Mathematics II – Examples
III. Double and triple intgral

III.1. Existence

Existence Theorem: Let D be a measurable (in the sense of Jordan) set in $\mathbb{E}_2$ (resp. in $\mathbb{E}_3$) and let f be a bounded function on D. Let the set of discontinuities has measure 0. Then f is integrable on D, i.e., there exists the integral

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad \left(\iiint_D f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \text{ resp.}\right)$$

Note: More detailed information you find in the textbook J. Neustupa, Mathematics II.

Example 237: Decide if exists the integral $\displaystyle\iint_D \frac{1}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, where

a) $D = \{[x,y] \in \mathbb{E}_2 : x^2 + (y-1)^2 \le \frac{1}{4}\}$;
b) $D = \{[x,y] \in \mathbb{E}_2 : x^2 + y^2 \le 1\}$;
c) $D = \{[x,y] \in \mathbb{E}_2 : x^2 + (y-1)^2 \le 2\}$;

Example 238: Consider the set $D = \{[x,y] \in \mathbb{E}_2 : x^2 + y^2 \le 4\}$. Check existence of the following integrals:

a) $\displaystyle\iint_D \frac{\sin(x^2+y^2)}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, b) $\displaystyle\iint_D \frac{x\,\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$, c) $\displaystyle\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2+1}$,

d) $\displaystyle\iint_D \frac{1}{1+xy}\,\mathrm{d}x\,\mathrm{d}y$, e) $\displaystyle\iint_D \frac{1}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y$.

Example 237: Decide if exists the following integrals:

a) $\displaystyle\iiint_W \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(1+x+z)^3}$, $W = \{[x,y,z] \in \mathbb{E}_3 : 0 \le x \le 2,\ -1 \le y \le 0,\ -2 \le z \le 2\}$;

b) $\displaystyle\iiint_W (x+yz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W = \{[x,y,z] \in \mathbb{E}_3 : x^2 \le y \le 2,\ z \ge 3\}$;

c) $\displaystyle\iiint_W \frac{1}{x^2+y^2+z^2-9}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $W = \{[x,y,z] \in \mathbb{E}_3 : 1 < x^2+y^2+z^2 < 4\}$;

III.2. Fubini's Theorem for double integral

The set $D = \{[x,y] \in \mathbb{E}_2;\ a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\,\}$, where $\phi_1(x)$ and $\phi_2(x)$ are continuous functions on $\langle a,b\rangle$ and $\phi_1(x) \le \phi_2(x)$, is called the elementary region relative to the x-axis.

Let D be an elemetary region relaive to the x-axis. Let the function $f(x,y)$ is continuous in D. Then

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x.$$

Note: The same assertion is valid relative to the y-axis.

- **Compute the following double integrals od given rectangular domains:**

Example 240: $I = \displaystyle\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{x^2-2xy+y^2},\quad D = \{[x,y] \in \mathbb{E}_2:\ 3 \le x \le 4,\ 0 \le y \le 2\,\}$

Example 241: $I = \displaystyle\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{(x-2y+3)^2},\quad D = \{[x,y] \in \mathbb{E}_2:\ 0 \le x \le 2,\ 0 \le y \le 1\,\}$

Example 242: $I = \displaystyle\iint_D y^2\sin^2 x\,\mathrm{d}x\,\mathrm{d},\quad D = \{[x,y] \in \mathbb{E}_2:\ 0 \le x \le \frac{\pi}{2},\ 1 \le y \le 2\,\}$

Hint: If f can be written in the form $f(x,y) = g(x)\cdot h(y)$ and the set D is a rectangle $D = \langle a,b\rangle \times \langle c,d\rangle$, then $\iint_D f(x,y)\,dx\,dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$.

Example 243: $I = \displaystyle\iint_D \frac{xye^{x^2}}{y^2+3}\,\mathrm{d}x\,\mathrm{d}y,\quad D = \{[x,y] \in \mathbb{E}_2:\ 0 \le x \le 2,\ 0 \le y \le 3\,\}$

- **Let the set M be bounded by given curves. Sketch this set and express them as an elementary region.**

Example 244: $2x - y = 1,\quad 2x - y = 5,\quad x = 0,\quad x = 2$

Example 245: $y = 0,\quad x = 2y,\quad x = 4$

Example 246: $y = 18 - x^2,\quad y = x^2$

Example 247: $xy = 4,\quad y = x,\quad y = 4x,\quad (x \ge 0)$

- **Change order of integration:**

Example 248: $I = \int_0^1 \left(\int_0^{1-y} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$

Example 249: $I = \int_0^1 \left(\int_{x^2}^{x} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$

Example 250: $I = \int_0^4 \left(\int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$

Example 251: $I = \int_0^4 \left(\int_0^{\frac{y}{2}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y + \int_4^6 \left(\int_0^{6-y} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$

- **Using Fubini's theorem transform the double integral $\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ to two-fold integrals in both directions of integration (relative to x-axis and/or relaive to y-axis), if the set $D \subset \mathbb{E}_2$ is bounded by the given curves:**

Example 252: $x = 0, \quad y = x^2, \quad x + y = 2, \quad (x \geq 0)$

Example 253: $x = y^2 - 4, \quad x = -3y$

- **Sketch the set $D \subset \mathbb{E}_2$ bounded by given curves and compute given integral:**

Example 254: $\iint_D \frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; xy = 1, \quad y = 4x, \quad x = 3$

Example 255: $\iint_D x^3 y^2\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; x^2 + y^2 = a^2, \quad x = 0 \;\; (x \geq 0)$

Example 256: $\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+1}, \quad D: \; y = 2x - x^2, \quad y = -x$

Example 257: $\iint_D (x+y)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; y^2 - x^2 \leq 1, \quad y \geq 0, \quad x \in \langle -2, 2 \rangle$

Example 258: $\iint_D (1+x)y\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; y = x^2 - 4, \quad y = -3x, \quad x \leq 0$

Example 259: $\iint_D (x+1)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; y = 2x, \quad 2y = x, \quad y = 2$

Example 260: $\iint_D \frac{1}{y}\,\mathrm{d}x\,\mathrm{d}y, \quad D: \; xy = 1, \quad y = x, \quad x = 4, \quad x \geq 0$

Example 261: $\iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{(x+y)^2}, \quad D: \; x = 3, \quad x = 4, \quad y = 1, \quad y = 2$

Example 262: $\iint_D \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \ x=0, \quad y=\pi, \quad y=x$

Example 263: $\iint_D (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \ y=0, \quad y=1-x, \quad y=1+x$

Example 264: $\iint_D (x+2y)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \ x=y^2-4, \quad x=5$

Example 265: $\iint_D xy\,\mathrm{d}x\,\mathrm{d}y, \quad D: \ y=x-4, \quad y^2=2x$

Example 266: $\iint_D \frac{1}{y+1}\,\mathrm{d}x\,\mathrm{d}y, \quad D: \ x=0, \quad y=2, \quad y=4, \quad y^2=x$

Example 267: $\iint_D \frac{x}{y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad D: \ y^2=x, \quad y^2=4x, \quad y=2$

Example 268: $\iint_D (xy+y)\,\mathrm{d}x\,\mathrm{d}y, \quad D: \ x=1, \quad x=2, \quad xy=4, \quad y=0$

Example 269: Transform double integral of the function $f(x,y)=1/x$ on set $D=\{[x,y]\in\mathbb{E}_2 : y\geq \ln x, \ x\geq 1, \ y\leq 1\}$ into two-fold integrals (i.e., use both orders of integration relative to both x-axis and y-axis) and compute the integral.