Mathematics II – Examples

IV. Line integral

IV.1. Parametrization of curves

Let $P(t) = [x(t),\ y(t),\ z(t)]$ be a morphism of interval $\langle a,\ b\rangle$ into $\mathbb{E}_3$. If

(1) $P(t)$ is continuous and simple on $\langle a, b\rangle$

(2) derivative $\dot{\mathbf{P}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ is bounded and continuous mapping of (a, b),

(3) $\dot{\mathbf{P}}(t) \neq \vec{0}$ for all $t \in (a, b)$,

then we will call the set $c = \{X \in \mathbb{E}_3 :\ X = \mathbf{P}(t),\ t \in \langle a, b\rangle\,\}$ **simple smooth curve** in $\mathbb{E}_3$ with **parametrization $\mathbf{P}(t)$**.

Analogously we define parametrization of a curve in $\mathbb{E}_2$.

The orientation of a curve determines a direction of move along the curve when value of parameter t increases. The orientation can be done by a unit tangent vector $\vec{\tau}$.

We say that a curve is oriented in accordance with its parametrization $P(t)$, if $P(a)$ is its initial point or if $\vec{\tau} = \dfrac{\dot{\mathbf{P}}(t)}{||\dot{\mathbf{P}}(t)||}$. If $\vec{\tau} = -\dfrac{\dot{\mathbf{P}}(t)}{||\dot{\mathbf{P}}(t)||}$ or if the initial point of a given curve is $P(b)$, then we say that the orienation of the given curve is in contrary with the parametrization $P(t)$.

A simple smooth closed curve c in $\mathbb{E}_2$ is positively oriented (with respect to its interior) if "*moving along the curve in its orientation, the interior is on the left hand*". This is in accordance with "*counterclockwise*" orientation. The opposite orientation is called negative.

Example 424: Consider the curve $c = \{[x, y] \in \mathbb{E}_2 :\ \ y = x^2,\ x \in \langle -4, 4\rangle\,\}$ with starting point $A = [-4,\ 16\,]$. Verify, that $P(t) = [x(t), y(t)]$ is a parametrization of a simple smooth curve c, if

a) $P(t) = [t, t^2],\ \ t \in \langle -4, 4\rangle$,

b) $P(t) = [t^2, t^4],\ \ t \in \langle -2, 2\rangle$,

c) $P(t) = [\sqrt{t}, t],\ \ t \in \langle 0, 16\rangle$.

Example 425: Consider a half circle $c = \{[x, y] \in \mathbb{E}_2 :\ x^2 + y^2 = a^2,\ y \geq 0\,\}$ with starting point $A = [-a,\ 0\,]$. Verify, if $P(t)$ is its parametrization, where

a) $P(t) = [a\cos t, a\sin t],\ \ t \in \langle 0, \pi\rangle$

b) $P(t) = [t, \sqrt{a^2 - t^2}],\ \ t \in \langle -a, a\rangle$

c) $P(t) = \left[\dfrac{at}{\sqrt{1+t^2}}, \dfrac{a}{\sqrt{1+t^2}}\right],\ \ t \in \mathbb{R}$

- Find a parametrization of a curve with initial point A and determine if its orientation is in accordance with the parametrization.

Example 426: The curve c is a line segment with initial point $A = [4,\ -1,\ 3]$ and terminal point $B = [3,\ 1,\ 5]$.

Example 427: $c = \{\,[x,\ y] \in \mathbb{E}_2 : (x+3)^2 + (y-2)^2 = 9,\ x \le -3\,\},\quad A = [-3, -1]$

Example 428: $c = \{\,[x,\ y] \in \mathbb{E}_2 : \dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1,\ y \ge 0\,\},\quad A = [3, 0]$

Example 429: $c = \{\,[x,\ y,\ z] \in \mathbb{E}_3 : x^2 + y^2 = 4x,\ y + z = 0,\ z \ge 0\,\},\quad A = [0, 0, 0]$

Example 430: $c = \{\,[x,\ y,\ z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2,\ x = y,\ x \ge 0\,\},\quad A = [0, 0, -a]$

- A curve is given $\mathbb{E}_2$ by its parametrization. Find an implicit formula and name this curve.

Example 431: $c = \{\,[x,\ y] \in \mathbb{E}_2 : x = 2t + 1,\ y = 3 - t,\ t \in \langle 1, 4\rangle\,\}$, the orientation is in accordance with parametrization. [*the line segment with initial point* $A = [3, 2] = P(1)$.]

