle\iint_D \sqrt{1+4x^2+9y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad D = \{[x,y] \in \mathbb{E}_2;\ 4x^2+9y^2 \leq 36,\ y \geq 0\}$

Example 283: $\displaystyle\iint_D (x-2y+3)\,\mathrm{d}x\,\mathrm{d}y, \quad D = \{[x,y] \in \mathbb{E}_2;\ x^2+y^2 \leq a^2\},\ b > 0$

Example 284: $\displaystyle\iint_D x\,\mathrm{d}x\,\mathrm{d}y, \quad D = \{[x,y] \in \mathbb{E}_2;\ (x-2)^2 + \frac{(y-1)^2}{4} \leq 1\}$
(*Hint: use transformation* $x = 2 + r\cos\varphi,\ y = 1 + 2r\sin\varphi$.)

Example 285: $\displaystyle\iint_D \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad D = \{[x,y] \in \mathbb{E}_2;\ x^2+y^2 \leq 2x\}$

Example 286: $\displaystyle\iint_D (x^2+y)\,\mathrm{d}x\,\mathrm{d}y, \quad D = \{[x,y] \in \mathbb{E}_2;\ xy = 1,\ xy = 4,\ y = x,\ y = 9x\}$
(*Hint: use transformation* $xy = u,\ \frac{y}{x} = v$.)

- **For given set D and function $f(x,y)$ compute integral $\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$:**

Example 287: $D = \{[x,y] \in \mathbb{E}_2;\ \frac{x^2}{9} + \frac{y^2}{4} \le 1,\ x \ge 0\},\quad f(x,y) = xy^2$

Example 288: $D = \{[x,y] \in \mathbb{E}_2;\ x^2 + 9y^2 \le 9,\ x \ge 0\},\quad f(x,y) = y^2$

Example 289: $D = \{[x,y] \in \mathbb{E}_2;\ x^2 + 4y^2 \le 4,\ y \ge 0\},\quad f(x,y) = y\sqrt{x^2 + 4y^2}$

Example 290: $D = \{[x,y] \in \mathbb{E}_2;\ 36x^2 + y^2 \le 9,\ x \ge 0,\ y \ge 0\},\quad f(x,y) = xy$

Example 291: $D = \{[x,y] \in \mathbb{E}_2;\ x^2 + y^2 \le 4x,\ y \ge 0\},\quad f(x,y) = xy$

Example 292: $D = \{[x,y] \in \mathbb{E}_2;\ x^2 + y^2 \le 4,\ x \ge 0\},\quad f(x,y) = \mathrm{e}^{-x^2-y^2}$

III.4. Applications of double integrals

- **Compute area of a planar region $D \subset \mathbb{E}_2$ bounded by given curves:**

Example 293: $y = x^2,\ x + 2y = 3,\ y = 0$

Example 294: $xy = 1,\ xy = 4,\ y = x,\ x = 8$

Example 295: Compute area of planar region bounded by x-axis and one arc of cycloid $x = a(t - \sin t),\ y = a(1 - \cos t)$.

Example 296: Compute area of planar region D bounded by the curve $x^{2/3} + y^{2/3} = a^{2/3}$ (so called "asteroide").

- **Compute area of a planar region $D \subset \mathbb{E}_2$ bounded by given closed curve:**

Example 297: $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ (Lemniscate of Bernoulli)

Example 298*: $(x^2 + 9y^2)^2 = x^2 y$

Example 299*: $(x^3 + y^3) = 3axy$ (Folium of Descartes)

Example 300: Compute area of a planar region $D \subset \mathbb{E}_2$ bounded by the curves $x^2 + y^2 + x = 0,\ x^2 + y^2 + 4x = 0, y = x,\ y = 0$.

Example 301: Consider a parabolic segment with chord perpendicular to its axis. Length of the chord is a, high of the segment is h, planar mass density $\rho = 1$. Compute:
a) moment of inertia of this segment relative to the chord,
b) centre of mass of this segment.

Example 302: Find a center of mass of the planar plate bounded by curves $x^2 + y^2 - 2x = 0$, $x^2 + y^2 - 4x = 0$ with planar mass density $\rho = 10$.

Example 303: Find coordinates of a center of mass of the circular sector (see picture in Czech version), if $\rho =$const.

Example 304: Compute moment of inertia relative to origin of coordinates of the planar plate bounded by curves $x^2 + y^2 = 1,\ x^2 + y^2 = 4$ with planar mass density $\rho = k$.

Example 305: Find a center of mass of the planar plate bounded by **cardioid** $r = a(1 + \cos\varphi)$, $\varphi \in \langle 0, 2\pi \rangle,\ a > 0,\ \rho = 1$.

- **Consider $\iint_D f(x, y)\, \mathrm{d}x\, \mathrm{d}y$, where $D \subset \mathbb{E}_2$ is a bounded region.**
  a) Sketch the region D.
  b) Verify assumptions for use of Fubini's theorem.
  c) Provide at least two examples of possible physical meaning of the given integral (if it can express mass, static moment, moment of inertia, under which mass density and relative to which object).
  d) Compute the integral.

