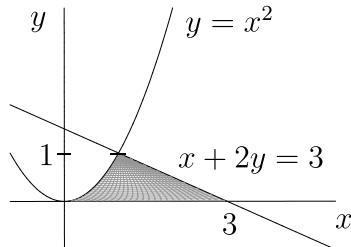


III.6. Aplikace dvojných integrálů

- Určete plošný obsah rovinného obrazce $D \subset \mathbb{E}_2$ ohraničeného danými křivkami :

Příklad 293. $y = x^2$, $x + 2y = 3$, $y = 0$

Rешение :

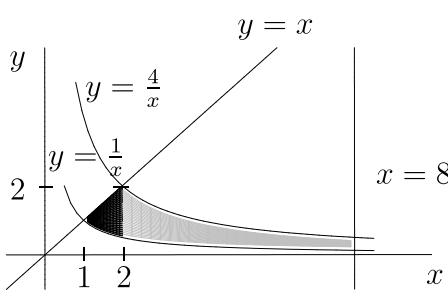


$$\begin{aligned} P &= \iint_D 1 \, dx \, dy = \int_0^1 \left(\int_{\sqrt{y}}^{3-2y} 1 \, dx \right) dy = \int_0^1 \left[x \right]_{\sqrt{y}}^{3-2y} dy = \\ &= \int_0^1 \left(3 - 2y - \sqrt{y} \right) dy = \left[3y - y^2 - \frac{2y\sqrt{y}}{3} \right]_0^1 = \frac{4}{3}. \end{aligned}$$

■

Příklad 294. $xy = 1$, $xy = 4$, $y = x$, $x = 8$

Rешение :

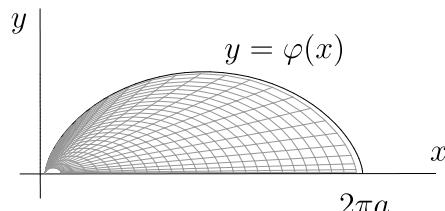


$$\begin{aligned} P &= \iint_D 1 \, dx \, dy = \int_1^2 \left(\int_{\frac{1}{x}}^x 1 \, dy \right) dx + \int_2^8 \left(\int_{\frac{1}{x}}^{\frac{4}{x}} 1 \, dy \right) dx = \\ &= \int_1^2 \left(x - \frac{1}{x} \right) dx + \int_2^8 \left(\frac{4}{x} - \frac{1}{x} \right) dx = \left[\frac{x^2}{2} - \ln|x| \right]_1^2 + \\ &\quad + 3 \left[\ln|x| \right]_2^8 = 2 - \ln 2 - \frac{1}{2} + 3(\ln 8 - \ln 2) = \frac{3}{2} + \ln \frac{8^3}{2 \cdot 2^3} = \\ &= \frac{3}{2} + \ln 32. \end{aligned}$$

■

Příklad 295. Určete plošný obsah rovinného obrazce omezeného osou x a jedním obloukem cykloidy o parametrických rovnicích $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Rешение :



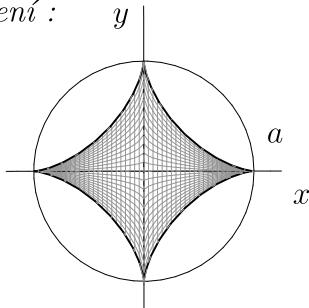
Jeden oblouk cykloidy opíše bod kružnice, která se kotádí po přímce $y = 0$, tj. $t \in \langle 0, 2\pi \rangle$.

$$\begin{aligned} P &= \iint_D 1 \, dx \, dy = \left| \begin{array}{l} 0 \leq x \leq 2\pi a \\ 0 \leq y \leq \varphi(x) \end{array} \right| = \int_0^{2\pi a} \left(\int_0^{\varphi(x)} 1 \, dy \right) dx = \int_0^{2\pi a} \varphi(x) \, dx = \\ &= \left| \begin{array}{l} \text{substituce :} \\ x = x(t) = a(t - \sin t) \Rightarrow dx = a(1 - \cos t) \, dt \\ y = y(t) = a(1 - \cos t) \\ x \in \langle 0, 2\pi a \rangle \Rightarrow t \in \langle 0, 2\pi \rangle \end{array} \right| = \int_0^{2\pi} a(1 - \cos t) \cdot a(1 - \cos t) \, dt = \\ &= a^2 \int_0^{2\pi} (1 - \cos t)^2 \, dt = a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) \, dt = a^2 \left[t - 2\sin t \right]_0^{2\pi} + \\ &+ a^2 \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt = a^2 \cdot 2\pi + \frac{a^2}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi} = 2\pi a^2 + a^2 \pi = 3\pi a^2. \end{aligned}$$

■

Příklad 296. Určete plošný obsah rovinného obrazce D omezeného asteroidou $x^{2/3} + y^{2/3} = a^{2/3}$. Použijte transformace: $x = r \cos^3 \varphi$, $y = r \sin^3 \varphi$.

