Mathematics II – Examples

II.7.* Derivative of composed function

Let the functions $z = f(x, y),\ x = x(u, v),\ y = y(u, v)$ are differentiable. Then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} \tag{1}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \tag{2}$$

- **Compute derivatives of a given differentiable functions:**

Example 160: For the following differentiable functions $z = f(x, y),\ x = u \cos v,\ y = u \sin v$

a) Compute the derivative of composed function $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$.

b) Calculate for the particular function $z = f(x, y) = \mathrm{e}^x \ln y$.

Example 161: $w = f(x^4 + y^4 - 2z^4)$. Compute $V = \frac{1}{4x^3}\frac{\partial w}{\partial x} + \frac{1}{4y^3}\frac{\partial w}{\partial y} + \frac{1}{4z^3}\frac{\partial w}{\partial z}$.

Example 162: Verify that the function $y = f(x + at) + g(x - at)$ satisfies the partial differential equation $\frac{\partial^2 y}{\partial t^2} - a^2 \frac{\partial^2 y}{\partial x^2} = 0$, where a is a constant in $\mathbb{R}$. Assume that both f and g have continuous partial derivative of second order. [Hint: $u = x + at,\ v = x - at,\ y = f(u) + g(v)$]

Example 163: Verify that the function $z = f(\frac{y}{x})$ satisfies the equation $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$. Assume that f is a differentiable function.

Example 164: Let us have $z = f(u, v),\ u = x^2 - y^2,\ v = \mathrm{e}^{xy}$. Compute the following differential expression $W = y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$, where f is a differentiable function.

Example 165: Let $f(u, v) = x^y$, where $x = u^2 + v^2,\ y = uv + v^2$. Compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ in the point A, which has coordinations $u = 1,\ v = -1$.

II.9. Implicit functions

Example 166: Prove, that the equation $x^3 + y^3 = 2x^2 + xy - 1$ defines unique implicit function $y = f(x)$ in a neighbourhood of the point $A = [1, 0]$. Compute $f'(1)$ and $f"(1)$.

Example 167: Write equations of both a tangent line t and a normal line n in the point $A = [1, 1]$ to the curve, which is defined by implicit way by the equation $F(x, y) \equiv x^3 y + y^3 x + x^2 y - 3 = 0$.

Example 168: By the equation $\ln\sqrt{x^2 + y^2} = \operatorname{arctg}\frac{y}{x}$ for $x \neq 0$ is defined an implicit function $y = f(x)$. Compute y' and $y"$.

Example 169: Show, that the equation $x^2 + 2xy + y^2 - 4x + 2y - 3 = 0$ define an implicit function $y = f(x)$, which graph is passing through the point $A = [0, 1]$. Check, if the function f is convex in some neighbourhood of the point $x_0 = 0$. Write an equation of a tangent line to the graph of this function in the point A

Example 170: Prove, that the equation $F(x, y, z) \equiv z^3 + 3x^2z - 2xy = 0$ defines unique implicit function $z = f(x, y)$ in a neighbourhood of the point $A = [-1, -2, 1]$. Compute $\text{grad}f(-1, -2)$.

Example 171: In a neighborhood of the point $[2, -2, 1]$ is given implicit function $z = f(x, y)$ by the equation $\ln z + x^2yz + 8 = 0$. Compute directional derivative $\dfrac{\partial z}{\partial \vec{s}}(A)$, where $\vec{s} = \overrightarrow{AB},\ A = [2, -2],\ B = [3, -3]$.

Example 172: Write equations of both a tangent plane τ and a normal line n to a surface $z = z(x, y)$ defined implicitly by the equation $F(x, y, z) \equiv x^2 + y^2 + z^2 - 25 = 0$ in the point $A = [3, 0, 4]$.

Example 173: Write an equation of a tangent plane to a surface defined implicitly by the equation $F(x, y, z) \equiv x(y + z) + z^2 - 5 = 0$ parallel with the plane $\rho:\ 3x - 3y + 6z = 2$.
[Note: There are two tangent planes satisfying the given condition.]

Example 174*: Write an equation of a tangent line in the point $A = [-2, 1, 6]$ to a curve in $\mathbb{E}_3$, which is defined implicitly as an intersection of surfaces $2x^2 + 3y^2 + z^2 = 47,\quad x^2 + 2y^2 = z$.

Example 175: a) Find a unit normal vector $\vec{n}^0$ to the surface, which is defined implicitly in the form $F(x, y, z) \equiv x^2 + y^2 + z^2 - 3 = 0$ in the point $A = [1, 1, 1]$.
b) Compute $\dfrac{\partial g}{\partial \vec{n}^0}$, where $g(x, y, z) = xy^2z^3$.

- **Write equation of both tangent line and normal line to a curve given implicitly by the equation $F(x, y) = 0$ in a point A:**

Example 176: $F(x, y) \equiv \arcsin x + xy^2 = 0, \quad A = [0, 2]$

Example 177: $F(x, y) \equiv x^2y + xy^2 - axy - a^3 = 0, \quad A = [a, a]$

Example 178: Prove, that the equation $\ln(x + y) + 2x + y = 0$ defines a function $f(x, y)$, which satisfies $f(-1) = 2$. Write equation of tangent line to the curve $y = f(x)$ in the point $A = [-1, 2]$.