Example 432: $c = \{\,[x,\ y] \in \mathbb{E}_2 : x = t^2 - 2t + 3,\ y = t^2 - 2t + 1,\ t \in \langle 0, 3\rangle\,\}$, the orientation is in contrary with parametrization. [*the line segment with initial point* $A = [6, 4] = P(3)$.]

Example 433: $c = \{\,[x,\ y] \in \mathbb{E}_2 : x = 2\sin^2 t,\ y = 4\cos^2 t,\ t \in \langle 0, \frac{\pi}{2}\rangle\,\}$, the orientation is in accordance with parametrization. [*the line segment with initial point* $A = [0, 4] = P(0)$.]

Example 434*: The curve is given in polar coordinates by equation $r(\varphi) = 4\sin\varphi,\ \varphi \in \langle \frac{\pi}{2},\ \pi\rangle\,\}$, the orientation is in contrary with parametrization. [*the circle*.]

- Verify that $c = c_1 \cup c_2$ is a simple, closed and smooth by parts curve. Find parametrization of curves c_1 and c_2. Sketch them and determine their orientation in a case when a point A is both an initial point of c_1 and a terminal point of c_2:

Example 435: $c_1,\ c_2 \in \mathbb{E}_2,\ A = [0, 0];\ c_1 = \{\,[x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4x,\ y \ge 0\,\},$
$c_2 = \{\,[x, y] \in \mathbb{E}_2 : y = 0,\ x \in \langle 0, 4\rangle\,\}.$

Example 436: $c_1,\ c_2 \in \mathbb{E}_2,\ A = [1, 8];\ c_1 = \{\,[x, y] \in \mathbb{E}_2 : xy = 8,\ x \in \langle 1, 4\rangle\,\},$
$c_2 = \{\,[x, y] \in \mathbb{E}_2 : y + 2x = 10,\ x \in \langle 1, 4\rangle\,\}.$

Example 437: $c_1,\ c_2 \in \mathbb{E}_2,\ A = [1, 1];\ c_1 = \{\,[x, y] \in \mathbb{E}_2 : y = \sqrt{x},\ x \in \langle 0, 1\rangle\,\},$
$c_2 = \{\,[x, y] \in \mathbb{E}_2 : y = x^2,\ x \in \langle 0, 1\rangle\,\}.$

Example 438: $c_1,\ c_2 \in \mathbb{E}_3,\ A = [3, 0, 2];\ c_1 = \{\,[x, y, z] \in \mathbb{E}_3 : x^2 + y^2 = 9,\ x - z = 1,\ y \ge 0\,\},$
$c_2 = \{\,[x, y, z] \in \mathbb{E}_3 : x - z = 1,\ y = 0\,\}.$

- Suggest a parametrization of a curve c with initial point A:

Example 439: $c = \{\, [x, y] \in \mathbb{E}_2 : 3x + y = 1,\ x \in \langle -1, 2 \rangle \,\},\quad A = [-1, 4]$

Example 440: $c = \{\, [x, y, z] \in \mathbb{E}_3 : 2x - y = 2,\ x + z = 3,\ y \in \langle 0, 2 \rangle \,\},\quad A = [2, 2, 1]$

Example 441: $c = \{\, [x, y, z] \in \mathbb{E}_3 : 4x^2 + z^2 = 4,\ y + z = 0,\ y \le 0 \,\},\quad A = [-1, 0, 0]$

IV.2. Line integral of a scalar function

- Check the existence of line integral $\int_c f\ \mathrm{d}s$. If this integral exists, compute it.

Example 442: $\displaystyle\int_c \frac{1}{x - 2y}\ \mathrm{d}s$; c is a line segment between points A, B, where
a) $A = [1, -2],\ B = [3, 0]$, b) $A = [1, -2],\ B = [3, 4]$.

Example 443: $\displaystyle\int_c \frac{x + 2}{\sqrt{x^2 + y^2}}\ \mathrm{d}s$; a) $c = \{\, [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4x \,\}$
b) $c = \{\, [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 4 \,\}$

Example 444: $\displaystyle\int_c x^2\ \mathrm{d}s$; $c = \{\, [x, y] \in \mathbb{E}_2 : y = \ln x,\ x \in \langle 1, 3 \rangle \}$

Example 445: $\displaystyle\int_c \frac{x^2}{y}\ \mathrm{d}s$; $c = \{\, [x, y] \in \mathbb{E}_2 : y^2 = 2x,\ y \in \langle \sqrt{2}, 2 \rangle \}$

Example 446: $\displaystyle\int_c (x^2 + y^2 + z^2)\ \mathrm{d}s$; c is the first thread of the helix $x = a\cos t,\ y = a\sin t,\ z = bt$.