Rешение:



$$P = \iint_D 1 \, dx \, dy$$

$$(r \cos^3 \varphi)^{2/3} + (r \sin^3 \varphi)^{2/3} = a^{2/3} \implies r^{2/3} = a^{2/3} \implies$$

$$\begin{vmatrix} 0 \leq r \leq a \\ 0 \leq \varphi \leq 2\pi \end{vmatrix}, \quad J = \begin{vmatrix} \cos^3 \varphi & -3r \cos^2 \varphi \sin \varphi \\ \sin^3 \varphi & 3r \sin^2 \varphi \cos \varphi \end{vmatrix} = 3r \sin^2 \varphi \cos^4 \varphi + 3r \sin^4 \varphi \cos^2 \varphi =$$

$$= 3r \sin^2 \varphi \cos^2 \varphi.$$

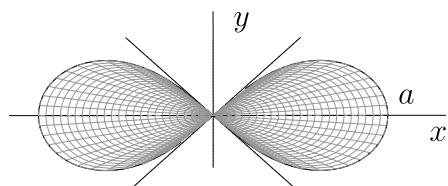
$$\begin{aligned} P &= \int_0^{2\pi} \left(\int_0^a 3r \sin^2 \varphi \cos^2 \varphi \, dr \right) d\varphi = 4 \int_0^{\frac{\pi}{2}} \sin^2 \varphi \cos^2 \varphi \, d\varphi \cdot 3 \int_0^a r \, dr = \\ &= 4 \int_0^{\frac{\pi}{2}} \frac{\sin^2 2\varphi}{4} \, d\varphi \cdot 3 \left[\frac{r^2}{2} \right]_0^a = \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\varphi}{2} \, d\varphi \cdot \frac{3}{2} a^2 = \frac{3}{4} a^2 \left[\varphi - \frac{\sin 4\varphi}{4} \right]_0^{\frac{\pi}{2}} = \\ &= \frac{3}{4} a^2 \cdot \frac{\pi}{2} = \frac{3}{8} \pi a^2. \end{aligned}$$

■

- Určete plošný obsah rovinného obrazce omezeného uzavřenou křivkou :

Příklad 297. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ (Bernoulliova lemniskáta)

Rешение:



$$P = \iint_D 1 \, dx \, dy = \begin{vmatrix} x = r \cos \varphi \\ y = r \sin \varphi \\ J = r \end{vmatrix}.$$

$$\begin{aligned} \text{Po dosazení do zadání dostáváme postupně: } r^4 &= a^2 r^2 (\cos^2 \varphi - \sin^2 \varphi), r^2 = a^2 \cos 2\varphi, \\ 0 \leq r &\leq a\sqrt{\cos 2\varphi} \implies \cos 2\varphi \geq 0, \quad \varphi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right). \end{aligned}$$

$$\begin{aligned} P &= 4 \int_0^{\frac{\pi}{4}} \left(\int_0^{a\sqrt{\cos 2\varphi}} r \, dr \right) d\varphi = 4 \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2} \right]_0^{a\sqrt{\cos 2\varphi}} d\varphi = 2 \int_0^{\frac{\pi}{4}} a^2 \cos 2\varphi \, d\varphi = \\ &= 2a^2 \left[\frac{\sin 2\varphi}{2} \right]_0^{\frac{\pi}{4}} = a^2. \end{aligned}$$

■

Příklad 298.* $(x^2 + 9y^2)^2 = x^2 y$

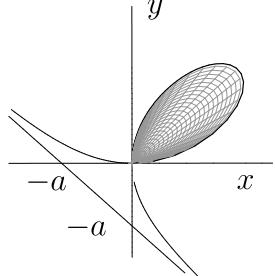
$$\begin{aligned} \text{Rешение: } P &= \iint_D 1 \, dx \, dy = \begin{vmatrix} x = r \cos \varphi \\ y = \frac{r}{3} \sin \varphi \\ J = \frac{1}{3} r \end{vmatrix} \end{aligned}$$

$$\begin{aligned} r^4 &= r^2 \cos^2 \varphi \cdot \frac{r}{3} \sin \varphi \\ 0 \leq r &\leq \frac{1}{3} \cos^2 \varphi \sin \varphi \\ \cos^2 \varphi \sin \varphi &\geq 0 \Rightarrow 0 \leq \varphi \leq \pi \end{aligned}$$

