Mathematics II – Examples

IV. Line integral

IV.1. Parametrization of curves

Let $P(t) = [x(t),\, y(t),\, z(t)]$ be a morphism of interval $\langle a,\, b\rangle$ into $\mathbb{E}_3$. If

(1) $P(t)$ is continuous and simple on $\langle a, b\rangle$

(2) derivative $\dot{\mathbf{P}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ is bounded and continuous mapping of (a, b),

(3) $\dot{\mathbf{P}}(t) \neq \vec{0}$ for all $t \in (a, b)$,

then we will call the set $c = \{X \in \mathbb{E}_3 :\ X = \mathbf{P}(t),\ t \in \langle a, b\rangle\,\}$ **simple smooth curve** in $\mathbb{E}_3$ with **parametrization $\mathbf{P}(t)$**.

Analogously we define parametrization of a curve in $\mathbb{E}_2$.

The orientation of a curve determines a direction of move along the curve when value of parameter t increases. The orientation can be done by a unit tangent vector $\vec{\tau}$.

We say that a curve is oriented in accordance with its parametrization $P(t)$, if $P(a)$ is its initial point or if $\vec{\tau} = \dfrac{\dot{\mathbf{P}}(t)}{||\dot{\mathbf{P}}(t)||}$. If $\vec{\tau} = -\dfrac{\dot{\mathbf{P}}(t)}{||\dot{\mathbf{P}}(t)||}$ or if the initial point of a given curve is $P(b)$, then we say that the orienation of the given curve is in contrary with the parametrization $P(t)$.

A simple smooth closed curve c in $\mathbb{E}_2$ is positively oriented (with respect to its interior) if "*moving along the curve in its orientation, the interior is on the left hand*". This is in accordance with "*counterclockwise*" orientation. The opposite orientation is called negative.

Example 424: Consider the curve $c = \{[x, y] \in \mathbb{E}_2 :\ \ y = x^2,\ x \in \langle -4, 4\rangle\,\}$ with starting point $A = [-4,\, 16\,]$. Verify, that $P(t) = [x(t), y(t)]$ is a parametrization of a simple smooth curve c, if

a) $P(t) = [t, t^2], \quad t \in \langle -4, 4\rangle$,

b) $P(t) = [t^2, t^4], \quad t \in \langle -2, 2\rangle$,

c) $P(t) = [\sqrt{t}, t], \quad t \in \langle 0, 16\rangle$.

Example 425: Consider a half circle $c = \{[x, y] \in \mathbb{E}_2 :\ x^2 + y^2 = a^2,\ y \geq 0\,\}$ with starting point $A = [-a,\, 0\,]$. Verify, if $P(t)$ is its parametrization, where

a) $P(t) = [a\cos t, a\sin t], \quad t \in \langle 0, \pi\rangle$

b) $P(t) = [t, \sqrt{a^2 - t^2}], \quad t \in \langle -a, a\rangle$

c) $P(t) = \left[\dfrac{at}{\sqrt{1 + t^2}}, \dfrac{a}{\sqrt{1 + t^2}}\right], \quad t \in \mathbb{R}$

- Find a parametrization of a curve with initial point A and determine if its orientation is in accordance with the parametrization.

Example 426: The curve c is a line segment with initial point $A = [4,\ -1,\ 3]$ and terminal point $B = [3,\ 1,\ 5]$.

Example 427: $c = \{\, [x,\ y] \in \mathbb{E}_2 : (x+3)^2 + (y-2)^2 = 9,\ x \leq -3 \,\},\quad A = [-3, -1]$

Example 428: $c = \{\, [x,\ y] \in \mathbb{E}_2 : \dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1,\ y \geq 0 \,\},\quad A = [3, 0]$

Example 429: $c = \{\, [x,\ y,\ z] \in \mathbb{E}_3 : x^2 + y^2 = 4x,\ y + z = 0,\ z \geq 0 \,\},\quad A = [0, 0, 0]$

Example 430: $c = \{\, [x,\ y,\ z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2,\ x = y,\ x \geq 0 \,\},\quad A = [0, 0, -a]$

- A curve is given $\mathbb{E}_2$ by its parametrization. Find an implicit formula and name this curve.

Example 431: $c = \{\, [x,\ y] \in \mathbb{E}_2 : x = 2t + 1,\ y = 3 - t,\ t \in \langle 1, 4 \rangle \,\}$, the orientation is in accordance with parametrization. [*the line segment with initial point* $A = [3, 2] = P(1)$.]