- Justify on which curve c the integral $\int_c f\ \mathrm{d}s$ exists. Compute the integral.

Example 447: $\displaystyle\int_c \frac{3 - y}{y - x + 2}\ \mathrm{d}s$, a) c is the circle $x^2 - 2x + y^2 = 0 \,\}$,
b) c is the line segment AB, where $A = [2, 3],\ B = [0, 1]$.
[*a) doesn't exists, b) exists*]

Example 448: $\displaystyle\int_c \frac{1}{x^2 + y^2}\ \mathrm{d}s$, a) $c = \{\, [x, y] \in \mathbb{E}_2 : x = t - 3,\ y = 3 - t,\ t \in \langle 1, 4 \rangle \,\}$,
b) $c = \{\, [x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2 \,\}$.
[*a) doesn't exists, b) exists*]

Example 449: $\displaystyle\int_c \frac{1}{x^2 - y}\ \mathrm{d}s$, a) $c = \{\, [x, y] \in \mathbb{E}_2 : y = 2x,\ x \in \langle 1, 3 \rangle \,\}$,
b) $c = \{\, [x, y] \in \mathbb{E}_2 : y = 9,\ x \in \langle 0, 2 \rangle \,\}$.
[*a) doesn't exists, b) exists*]

Example 450: $\displaystyle\int_c xy\ \mathrm{d}s, \quad c = \{\, [x,y] \in \mathbb{E}_2 : x^2 + y^2 = a^2,\ x \le 0,\ y \ge 0 \,\},$

Example 451: $\displaystyle\int_c \sqrt{2y}\ \mathrm{d}s, \quad c = \{\, [x,y] \in \mathbb{E}_2 : x = a(t - \sin t),\ y = a(1 - \cos t), t \in \langle 0, 2\pi \rangle \}$

Example 452: $\displaystyle\int_c \sqrt{x}\ \mathrm{d}s, \quad c = \{\, [x,y] \in \mathbb{E}_2 : y = \sqrt{x}, x \in \langle 1, 2 \rangle \}$

Example 453: $\displaystyle\int_c (xy + 2)\ \mathrm{d}s, \quad c = \{\, [x,y] \in \mathbb{E}_2 : x = \cos t,\ y = 2\sin t,\ t \in \langle 0, 2\pi \rangle \}$

Example 454: $\displaystyle\int_c z\ \mathrm{d}s, \quad c = \{\, [x,y,z] \in \mathbb{E}_3 : x = t\cos t,\ y = t\sin t,\ z = t,\ t \in \langle 0, \pi \rangle \}$

Example 455: $\displaystyle\int_c (x+y)\ \mathrm{d}s,\ c = \{\, [x,y,z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2,\ y = x,\ x \ge 0,\ y \ge 0,\ z \ge 0 \,\}$

Example 456: $\displaystyle\int_c xyz\ \mathrm{d}s, \quad c = \{\, [x,y,z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 = \tfrac{a^2}{4}$, in the first octant $\}$

- Consider a scalar function f and a curve c, which is intersection of two given surfaces.

  a) Suggest a parametrization of the curve c.

  b) Write the vector $\dot{\mathbf{P}}(t)$ and compute its length $||\dot{\mathbf{P}}(t)||$.

  c) Compute line integral of a given scalar function f.

Example 457: $f(x,y,z) = y^2 + 2z^2$, curve c is intersection of the planes $x + y + 2z = 5$, $2x + 5y - 2z = 4$ in the first octant.

Example 458: $f(x,y,z) = z^2$, curve c is cut cylindrical surface $\frac{x^2}{9} + \frac{y^2}{25} = 1$ and the plane $4x - 3z = 0$.

IV.3. Application of line integral of a scalar function

- Compute length l of curve c.