$$\begin{aligned}
 P &= \int_0^\pi \left(\int_0^{\frac{1}{3}\cos^2 \varphi \sin \varphi} \frac{1}{3} r dr \right) d\varphi = \frac{1}{3} \int_0^\pi \left[\frac{r^2}{2} \right]_0^{\frac{1}{3}\cos^2 \varphi \sin \varphi} d\varphi = \\
 &= \frac{1}{6} \int_0^\pi \frac{1}{9} \cos^4 \varphi \sin^2 \varphi d\varphi = \frac{1}{54} \cdot 2 \cdot \int_0^{\frac{\pi}{2}} \cos^4 \varphi (1 - \cos^2 \varphi) d\varphi = \\
 &= \frac{1}{27} \int_0^{\frac{\pi}{2}} (\cos^4 \varphi - \cos^6 \varphi) d\varphi = \text{(Wallisova formule)} = \frac{1}{27} \left(\frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} - \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \right) = \frac{\pi}{864}. \blacksquare
 \end{aligned}$$

Příklad 299.* $(x^3 + y^3) = 3axy$ (Descartesův list)

Řešení :



$$P = \iint_D 1 dx dy = \left| \begin{array}{l} x = r \cos \varphi \\ y = r \sin \varphi \\ J = r \end{array} \quad \begin{array}{l} 0 \leq r \leq r(\varphi) \\ \varphi_1 \leq \varphi \leq \varphi_2 \end{array} \right| =$$

$$= \int_{\varphi_1}^{\varphi_2} \left(\int_0^{r(\varphi)} r dr \right) d\varphi = \frac{1}{2} \int_{\varphi_1}^{\varphi_2} r^2(\varphi) d\varphi.$$

Nyní určíme $r(\varphi)$ a dosadíme do posledního integrálu.

$$x^3 + y^3 = 3axy \implies r^3 \cos^3 \varphi + r^3 \sin^3 \varphi = 3ar^2 \cos \varphi \sin \varphi, \quad \text{takže}$$

$$r(\varphi) = \frac{3a \cos \varphi \sin \varphi}{\cos^3 \varphi + \sin^3 \varphi}, \quad 0 \leq \varphi \leq \frac{\pi}{2}, \quad r(0) = r\left(\frac{\pi}{2}\right) = 0.$$

$$P = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(\frac{3a \cos \varphi \sin \varphi}{\cos^3 \varphi + \sin^3 \varphi} \right)^2 d\varphi = \frac{9a^2}{2} \int_0^{\frac{\pi}{2}} \frac{\cos^2 \varphi \sin^2 \varphi}{\left(\cos^3 \varphi + \sin^3 \varphi \right)^2} d\varphi =$$

(čitatel a jmenovatel vydělíme $\cos^6 \varphi$)

$$= \frac{9a^2}{2} \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^2 \varphi}{\left(\operatorname{tg}^3 \varphi + 1 \right)^2} \cdot \frac{d\varphi}{\cos^2 \varphi} = \left| \begin{array}{l} \operatorname{tg} \varphi = u \\ \frac{1}{\cos^2 \varphi} d\varphi = du \end{array} \right| = \frac{9a^2}{2 \cdot 3} \int_0^{\infty} \frac{3u^2}{(u^3 + 1)^2} du =$$

$$= \frac{3}{2} a^2 \lim_{C \rightarrow +\infty} \int_0^C \frac{3u^2}{(u^3 + 1)^2} du = \frac{3}{2} a^2 \lim_{C \rightarrow +\infty} \left[\frac{-1}{u^3 + 1} \right]_0^C = \frac{3}{2} a^2 \lim_{C \rightarrow +\infty} \left(\frac{-1}{C^3 + 1} + 1 \right) =$$

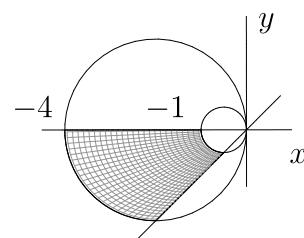
$$= \frac{3}{2} a^2. \blacksquare$$

Příklad 300. Určete plošný obsah rovinného obrazce omezeného křivkami

$$x^2 + y^2 + x = 0, \quad x^2 + y^2 + 4x = 0, \quad y = x, \quad y = 0.$$

Řešení :

$$\begin{aligned}
 x^2 + y^2 + x = 0 &\implies \left(x + \frac{1}{2} \right)^2 + y^2 = \frac{1}{4} \\
 x^2 + y^2 + 4x = 0 &\implies (x + 2)^2 + y^2 = 4
 \end{aligned}$$



$$P = \iint_D 1 dx dy = \left| \begin{array}{l} x = r \cos \varphi \\ y = r \sin \varphi \\ J = r \end{array} \quad \begin{array}{l} x^2 + y^2 + x = 0 \implies r = -\cos \varphi \\ x^2 + y^2 + 4x = 0 \implies r = -4 \cos \varphi \\ -\cos \varphi \leq r \leq -4 \cos \varphi \\ \pi \leq \varphi \leq \frac{5}{4}\pi \end{array} \right| =$$

