Mathematics II – Examples

II.10. Extremes

Necessary condition for existence of local extreme: Let the functions $f(x,y)$ is differentiable at a point A. If f has an local extreme at the point A, then $\text{grad} f(A) = \vec{0}$.

Let us denote

$$\triangle_1(A) = \frac{\partial^2 f}{\partial x^2}(A), \quad \triangle_2(A) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(A), & \frac{\partial^2 f}{\partial x \partial y}(A) \\ \frac{\partial^2 f}{\partial y \partial x}(A), & \frac{\partial^2 f}{\partial y^2}(A) \end{vmatrix}$$

(The determinant $\triangle_2$ is called Hessian.)

Sufficient condition for existence of local extreme: Let the functions $f(x,y)$ has continuous partial derivatives of second order at a point A. Let the necessary condition $\text{grad} f(A) = \vec{0}$ holds. Then

If $\triangle_1(A) > 0$ and $\triangle_2(A) > 0$, then the function f has sharp local minimum in point A.
If $\triangle_1(A) > 0$ and $\triangle_2(A) < 0$, then the function f has sharp local maximum in point A.
If $\triangle_1(A) < 0$, then the function f has no local extreme in point A.

Procedure for calculating local extremes:

1st step: We determine so called "critical points", which satisfy the necessary condition, i.e.

a) points, in which the function $f(x,y)$ is not differentiable,

b) points, which are solution of the system of equations $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$.

2nd step: For the points from b) we verify sufficient condition for existence of extreme.

- **Find all local extremes of a given function:**

Example 197: $z(x,y) = \sqrt{x^2 + y^2}$

Example 198: $f(x,y) = x^3 - x^2 + \frac{1}{2}y^2 - xy - 6x$

Example 199: $z(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$

Example 200: Implicit function $y = f(x)$ is defined by $F(x,y) = x^2 + 2xy - y^2 + 8$.

Example 201*: $f(x,y,z) = x^3 + y^2 + z^2 + 12xy + 2z$

Example 202: $z = 2(x^2 + y^2)$, if $x + y = 2$

Example 203: $z = \dfrac{x}{3} + \dfrac{y}{4}$, if $x^2 + y^2 = 1$

Example 204: $z = \sin^2 x + \sin^2 y$, if $y = x - \frac{\pi}{4}$

Example 205*: $z = x^2 + 2y^2$, if $x^2 - 2x + 2y^2 + 4y = 0$ [Hint: use Lagrange function.]

Example 206: Find a point in the plane $x + 2y - z + 3 = 0$, which has the smallest sum of square distances from points $A = [1,1,1]$ and $B = [2,2,2]$.

Sufficient conditions for existence of global extremes on a given set: Let the functions $f(x, y)$ is continuous on nonempty bounded and closed set M. Then f has both global minimum and global maximum on this set.

A special case is **constrained exreme**. This is the case when $M = \{[x, y] \in \mathbb{E}_2 : \ g(x, y) = 0\}$ for some given function $g(x, y)$. Usually, we express one variable from the formula $g(x, y) = 0$ (it is it possible) and putting it into given function f, we obtain a simpler problem to find extremes of function of one variable only.

Procedure for calculating global extrema on a nonempty, bounded and closed set M: Denote M^0 an interior of set M, ∂M is a boundary of M.

1st step: Verify fulfillment of sufficient conditions (see the theorem above) to justify existence of extremes.

2nd step: Find all critical points X of function f on M^0.

3rd step: Find all points $X \in \partial M$, in which function f can have global extremes.

4th step: Compute values of function f in all obtained points X. By comparation determine $\min_M f$ and $\max_M f$.