Example 459: $c = \{\, [x,y] \in \mathbb{E}_2 : y = 2 - \ln(\cos x),\ x \in \langle 0, \dfrac{\pi}{4} \rangle \,\}$

Example 460: $c = \{\, [x,y] \in \mathbb{E}_2 : x = t^2,\ y = t - \dfrac{t^3}{3},\ t \in \langle -\sqrt{3}, \sqrt{3} \rangle \,\}$

Example 461*: $c = \{\, [x,y] \in \mathbb{E}_2 : x = a\cos^3 t,\ y = a\sin^3 t,\ t \in \langle 0, 2\pi \rangle \,\}$

Example 462*: c is a part of logarithmic spiral $r = ae^{k\varphi}$ inside a circle $r = a,\ k \geq 0,\ a \geq 0.$

Example 463*: $c = \{\, [x, y, z] \in \mathbb{E}_3 : x = \displaystyle\int_1^t \frac{\cos z}{z}\,\mathrm{d}z, y = \int_1^t \frac{\sin z}{z}\,\mathrm{d}z,\ t \in \mathbb{R}\,\}$. Compute the distance between origin and the closest point, in which a tangent is parallel with y-axis.

- Compute area of given cylindrical surfaces between xy-plane and given surfaces.

Example 464*: $y^2 = 4x,\ z = 2\sqrt{x - x^2}$

Example 465*: $x^2 + y^2 = \dfrac{1}{4},\ z = xy,\ x \geq 0,\ y \geq 0$

- Compute mass of a curve c with longitudinal mass density $\rho = \rho(x, y)$ resp. $\rho(x, y, z)$:

Example 466: $c = \{\, [x, y] \in \mathbb{E}_2 : x^2 + y^2 = a^2,\ x \geq 0,\ y \geq 0\,\},\quad \rho(x, y) = x$

Example 467: $c = \{\, [x, y, z] \in \mathbb{E}_3 : x = at,\ y = \dfrac{a}{\sqrt{2}}t^2,\ x = \dfrac{a}{3}t^3,\ t \in \langle 0, 1\rangle\,\},\quad \rho = \sqrt{\dfrac{2y}{a}}$

Example 468: The curve c is the first thread of the helix $x = a\cos t,\ y = a\sin t,\ z = at$ and the longitudinal density equals to square distance from z-axis.

Example 469: $c = c_1 \cup c_2;\quad c_1 = \{\, [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 2ax,\ y \geq 0\,\},$
$c_2 = \{\, [x, y] \in \mathbb{E}_2 : y = 0,\ x \in \langle 0, 2a\rangle,\ a > 0\,\},\quad \rho(x, y) = x^2 + y^2$

- Find a center of mass of a curve c with mass density $\rho(x, y)$ or $\rho(x, y, z)$ respectively:

Example 470: $c = \{\, [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 1,\ y \geq 0\,\},\quad \rho(x, y) = a(1 - y),\ a > 0$

Example 471: $c = \{\, [x, y] \in \mathbb{E}_2 :\ x = a(t - \sin t),\ y = a(1 - \cos t),\ a > 0,\ t \in \langle 0, 2\pi\rangle\,\},\ \ \rho(x, y) = 1$

Example 472: $c = c_1 \cup c_2 \cup c_3;\quad c_1 = \{\, [x, y, z] \in \mathbb{E}_3 :\ x^2 + y^2 = a^2,\ z = 0,\ x \geq 0,\ y \geq 0\,\},$
$c_2 = \{\, [x, y, z] \in \mathbb{E}_3 :\ x^2 + z^2 = a^2,\ y = 0,\ x \geq 0,\ z \geq 0\,\},$
$c_3 = \{\, [x, y, z] \in \mathbb{E}_3 :\ y^2 + z^2 = a^2,\ x = 0,\ y \geq 0,\ z \geq 0\,\},\quad \rho(x, y) = 1$

Example 473: Compute moment of inertia relative to yz-plane of the curve
$c = \{\, [x, y, z] \in \mathbb{E}_3 :\ x = a\cos t,\ y = a\sin t,\ z = bt,\ t \in \langle 0, 2\pi\rangle\,\}$, if $\rho(x, y, z) = x^2 + y^2$.

Example 474: Compute moment of inertia relative to z-axis of the curve $c \subset \mathbb{E}_3$:
$c = \{\, [x, y, z] \in \mathbb{E}_3 :\ 2x^2 + y^2 = 2,\ x + z = 1\,\}$, if $\rho(x, y, z) = z$.

- Consider a curve c, which has a longitudinal mass density ρ.
  a) Suggest a parametrization $X = P(t),\ t \in \langle a, b\rangle$ of the given curve c and compute length of vector $\dot{P}(t)$.
  b) Compute mass of the curve c if a longitudinal mass density is ρ.
  c) Indicate, what can be a meaning of the given integral, e.g., static moment or moment of inertia, under which density and relative to which object (point, line or plane).