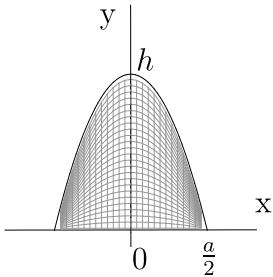
$$\begin{aligned}
 &= \int_{\pi}^{\frac{5}{4}\pi} \left(\int_{-\cos\varphi}^{-4\cos\varphi} r dr \right) d\varphi = \frac{1}{2} \int_{\pi}^{\frac{5}{4}\pi} [r^2]_{-\cos\varphi}^{-4\cos\varphi} d\varphi = \frac{1}{2} \int_{\pi}^{\frac{5}{4}\pi} (16\cos^2\varphi - \cos^2\varphi) d\varphi = \\
 &= \frac{15}{2} \int_{\pi}^{\frac{5}{4}\pi} \cos^2\varphi d\varphi = \frac{15}{2} \int_{\pi}^{\frac{5}{4}\pi} \frac{1 + \cos 2\varphi}{2} d\varphi = \frac{15}{4} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_{\pi}^{\frac{5}{4}\pi} = \\
 &= \frac{15}{4} \left(\frac{5}{4}\pi + \frac{\sin \frac{5}{2}\pi}{2} - \pi - \frac{\sin 2\pi}{2} \right) = \frac{15}{4} \left(\frac{\pi}{4} + \frac{1}{2} \right) = \frac{15(\pi+2)}{16}.
 \end{aligned}$$

■

Příklad 301. Je dána parabolická úseč s tětivou kolmou k ose. Délka tětivy je a , výška úseče h , a plošná hustota $\varrho = 1$. Určete :

- a) moment setrvačnosti úseče vzhledem k tětivě, b) těžiště úseče.

Rешení :



Analytické vyjádření této paraboly bude $y - h = px^2$.

$$\begin{aligned}
 &\text{Použijeme-li bod } \left[\frac{a}{2}, 0 \right], \text{ pak } -h = p \frac{a^2}{4} \implies p = \frac{-4h}{a^2} \implies \\
 &y = h - \frac{4h}{a^2} x^2.
 \end{aligned}$$

a) Moment setrvačnosti k tětivě je nyní momentem setrvačnosti vzhledem k ose x .

$$\begin{aligned}
 J_x &= \iint_D y^2 dx dy = \left| \begin{array}{l} D : 0 \leq y \leq h - \frac{4h}{a^2} x^2 \\ -\frac{a}{2} \leq x \leq \frac{a}{2} \end{array} \right| = \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\int_0^{h - \frac{4h}{a^2} x^2} y^2 dy \right) dx = \\
 &= \frac{1}{3} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left[y^3 \right]_0^{h - \frac{4h}{a^2} x^2} dx = \frac{1}{3} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(h - \frac{4h}{a^2} x^2 \right)^3 dx = \frac{h^3}{3} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(1 - \frac{4x^2}{a^2} \right)^3 dx = \\
 &= \frac{2h^3}{3} \int_0^{\frac{a}{2}} \left(1 - \frac{12x^2}{a^2} + \frac{48x^4}{a^4} - \frac{64x^6}{a^6} \right) dx = \frac{2h^3}{3} \left[x - \frac{4x^3}{a^2} + \frac{48x^5}{5a^4} - \frac{64x^7}{7a^6} \right]_0^{\frac{a}{2}} = \\
 &= \frac{2h^3}{3} \left(\frac{a}{2} - \frac{a}{2} + \frac{3a}{10} - \frac{a}{14} \right) = \frac{h^3 a}{3} \left(\frac{3}{5} - \frac{1}{7} \right) = \frac{16h^3 a}{105}.
 \end{aligned}$$

$$b) T = [0, y_T], \quad y_T = \frac{M_x}{m}$$

$$\begin{aligned}
 m &= \iint_D dx dy = \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\int_0^{h - \frac{4h}{a^2} x^2} dy \right) dx = 2 \int_0^{\frac{a}{2}} \left(h - \frac{4h}{a^2} x^2 \right) dx = \\
 &= 2 \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(1 - \frac{4x^2}{a^2} \right) dx = 2h \left[x - \frac{4x^3}{3a^2} \right]_0^{\frac{a}{2}} = 2h \left(\frac{a}{2} - \frac{a}{6} \right) = \frac{2}{3} ha,
 \end{aligned}$$

$$M_x = \iint_D y dx dy = \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\int_0^{h - \frac{4h}{a^2} x^2} y dy \right) dx = \dots = \frac{2}{5} h^2 a,$$

$$y_T = \frac{\frac{2}{5} h^2 a}{\frac{2}{3} ha} = \frac{3}{5} h,$$

$$T = \left[0, \frac{3}{5} h \right].$$

■

Příklad 302. Určete těžiště rovinné desky omezené křivkami $x^2 + y^2 - 2x = 0$, $x^2 + y^2 - 4x = 0$, je-li plošná hustota $\varrho = 10$.