Example 432: $c = \{\, [x,\ y] \in \mathbb{E}_2 : x = t^2 - 2t + 3,\ y = t^2 - 2t + 1,\ t \in \langle 0, 3 \rangle \,\}$, the orientation is in contrary with parametrization. [*the line segment with initial point* $A = [6, 4] = P(3)$.]

Example 433: $c = \{\, [x,\ y] \in \mathbb{E}_2 : x = 2\sin^2 t,\ y = 4\cos^2 t,\ t \in \langle 0, \frac{\pi}{2} \rangle \,\}$, the orientation is in accordance with parametrization. [*the line segment with initial point* $A = [0, 4] = P(0)$.]

Example 434*: The curve is given in polar coordinates by equation $r(\varphi) = 4\sin\varphi,\ \varphi \in \langle \frac{\pi}{2},\ \pi \rangle \,\}$, the orientation is in contrary with parametrization. [*the circle.*]

- Verify that $c = c_1 \cup c_2$ is a simple, closed and smooth by parts curve. Find parametrization of curves c_1 and c_2. Sketch them and determine their orientation in a case when a point A is both an initial point of c_1 and a terminal point of c_2:

Example 435: $c_1,\ c_2 \in \mathbb{E}_2,\ A = [0, 0];\ c_1 = \{\, [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4x,\ y \geq 0 \,\}$,
$c_2 = \{\, [x, y] \in \mathbb{E}_2 : y = 0,\ x \in \langle 0, 4 \rangle \,\}$.

Example 436: $c_1,\ c_2 \in \mathbb{E}_2,\ A = [1, 8];\ c_1 = \{\, [x, y] \in \mathbb{E}_2 : xy = 8,\ x \in \langle 1, 4 \rangle \,\}$,
$c_2 = \{\, [x, y] \in \mathbb{E}_2 : y + 2x = 10,\ x \in \langle 1, 4 \rangle \,\}$.

Example 437: $c_1,\ c_2 \in \mathbb{E}_2,\ A = [1, 1];\ c_1 = \{\, [x, y] \in \mathbb{E}_2 : y = \sqrt{x},\ x \in \langle 0, 1 \rangle \,\}$,
$c_2 = \{\, [x, y] \in \mathbb{E}_2 : y = x^2,\ x \in \langle 0, 1 \rangle \,\}$.

Example 438: $c_1,\ c_2 \in \mathbb{E}_3,\ A = [3, 0, 2];\ c_1 = \{\, [x, y, z] \in \mathbb{E}_3 : x^2 + y^2 = 9,\ x - z = 1,\ y \geq 0 \,\}$,
$c_2 = \{\, [x, y, z] \in \mathbb{E}_3 : x - z = 1,\ y = 0 \,\}$.

- Suggest a parametrization of a curve c with initial point A:

Example 439: $c = \{ [x, y] \in \mathbb{E}_2 : 3x + y = 1,\ x \in \langle -1, 2 \rangle \},\quad A = [-1, 4]$

Example 440: $c = \{ [x, y, z] \in \mathbb{E}_3 : 2x - y = 2,\ x + z = 3,\ y \in \langle 0, 2 \rangle \},\quad A = [2, 2, 1]$

Example 441: $c = \{ [x, y, z] \in \mathbb{E}_3 : 4x^2 + z^2 = 4,\ y + z = 0,\ y \le 0 \},\quad A = [-1, 0, 0]$

IV.2. Line integral of a scalar function

- Check the existence of line integral $\int_c f \,\mathrm{d}s$. If this integral exists, compute it.

Example 442: $\displaystyle\int_c \frac{1}{x - 2y}\,\mathrm{d}s$; c is a line segment between points A, B, where
a) $A = [1, -2],\ B = [3, 0]$, b) $A = [1, -2],\ B = [3, 4]$.

Example 443: $\displaystyle\int_c \frac{x + 2}{\sqrt{x^2 + y^2}}\,\mathrm{d}s$; a) $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4x \}$
b) $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4 \}$

Example 444: $\displaystyle\int_c x^2\,\mathrm{d}s$; $c = \{ [x, y] \in \mathbb{E}_2 : y = \ln x,\ x \in \langle 1, 3 \rangle \}$

Example 445: $\displaystyle\int_c \frac{x^2}{y}\,\mathrm{d}s$; $c = \{ [x, y] \in \mathbb{E}_2 : y^2 = 2x,\ y \in \langle \sqrt{2}, 2 \rangle \}$

Example 446: $\displaystyle\int_c (x^2 + y^2 + z^2)\,\mathrm{d}s$; c is the first thread of the helix $x = a\cos t,\ y = a\sin t,\ z = bt$.