- **Let the equation $F(x, y) = 0$ defines implicit function of one variable $y = f(x)$**
  - **a) Compute partial derivatives of first order of the function F.**
  - **b) Verify, that the equation $F(x, y) = 0$ defines implicit function $y = f(x)$ in a neighborhood of a point $[x_0, y_0]$, which has a continuous derivative $f'(x)$ in some neighborhood of a point x_0.**
  - **c) Compute $f'(x_0)$ and write equation of both tangent line and normal line to a graph of the function f in touch point $[x_0, y_0]$. Describe behavior of this function in point x_0, i.e. if the function is increasing, decreasing.**
  - **d) Use the equatiuon of tangent line for approximation of a value $y = f(x)$ in a given point x_1. Sketch the tangent line.**

Example 179: $F(x,y) = x^2y - x^3 - 2\sqrt{y} + 1, \quad [x_0, y_0] = [1, 4], \quad x_1 = 0.8$

Example 180: $F(x,y) = y\mathrm{e}^x + y^2 - 2x^2y - 2, \quad [x_0, y_0] = [0, 1], \quad x_1 = -0.3$

Example 181: $F(x,y) = \ln(x - y) + 2x + y^2 - 2, \quad [x_0, y_0] = [1, 0], \quad x_1 = 1.1$

Example 182: $F(x,y) = xy\mathrm{e}^{x-y} - \dfrac{2}{\mathrm{e}}, \quad [x_0, y_0] = [1, 2], \quad x_1 = 1.1$

- **Consider a function $y = f(x)$ of one variable, which is defined implicitly by an equation $F(x,y) = 0$.**
  - a) **Verify all assumptions, under which the equation $F(x,y) = 0$ defines implicit function $y = f(x)$, whose graph is passing through a point $[x_0, y_0]$ and which has continuous 1st and 2nd derivative in a neighborhood of x_0.**
  - b) **Compute $f'(x_0)$ and $f''(x_0)$.(Computation of second derivative is not requested in exams of level B.)**
  - c) **Write Taylor's polynomial of second order $T_2(x)$ of the function f with center in point x_0. Using $T_2(x)$ compute approximate value of $f(x_1)$.**
  - d) **Sketch a graph of function f in a neighborhood of $[x_0, y_0]$.**

Example 183: $F(x,y) = x^3 + 2x^2y + y^4 - 1 = 0, \quad [x_0, y_0] = [2, -1], \quad x_1 = 2.2$

Example 184: $F(x,y) = x^3 + \dfrac{y^2}{2} - x^2y - 1 = 0, \quad [x_0, y_0] = [1, 2], \quad x_1 = 1.2$

Example 185: $F(x,y) = x^3 + xy^2 + xy - 7 = 0, \quad [x_0, y_0] = [1, 2], \quad x_1 = 0.9$

Example 186: $F(x,y) = x^2 + 2y^2 - 2xy - y = 0, \quad [x_0, y_0] = [1, 1], \quad x_1 = 1.2$

- **Write equations of both tangent plane τ and normal line n to surface $F(x,y,z) = 0$ in a point A:**

Example 187: $x^2 + 2y^2 + 3z^2 - 21 = 0, \quad A = [1, 2, 2]$

Example 188: $x^3 + y^3 + z^3 + xyz - 6 = 0, \quad A = [1, 2, -1]$

Example 189: $xyz^2 - x - y - z = 0, \quad A = [1, -1, -1]$

- **Write equation of tangent plane to surface $F(x,y,z) = 0$, which is parallel with plane ρ:**

Example 190: $x^2 + y^2 + z^2 - 1 = 0, \quad \rho: \; x + 2y + z = 0$

Example 191: $x^2 + 2y^2 + 3z^2 - 21 = 0, \quad \rho: \; x + 4y + 6z = 0$

Example 192: Let us have the implicit function $f(x)$ given by $\mathrm{e}^z - xyz = \mathrm{e}$. Compute $f_x(A), \; f_y(A), \; f_{xy}(A)$, where $A = [0, \mathrm{e}, 1]$.

Example 193: Consider two surfaces given by $x+2y-\ln z+4=0$ and $x^2-xy-8x+\frac{1}{2}z^2+\frac{11}{2}=0$. Determine relative position of two tangent planes to both surfaces in a common touch point $T=[2,-3,1]$.

- **Consider a function $y=f(x,y)$ of two variables, which is defined implicitly by an equation $F(x,y,z)=0$.**
  - **a) Verify all assumptions, under which the equation $F(x,y,z)=0$ defines implicit function $z=f(x,y)$, whose graph is passing through a point $[x_0,y_0,z_0]$ and which has continuous partial derivatives of first order in a neighborhood of a point $[x_0,y_0]$.**
  - **b) Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and grad$f(x_0,y_0)$.**
  - **c) Write equation of both tangent plane τ and normal line n to graph of the function f in point A.**
  - **d) Compute differential of function f in point $[x_0,y_0]$. Compute approximate value $f(x_1,y_1)$.**
  - **d) Compute directional derivative of f in point $[x_0,y_0]$ and in direction $\vec{u}$. Find direction $\vec{s}$ in which function f grows most rapidly. Compute directional derivative of f in point $[x_0,y_0]$ and in direction $\vec{s}$.**

  (Examples with implicit function of two variables do not belong to requests for exams of level B.

Example 194: $F(x,y,z)=2x^3yz-x+y^3+z^3-2=0$, $A=[1,2,-1]$, $[x_1,y_1]=[0.9,2.1]$, $\vec{u}=(2,4)$

Example 195: $F(x,y,z)=x^3+3xy^2z+2yz^3=0$, $A=[1,-1,-1]$, $[x_1,y_1]=[0.9,-1.1]$, $\vec{u}=(1,-2)$

Example 196: $F(x,y,z)=z\mathrm{e}^{x+y}-xy+z^2-6=0$, $A=[0,0,2]$, $[x_1,y_1]=[0.2,0.1]$, $\vec{u}=(1,2)$