Rешение:

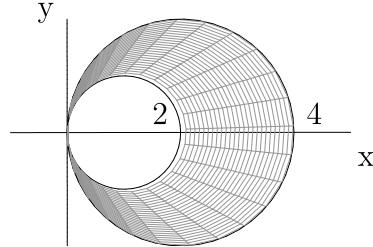
$$x^2 + y^2 - 2x = 0 \implies (x-1)^2 + y^2 = 1$$

$$x^2 + y^2 - 4x = 0 \implies (x-2)^2 + y^2 = 4$$

$$T = [x_T, 0], \quad x_T = \frac{M_y}{m}, \quad m = \varrho \cdot P$$

$$m = 10 \cdot P = 10(\pi \cdot 2^2 - \pi \cdot 1^2) = 30\pi,$$

kde P = je plocha dané desky



$$\begin{aligned} M_y &= \iint_D x \varrho dx dy = \iint_D 10x dx dy = \left| \begin{array}{l} x = r \cos \varphi \\ y = r \sin \varphi \\ J = r \end{array} \right| \left| \begin{array}{l} x^2 + y^2 \geq 2x \implies r \geq 2 \cos \varphi \\ x^2 + y^2 \leq 4x \implies r \leq 4 \cos \varphi \\ 2 \cos \varphi \leq r \leq 4 \cos \varphi \\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \end{array} \right| = \\ &= 10 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_{2 \cos \varphi}^{4 \cos \varphi} r^2 \cos \varphi dr \right) d\varphi = 10 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \cdot \left[\frac{r^3}{3} \right]_{2 \cos \varphi}^{4 \cos \varphi} d\varphi = \frac{10}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 56 \cos^4 \varphi d\varphi = \\ &= \frac{20}{3} \cdot 56 \int_0^{\frac{\pi}{2}} \cos^4 \varphi d\varphi = \frac{1120}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = 70\pi, \end{aligned}$$

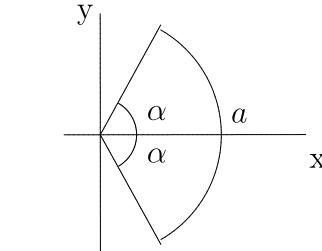
$$x_T = \frac{70\pi}{30\pi},$$

$$T = \left[\frac{7}{3}, 0 \right].$$

■

Příklad 303. Určete souřadnice těžiště kruhové výseče (viz obrázek), je-li $\varrho = \text{konst}$.

Rешение:



$$T = [x_T, 0], \quad m = \frac{\pi a^2}{2\pi} \cdot 2\alpha \varrho = a^2 \alpha \varrho,$$

$$M_y = \iint_D x \varrho dx dy =$$

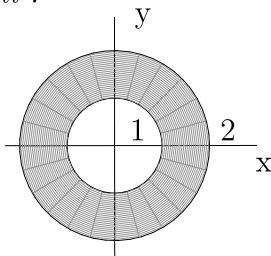
$$\begin{aligned} &= \left| \begin{array}{l} x = r \cos \varphi \\ y = r \sin \varphi \\ J = r \end{array} \right| \left| \begin{array}{l} 0 \leq r \leq a \\ -\alpha \leq \varphi \leq \alpha \end{array} \right| = \varrho \int_{-\alpha}^{\alpha} \left(\int_0^a r^2 \cos \varphi dr \right) d\varphi = \varrho \left[\frac{r^3}{3} \right]_0^a \cdot [\sin \varphi]_{-\alpha}^{\alpha} = \\ &= \frac{2}{3} \varrho a^3 \sin \alpha, \quad x_T = \frac{2}{3} \cdot \frac{a \sin \alpha}{\alpha} \end{aligned}$$

$$T = \left[\frac{2}{3} \cdot \frac{a \sin \alpha}{\alpha}, 0 \right]$$

■

Příklad 304. Určete moment setrvačnosti vzhledem k počátku soustavy souřadnic homogenní rovinné desky s plošnou hustotou $\varrho = k$ omezené křivkami $x^2 + y^2 = 1$, $x^2 + y^2 = 4$.

Rешение:

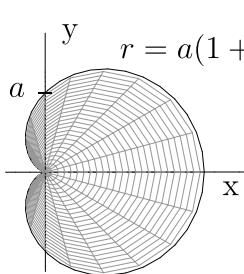


$$\begin{aligned} J_0 &= \iint_D (x^2 + y^2) \varrho(x, y) dx dy = k \iint_D (x^2 + y^2) dx dy = \\ &= (\text{polární souřadnice}) = k \int_0^{2\pi} \int_1^2 r^3 dr d\varphi = \frac{15}{2} k \pi. \end{aligned}$$

■

Příklad 305. Určete polohu těžiště obrazce omezeného **kardiodou** $r = a(1 + \cos \varphi)$, $\varphi \in \langle 0, 2\pi \rangle$, $a > 0$, $\varrho = 1$.