- Justify on which curve c the integral $\int_c f \,\mathrm{d}s$ exists. Compute the integral.

Example 447: $\displaystyle\int_c \frac{3 - y}{y - x + 2}\,\mathrm{d}s$, a) c is the circle $x^2 - 2x + y^2 = 0 \}$,
b) c is the line segment AB, where $A = [2, 3],\ B = [0, 1]$.
[*a) doesn't exists, b) exists*]

Example 448: $\displaystyle\int_c \frac{1}{x^2 + y^2}\,\mathrm{d}s$, a) $c = \{ [x, y] \in \mathbb{E}_2 : x = t - 3,\ y = 3 - t,\ t \in \langle 1, 4 \rangle \}$,
b) $c = \{ [x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2 \}$.
[*a) doesn't exists, b) exists*]

Example 449: $\displaystyle\int_c \frac{1}{x^2 - y}\,\mathrm{d}s$, a) $c = \{ [x, y] \in \mathbb{E}_2 : y = 2x,\ x \in \langle 1, 3 \rangle \}$,
b) $c = \{ [x, y] \in \mathbb{E}_2 : y = 9,\ x \in \langle 0, 2 \rangle \}$.
[*a) doesn't exists, b) exists*]

Example 450: $\displaystyle\int_c xy \; \mathrm{d}s, \quad c = \{\,[x,y] \in \mathbb{E}_2 : x^2 + y^2 = a^2,\ x \le 0,\ y \ge 0\,\},$

Example 451: $\displaystyle\int_c \sqrt{2y} \; \mathrm{d}s, \quad c = \{\,[x,y] \in \mathbb{E}_2 : x = a(t - \sin t),\ y = a(1 - \cos t), t \in \langle 0, 2\pi\rangle\}$

Example 452: $\displaystyle\int_c \sqrt{x} \; \mathrm{d}s, \quad c = \{\,[x,y] \in \mathbb{E}_2 : y = \sqrt{x}, x \in \langle 1, 2\rangle\}$

Example 453: $\displaystyle\int_c (xy + 2) \; \mathrm{d}s, \quad c = \{\,[x,y] \in \mathbb{E}_2 : x = \cos t,\ y = 2\sin t,\ t \in \langle 0, 2\pi\rangle\}$

Example 454: $\displaystyle\int_c z \; \mathrm{d}s, \quad c = \{\,[x,y,z] \in \mathbb{E}_3 : x = t\cos t,\ y = t\sin t,\ z = t,\ t \in \langle 0, \pi\rangle\}$

Example 455: $\displaystyle\int_c (x + y) \; \mathrm{d}s, \ c = \{\,[x,y,z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2,\ y = x,\ x \ge 0,\ y \ge 0,\ z \ge 0\,\}$

Example 456: $\displaystyle\int_c xyz \; \mathrm{d}s, \quad c = \{\,[x,y,z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 = \tfrac{a^2}{4}, \text{in the first octant}\,\}$

- Consider a scalar function f and a curve c, which is intersection of two given surfaces.
  a) Suggest a parametrization of the curve c.
  b) Write the vector $\dot{\mathbf{P}}(t)$ and compute its length $||\dot{\mathbf{P}}(t)||$.
  c) Compute line integral of a given scalar function f.

Example 457: $f(x,y,z) = y^2 + 2z^2$, curve c is intersection of the planes $x + y + 2z = 5$, $2x + 5y - 2z = 4$ in the first octant.

Example 458: $f(x,y,z) = z^2$, curve c is cut cylindrical surface $\frac{x^2}{9} + \frac{y^2}{25} = 1$ and the plane $4x - 3z = 0$.

IV.3. Application of line integral of a scalar function

- Compute length l of curve c.

Example 459: $c = \{\,[x,y] \in \mathbb{E}_2 : y = 2 - \ln(\cos x),\ x \in \langle 0, \dfrac{\pi}{4}\rangle\,\}$

Example 460: $c = \{\,[x,y] \in \mathbb{E}_2 : x = t^2,\ y = t - \dfrac{t^3}{3},\ t \in \langle -\sqrt{3}, \sqrt{3}\rangle\,\}$