Example 306: D is bounded by curves $x = 1,\ x = y^2 + 2,\ y = 0,\ y = 2, \quad f(x, y) = y/\sqrt{x}$

Example 307: D is bounded by curves $y = x^2,\ y = \sqrt{x}, \quad f(x, y) = x$

Example 308: D is bounded by curves $x = y^2,\ x - y - 2 = 0, \quad f(x, y) = y^2$

Example 309: $D = \{[x, y] \in \mathbb{E}_2;\ x \geq 0,\ y \leq x + 2,\ y \geq x^2\}, \quad f(x, y) = 2x(y + 1)$

Example 310: D is bounded by curves $y = 2x,\ y = 2/x,\ x = 2, \quad f(x, y) = x^2 y$

Example 311: D is bounded by curves $y = x,\ y = 1/x,\ x = 3, \quad f(x, y) = \sqrt{x}$

Example 312: $D = \{[x, y] \in \mathbb{E}_2;\ x^2 + y^2 \leq 4,\ x \geq 0,\ y \geq 0\}, \quad f(x, y) = xy$

- **Compute area of planar region $D \subset \mathbb{E}_2$ bounded by given curves:**

Example 313: $x = y^2,\ 8x = y^2,\ y = 5$

Example 314: $y = x^2,\ x - y + 2 = 0,\ x = 0,\ x = 1$

Example 315: $x = y^2,\ xy = 1,\ x = 4y,\ (xy \geq 1)$

Example 316: $\left(4x^2 + \dfrac{y^2}{9}\right)^2 = xy$

Example 317: $y = \ln x,\ x - y = 1,\ y = -1$

Example 318: $y = \dfrac{x^2}{4},\ y = \dfrac{8}{4 + x^2}$

- **Compute mass m of planar plate $D \subset \mathbb{E}_2$ bounded by given curves, if mass density is $\rho(x, y)$:**

Example 319: $y = x^2,\ x - y + 2 = 0,\ \ \rho(x, y) = xy$

Example 320: $x^2 + y^2 = 2ax,\ \ \rho(x, y) = \sqrt{x^2 + y^2},\ a > 0$

Example 321: $x = y^2,\ xy = 1,\ x = 4,\ \ \rho(x, y) = 2x$

Example 322: $x^2 + y^2 = 1,\ x + y \geq 1,\ \ \rho(x, y) = y$

Example 323: $x^2 + y^2 - 2x = 0,\ x^2 + y^2 - 4x = 0,\ y = x,\ y = 0,\ \ \rho(x, y)$ equals to the distance of $[x, y]$ from origin.

- **Compute mass m of planar plate $D \subset \mathbb{E}_2$ with mass density $\rho(x, y)$:**

Example 324: $D = \{[x, y] \in \mathbb{E}_2;\ y \leq x + 2,\ y \geq x^2,\ x \geq 0\},\ \ \rho(x, y) = xy$

Example 325: $D = \{[x, y] \in \mathbb{E}_2;\ x \leq 4,\ x \geq y^2,\ y \geq 1/x\},\ \ \rho(x, y) = 2x$

Example 326: $D = \{[x, y] \in \mathbb{E}_2;\ x^2 + y^2 \leq 1,\ x + y \geq 1\},\ \ \rho(x, y) = y$

- **Find center of mass T of planar plate bounded by given curves:**

Example 327: $y = 2x - 3x^2,\ y = -x,\ \ \rho(x, y) = 1$

Example 328: $y = \sin x,\ y = 0,\ x \in \langle 0, \pi \rangle,\ \ \rho(x, y) = 1$

Example 329: $y^2 = 4x + 4,\ y^2 = -2x + 4, \quad \rho(x, y) = 1$

Example 330: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},\ x \geq 0,\ y \geq 0, \quad \rho(x, y) = 1$
(*Hint: use transformation* $x = r\cos^3\varphi,\ y = r\sin^3\varphi$.)

- **Compute moment of inertia**

Example 331: of a circle with radius a relative to its tangent, $rho(x, y) = 1$,

Example 332: of elipse $4x^2 + y^2 \leq 1$ relative to y-axis, $\rho(x, y) = y$,

Example 333: of a quarter circle with radius a relative to its axis of symmetry, $\rho(x, y) = 1$
(*Hint: locate the circle so that the axis of symmetry will be the* x*-axis.*)

Example 334: of part of annulus $x^2 + y^2 = 1,\ x^2 + y^2 = 4$, bounded by lines $y = x,\ y = 0$ in the I. quadrant with density $\rho(x, y) = k,\ (k > 0)$ relative to center of the annulus.

- **Consider $\iint_D f(x, y)\ \mathrm{d}x\,\mathrm{d}y$, where $D \subset \mathbb{E}_2$ is a bounded region.**
  a) Sketch a solid in $\mathbb{E}_3$ with volume equal to given integral.
  b) Sketch a projection of the solid to plane $z = 0$.
  c) Name the surface $z = f(x, y)$.
  d) Compute $\iint_D f(x, y)\ \mathrm{d}x\,\mathrm{d}y$.

Example 336: D is bounded by curves $y = x,\ y = 2x,\ x = 2, \quad f(x, y) = x + y$.
[c) plane]

Example 337: $D = \{[x, y] \in \mathbb{E}_2;\ x + y \leq 1,\ x + 1 \geq y \geq 0\}, \quad f(x, y) = x^2 + y^2$.
[c) rotational paraboloid]

Example 338: $D = \{[x, y] \in \mathbb{E}_2;\ y^2 - x^2 \leq 1,\ 0 \leq x \leq 2,\ y \geq 0\}, \quad f(x, y) = y$
[c) plane]

Example 339: $D = \{[x, y] \in \mathbb{E}_2;\ y \geq 0,\ y \leq 2 - x,\ x \geq y^2\}, \quad f(x, y) = y^2$
[c) cylindrical surface]

Example 340: $D = \{[x, y] \in \mathbb{E}_2;\ x \geq 0,\ x + y \leq 2,\ x \leq v^2\}, \quad f(x, y) = xy$
[c) hyperbolic paraboloid]

Example 341: $D = \{[x, y] \in \mathbb{E}_2;\ x^2 + y^2 \leq 9,\ y \geq 0\}, \quad f(x, y) = \sqrt{9 - x^2 - y^2}$
[c) sphere]