Rешение:

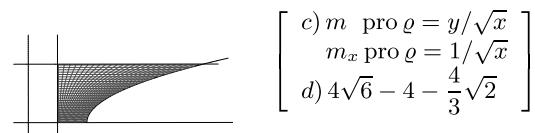


$$\begin{aligned}
 r &= a(1 + \cos \varphi) & T &= [x_T, 0], \quad x_T = \frac{M_y}{m} \\
 m &= \iint_D dx dy = \int_{\varphi_1}^{\varphi_2} \left(\int_0^{r(\varphi)} 1 \cdot r dr \right) d\varphi = \\
 &= \int_0^{2\pi} \left(\int_0^{a(1+\cos\varphi)} r dr \right) d\varphi = \frac{1}{2} \int_0^{2\pi} [r^2]_0^{a(1+\cos\varphi)} d\varphi = \\
 &= \frac{1}{2} a^2 \int_0^{2\pi} (1 + \cos \varphi)^2 d\varphi = \frac{1}{2} a^2 \int_0^{2\pi} (1 + 2 \cos \varphi + \cos^2 \varphi) d\varphi = \\
 &= \frac{1}{2} a^2 [\varphi + 2 \sin \varphi]_0^{2\pi} + \frac{1}{2} a^2 \int_0^{2\pi} \frac{1 + \cos 2\varphi}{2} d\varphi = a^2 \pi + a^2 \frac{\pi}{2} = \frac{3}{2} a^2 \pi, \\
 M_y &= \iint_D x dx dy = [\text{polární souř.}] = \int_0^{2\pi} \left(\int_0^{a(1+\cos\varphi)} r^2 \cos \varphi dr \right) d\varphi = \\
 &= \frac{1}{3} \int_0^{2\pi} \cos \varphi \cdot a^3 (1 + \cos \varphi)^3 d\varphi = \frac{a^3}{3} \int_0^{2\pi} (\cos \varphi + 3 \cos^2 \varphi + 3 \cos^3 \varphi + \cos^4 \varphi) d\varphi = \\
 &= \frac{5}{4} a^3 \pi, \quad x_T = \frac{5a}{6}, \quad \boxed{T = \left[\frac{5}{6} a, 0 \right]} \quad \blacksquare
 \end{aligned}$$

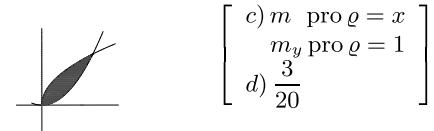
- Je dána omezená množina $D \subset \mathbb{E}_2$ a funkce $f(x, y)$

- Načrtněte množinu D .
- Ověřte splnění předpokladů pro použití Fubiniové věty
- Uveďte alespoň dva příklady možného fyzikálního významu daného integrálu.
Uveďte, zda se jedná o hmotnost (při jaké hustotě), statický moment nebo moment setrvačnosti (při jaké hustotě a vzhledem k jakému bodu nebo přímce).
- Vypočítejte $\iint_D f(x, y) dx dy$.

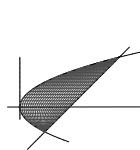
306. D je ohraničena křivkami: $x = 1$, $x = y^2 + 2$, $y = 0$, $y = 2$, $f(x, y) = y/\sqrt{x}$



307. D je ohraničena křivkami: $y = x^2$, $y = \sqrt{x}$, $f(x, y) = x$,

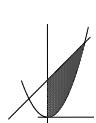


308. D je ohraničena křivkami: $x = y^2$, $x - y - 2 = 0$, $f(x, y) = y^2$



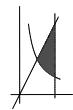
$$\begin{bmatrix} c) m \text{ pro } \varrho = y^2 \\ m_x \text{ pro } \varrho = y \\ J_x \text{ pro } \varrho = 1 \\ d) \frac{29}{60} \end{bmatrix}$$

309. $D = \{[x, y] \in \mathbb{E}_2; x \geq 0, y \leq x + 2, y \geq x^2\}$, $f(x, y) = 2x(y + 1)$



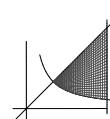
$$\begin{bmatrix} c) m \text{ pro } \varrho = 2x(y + 1) \\ m_y \text{ pro } \varrho = 2(y + 1) \\ d) \frac{52}{3} \end{bmatrix}$$

310. D je ohraničena křivkami: $y = 2x$, $y = 2/x$, $x = 2$, $f(x, y) = x^2 y$



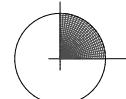
$$\begin{bmatrix} c) m \text{ pro } \varrho = x^2 y \\ m_x \text{ pro } \varrho = x^2 \\ m_y \text{ pro } \varrho = xy \\ J_y \text{ pro } \varrho = y \\ d) \frac{52}{5} \end{bmatrix}$$