- **Find global extremes of a given function $z = f(x, y)$ on a set M:**

Example 207: $f(x, y) = x^2 + y^2 + xy - x - y, \quad M = \{[x, y] \in \mathbb{E}_2 : \ x \geq 0, \ y \geq 0, \ x + y \leq 1\}$

Example 208: $f(x, y) = x^2 + y^2 - 6x + 6y, \quad M = \{[x, y] \in \mathbb{E}_2 : \ x^2 + y^2 \leq 4\}$

- **There is given function $f(x, y)$**
  - **a) Write necessary condition for existence of local extreme of a differentiable function of n variables at a point A.**
  - **b) Write sufficient condition for existence of local minimum, resp. maximum, of a function $f(x, y)$ at a point A.**
  - **c) Find all local extremes of the given function f, i.e. determine their location, type and value.**

Example 209: $f(x, y) = x^2 + 12y^2 - 6xy + 4z$

Example 210: $f(x, y) = x^3 + y^2 - 12x + 6y$

Example 211: $f(x, y) = 2y - y^2 - x\mathrm{e}^x$

Example 212: $f(x, y) = x^3 + y^2 - 2x^2 - 2xy + 6$

Example 213: $f(x, y) = x^3 + 3y^2 - 6xy - 9x + 8$

Example 214: $f(x, y) = x^2 + 2x + y^4 - 4y + 7$

Example 215: $f(x, y) = xy + \dfrac{50}{x} + \dfrac{20}{y}$

Example 216: $f(x, y) = x^2 + xy + y^2 - 6 \ln x$

- **Find all local extremes of a given function:**

Example 217: $z = x^2 + y - x\sqrt{y} - 6x + 12$

Example 218: $z = x^2 + y^2 + 6x - 4y$

Example 219: $z = \ln(1 - x^2 - y^2)$

Example 220: $z = x^3 + 8y^3 - 6xy + 5$

Example 221: $z = \mathrm{e}^{x/2}(x^2 + y^2)$

Example 222: $z = x^3 + y^3 + \frac{9}{2}x^2 - 3y - 12x$

Example 223*: $x^2 + y^2 + z^2 - 4x + 6y - 2z + 13 = 0$

Example 224*: $x^2 + xy - z^2 + z + y + 5 = 0$

Example 225*: $f(x, y, z) = x^2 + y^2 + z^2 - 6x + 2y + 6z$

Example 226*: $f(x, y, z) = x^2 + 2y^2 + z^3 - 12yz - 6x$

Example 227*: $z = \ln(xy) + x^2 + y^2 - 4x + 7$

Example 228: a) Find all local extremes of the function $f(x, y) = 2x^2 + y^2 - xy + 3x + y + 1$.
b) Justify existence and find global extremes of this function on a line segment AB, where $A = [0, 2],\ B = [1, 1]$.

- **There is given function f and a set M.**
  - **a) Justify existence of global extremes of the function f on the given set M.**
  - **b) Find the global extremes, i.e. determine their location and compute value of both maximum and minimum of the function f on set M.**

Example 229: $f(x, y) = x + \ln x - y^2,\ M = \{[x, y] \in \mathbb{E}_2 :\ y = x + 1,\ 1/4 \le x \le 1\}$

Example 230: $f(x, y) = x^2 + xy - 3x - y,\ M = \{[x, y] \in \mathbb{E}_2 :\ x + y \le 3,\ x \ge 0,\ y \ge 0\}$

Example 231: $f(x, y) = x^2 - 2x + y^2,\ M = \{[x, y] \in \mathbb{E}_2 :\ x^2 + y^2 \le 9,\ y \ge 0\}$

- **Find global extremes of function $f(x, y)$ on a set M:**

Example 232: $f(x, y) = x^2 + y^2 - xy,\ M = \{[x, y] \in \mathbb{E}_2 :\ |x| + |y| \le 1\}$

Example 233: $f(x, y) = x^2 - y^2,\ M = \{[x, y] \in \mathbb{E}_2 :\ x2 + y^2 \le 4\}$

Example 234: $f(x, y) = xy(x - a)(y - b),\ M = \{[x, y] \in \mathbb{E}_2 :\ 0 \le x \le a,\ 0 \le y \le b\}$

Example 235: $f(x, y) = xy(2 - x - y),\ M = \{[x, y] \in \mathbb{E}_2 :\ x \ge 0,\ y \ge 0,\ x + y \le 1\}$

Example 236: $f(x, y) = x^2 + y^2 - xy + x + y - 1,\ M = \{[x, y] \in \mathbb{E}_2 :\ x \le 0,\ y \ge 0,\ x - y + 3 \ge 0\}$