Mathematics II – Examples

II.7.* Derivative of composed function

Let the functions $z = f(x,y),\ x = x(u,v),\ y = y(u,v)$ are differentiable. Then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} \tag{1}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} \tag{2}$$

Example 160: For the following differentiable functions $z = f(x,y),\ x = u\cos v,\ y = u\sin v$
a) Compute the derivative of composed function $\frac{\partial z}{\partial u},\ \frac{\partial z}{\partial v}$.
b) Calculate for the particular function $z = f(x,y) = \mathrm{e}^x \ln y$.

Example 161: $w = f(x^4 + y^4 - 2z^4)$. Compute $V = \frac{1}{4x^3}\frac{\partial w}{\partial x} + \frac{1}{4y^3}\frac{\partial w}{\partial y} + \frac{1}{4z^3}\frac{\partial w}{\partial z}$.

Example 162: Verify that the function $y = f(x+at) + g(x-at)$ satisfies the partial differential equation $\frac{\partial^2 y}{\partial t^2} - a^2\frac{\partial^2 y}{\partial x^2} = 0$, where a is a constant in $\mathbb{R}$. Assume that both f and g have continuous partial derivative of second order. [Hint: $u = x + at,\ v = x - at,\ y = f(u) + g(v)$]

Example 163: Verify that the function $z = f(\frac{y}{x})$ satisfies the equation $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$. Assume that f is a differentiable function.

Example 164: Let us have $z = f(u,v),\ u = x^2 - y^2,\ v = \mathrm{e}^{xy}$. Compute the following differential expression $W = y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$, where f is a differentiable function.

Example 165: Let $f(u,v) = x^y$, where $x = u^2 + v^2,\ y = uv + v^2$. Compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ in the point A, which has coordinations $u = 1,\ v = -1$.

II.9. Implicit functions

Example 166: Prove, that the equation $x^3 + y^3 = 2x^2 + xy - 1$ defines unique implicit function $y = f(x)$ in a neighbourhood of the point $A = [1,0]$. Compute $f'(1)$ and $f"(1)$.

Example 167: Write equations of both a tangent line t and a normal line n in the point $A = [1,1]$ to the curve, which is defined by implicit way by the equation $F(x,y) \equiv x^3y + y^3x + x^2y - 3 = 0$.

Example 168: By the equation $\ln\sqrt{x^2+y^2} = \mathrm{arctg}\frac{y}{x}$ for $x \neq 0$ is defined an implicit function $y = f(x)$. Compute y' and $y"$.

Example 169: Show, that the equation $x^2 + 2xy + y^2 - 4x + 2y - 3 = 0$ define an implicit function $y = f(x)$, which graph is passing through the point $A = [0,1]$. Check, if the function f is convex in some neighbourhood of the point $x_0 = 0$. Write an equation of a tangent line to the graph of this function in the point A

Example 170: Prove, that the equation $F(x,y,z) \equiv z^3 + 3x^2z - 2xy = 0$ defines unique implicit function $z = f(x,y)$ in a neighbourhood of the point $A = [-1,-2,1]$. Compute $\mathrm{grad} f(-1,-2)$.

Example 171: In a neighborhood of the point $[2,-2,1]$ is given implicit function $z = f(x,y)$ by the equation $\ln z + x^2yz + 8 = 0$. Compute directional derivative $\frac{\partial z}{\partial \vec{s}}(A)$, where $\vec{s} = \overrightarrow{AB},\ A = [2,-2],\ B = [3,-3]$.

Example 172: Write equations of both a tangent plane τ and a normal line n to a surface $z = z(x,y)$ defined implicitly by the equation $F(x,y,z) \equiv x^2 + y^2 + z^2 - 25 = 0$ in the point $A = [3,0,4]$.

Example 173: Write an equation of a tangent plane to a surface defined implicitly by the equation $F(x,y,z) \equiv x(y+z) + z^2 - 5 = 0$ parallel with the plane $\rho:\ 3x - 3y + 6z = 2$.
[Note: There are two tangent planes satisfying the given condition.]