311. D je ohraničena křivkami: $y = x$, $y = 1/x$, $x = 3$, $f(x, y) = \sqrt{x}$



$$\begin{bmatrix} c) m \text{ pro } \varrho = \sqrt{x} \\ d) \frac{8(1 + \sqrt{3})}{5} \end{bmatrix}$$

312. $D = \{[x, y] \in \mathbb{E}_2; x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$, $f(x, y) = xy$



$$\begin{bmatrix} c) m \text{ pro } \varrho = xy \\ m_x \text{ pro } \varrho = x \\ m_y \text{ pro } \varrho = y \\ d) 2 \end{bmatrix}$$

- Určete plošný obsah P rovinného obrazce $D \subset \mathbb{E}_2$ ohraničeného danými křivkami :

313. $x = y^2$, $8x = y^2$, $y = 5$ $\left[\frac{875}{24}\right]$

314. $y = x^2$, $x - y + 2 = 0$, $x = 0$, $x = 1$ $\left[\frac{13}{6}\right]$

315. $x = y^2$, $xy = 1$, $x = 4y$, $(xy \geq 1)$ $\left[\frac{21}{2} - \ln 2\right]$

316. $\left(4x^2 + \frac{y^2}{9}\right)^2 = xy$ $\left[\frac{9}{8}\right]$

317. $y = \ln x$, $x - y = 1$, $y = -1$ $\left[\frac{1}{2} - \frac{1}{e}\right]$

318. $y = \frac{x^2}{4}$, $y = \frac{8}{4+x^2}$ $\left[2\left(\pi - \frac{2}{3}\right)\right]$

- Určete hmotnost m rovinné desky omezené křivkami :

319. $y = x^2$, $x - y + 2 = 0$, je-li hustota $\varrho(x, y) = xy$ $\left[\frac{45}{8}\right]$

320. $x^2 + y^2 = 2ax$, je-li $\varrho(x, y) = \sqrt{x^2 + y^2}$, $a > 0$ $\left[\frac{32}{9}a^3\right]$

321. $x^2 + y^2 = 1$, $y = 0$, ($y \geq 0$), je-li $\varrho(x, y) = y$ $\left[\frac{2}{3}\right]$

322. $x^2 + y^2 - 2x = 0$, $x^2 + y^2 - 4x = 0$, $y = x$, $y = 0$, je-li hustota $\varrho(x, y)$ v libovolném bodě rovna vzdálenosti tohoto bodu od počátku soustavy souřadnic. $\left[\frac{70\sqrt{2}}{9}\right]$

- Určete hmotnost m rovinné desky D při dané plošné hustotě $\varrho(x, y)$.

323. $D = \{[x, y] \in \mathbb{E}_2 ; y \leq x + 2, y \geq x^2, x \geq 0\}$, $\varrho(x, y) = xy$ [6]

324. $D = \{[x, y] \in \mathbb{E}_2 ; x \leq 4, x \geq y^2, y \geq 1/x\}$, $\varrho(x, y) = 2x$ $\left[\frac{94}{5}\right]$

325. $D = \{[x, y] \in \mathbb{E}_2 ; x \leq 4, x \geq y^2, y \leq 1/x\}$, $\varrho(x, y) = 2x$ $\left[\frac{162}{5}\right]$

326. $D = \{[x, y] \in \mathbb{E}_2 ; x^2 + y^2 \leq 1, x + y \geq 1\}$, $\varrho(x, y) = y$ $\left[\frac{1}{6}\right]$

- Určete těžiště T rovinné desky omezené křivkami :

327. $y = 2x - 3x^2$, $y = -x$, je-li $\varrho(x, y) = 1$ $\left[T = \left[\frac{1}{2}, -\frac{1}{5}\right]\right]$

328. $y = \sin x$, $y = 0$, $x \in \langle 0, \pi \rangle$, je-li $\varrho(x, y) = 1$ $\left[T = \left[\frac{\pi}{2}, \frac{\pi}{8}\right]\right]$

329. $y^2 = 4x + 4$, $y^2 = -2x + 4$, je-li $\varrho(x, y) = 1$ $\left[T = \left[\frac{2}{5}, 0\right]\right]$

330. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, $x \geq 0$, $y \geq 0$, je-li $\varrho(x, y) = 1$ (jde o čtvrtinu asteroidy ležící v I. kvadrantu, použijte souřadnice $x = r \cos^3 \varphi$, $y = r \sin^3 \varphi$) $\left[x_T = y_T = \frac{256a}{315\pi}\right]$

- Určete moment setrvačnosti :

331. kruhu o poloměru a vzhledem k jeho tečně, $\varrho(x, y) = 1$, $\left[\frac{5}{4}\pi a^4\right]$

332. množiny ohraničené elipsou $4x^2 + y^2 \leq 1$ vzhledem k ose y , $\varrho(x, y) = y$, $\left[\frac{1}{30}\right]$

333. čtvrtiny kruhu o poloměru a vzhledem k jeho ose souměrnosti, $\varrho(x, y) = 1$,
(Zvolte polohu tak, aby osa x byla osou souměrnosti.) $\left[\frac{a^4(\pi - 2)}{16}\right]$

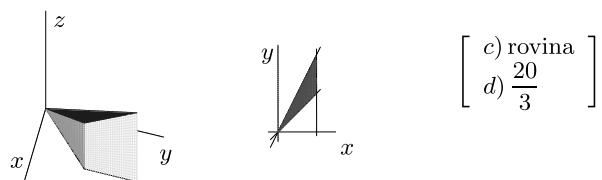
334. čtverce o straně a vzhledem k jeho vrcholu, $\varrho(x, y) = 1$, $\left[\frac{2}{3}a^4\right]$

335. části mezikruží $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, omezeného přímkami $y = x$, $y = 0$
v I. kvadrantu s hustotou $\varrho(x, y) = k$, ($k > 0$) vzhledem ke středu mezikruží. $\left[\frac{15k\pi}{16}\right]$

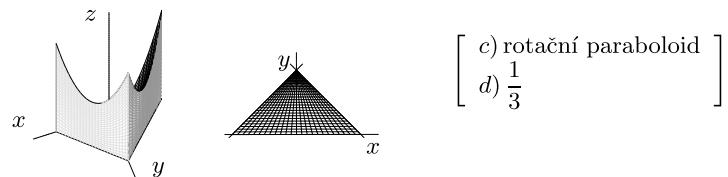
- Je dána omezená množina $D \subset \mathbb{E}_2$ a funkce $f(x, y)$

- Načrtněte těleso, jehož objem bude roven hodnotě spočítaného integrálu.
- Načrtněte průmět tělesa do roviny $z = 0$.
- Napište název plochy $z = f(x, y)$.
- Vypočítejte $\iint_D f(x, y) dx dy$.

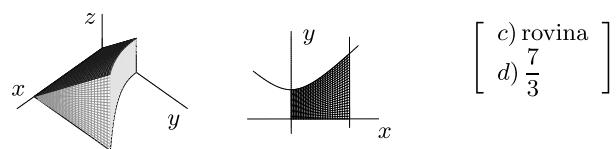
336. D je ohraničena křivkami: $y = x$, $y = 2x$, $x = 2$, $f(x, y) = x + y$



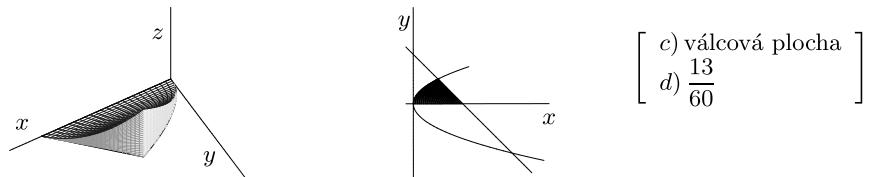
337. $D = \{[x, y] \in \mathbb{E}_2 ; x + y \leq 1, x + 1 \geq y \geq 0\}$, $f(x, y) = x^2 + y^2$



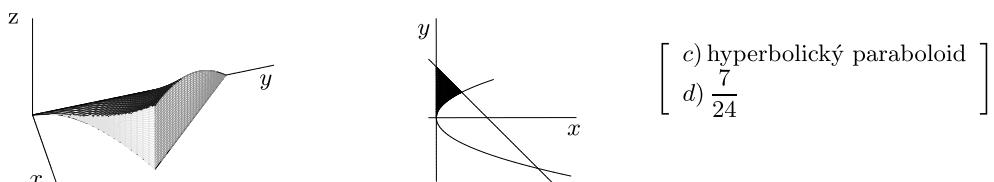
338. $D = \{[x, y] \in \mathbb{E}_2 ; y^2 - x^2 \leq 1, 0 \leq x \leq 2, y \geq 0\}$, $f(x, y) = y$



339. $D = \{[x, y] \in \mathbb{E}_2 ; y \geq 0, y \leq 2 - x, x \geq y^2\}$, $f(x, y) = y^2$



340. $D = \{[x, y] \in \mathbb{E}_2 ; x \geq 0, x + y \leq 2, x \leq y^2\}$, $f(x, y) = xy$



341. $D = \{[x, y] \in \mathbb{E}_2 ; x^2 + y^2 \leq 9, y \geq 0\}$, $f(x, y) = \sqrt{9 - x^2 - y^2}$