Example 475: The curve is line segment AB, where $A = [1, 0],\ B = [2, 3],\ \rho(x, y) = x^2 + y^2$.

Example 476: The curve is line segment AB, where $A = [1, 0],\ B = [1, 2],\ \rho(x, y) = x^2 y$.

Example 477: $c = \{\, [x, y] \in \mathbb{E}_2 : x^2 + y^2 = 9,\ x \geq 0 \,\},\quad \rho(x, y) = x$

Example 478: $c = \{\, [x, y, z] \in \mathbb{E}_3 : x = 3\cos t,\ y = 3\sin t,\ z = t/3,\ t \in \langle 0, 3\rangle \,\}$,
$\rho(x, y, z) = x^2 + y^2 + z^2$

Example 479: $c = \{\, [x, y, z] \in \mathbb{E}_3 : x = 2\cos t,\ y = 2\sin t,\ z = t/4,\ t \in \langle 0, 2\pi\rangle \,\}$,
$\rho(x, y, z) = z^2/(x^2 + y^2)$

Example 480: $c = \{\, [x, y] \in \mathbb{E}_2 : y = \frac{x^2}{2} + 2$ between points $A = [0, 2],\ B = [2, 4] \,\},\quad \rho(x, y) = x$

Example 481: $c = \{\, [x, y, z] \in \mathbb{E}_3 : x = t\cos t,\ y = t\sin t,\ z = t,\ t \in \langle 0, 1\rangle \,\},\quad \rho(x, y, z) = z$

- Compute length l of a given curve c:

Example 482: $c = \{\, [x, y] \in \mathbb{E}_2 : y = \frac{1}{3}x\sqrt{x},\ x \in \langle 0, 5\rangle \,\}$

Example 483: $c = \{\, [x, y, z] \in \mathbb{E}_3 : x = 3t,\ y = 3t^2,\ z = 2t^3,\ t \in \langle 0, 1\rangle \,\}$

Example 484: $c = \{\, [\varphi, r] \in \mathbb{E}_2 : r(\varphi) = a(1 + \cos\varphi),\ \varphi \in \langle 0, \pi\rangle,\ a > 0 \,\}$
[*upper half of cardioide*]

Example 485: $c = \{\, [\varphi, r] \in \mathbb{E}_2 : r(\varphi) = \sin^3 \frac{\varphi}{3},\ \varphi \in \langle 0, 3\pi\rangle \,\}$

Example 486: $c = \left\{\, [x, y] \in \mathbb{E}_2 : y = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x}\, \mathrm{d}x \,\right\}$ [*use* $\cos x \geq 0 \Rightarrow x \in (-\pi/2, \pi/2)$]

Example 487: $c = \{\, [x, y, z] \in \mathbb{E}_3 : x = 2\cos t,\ y = 2\sin t, z = t/2,\ t \in \langle 0, 2\pi\rangle \,\}$

Example 488: $c = \{\, [x, y] \in \mathbb{E}_2 : y = \frac{1}{3}x\sqrt{x},\ x \in \langle 0, 5\rangle \,\}$

Example 489: $c = \{\, [x, y, z] \in \mathbb{E}_3 : x = R\cos t,\ y = R\sin t, z = at,\ t \in \langle 0, 2\pi\rangle \,\}$

- Compute mass m of a curve c with longitudinal mass density $\rho(x,y)$ or $\rho(x,y,z)$ resp.

Example 490: $\rho = x(x^2+z^2), \quad c = \{\, [x,y,z] \in \mathbb{E}_3 : y^2 + 2z^2 = 4,\ x = z,\ x \geq 0 \,\}$

Example 491: $\rho = x(y+2), \quad c = \{\, [x,y] \in \mathbb{E}_2 : x^2 + y^2 = 4,\ x \geq 0 \,\}$

Example 492: $\rho = x^{4/3} + y^{4/3}, \quad c = \{\, [x,y] \in \mathbb{E}_2 : x = a\cos^3 t,\ y = a\sin^3 t,\ t \in \langle 0, \pi/2 \rangle \,\}$

Example 493: $\rho = \mathrm{e}^{x^2+y^2}, \quad c = \{\, [x,y] \in \mathbb{E}_2 : x^2 + y^2 = a^2,\ x \geq 0,\ y \geq 0 \,\}$

Example 494: Compute moment of inertia of homogeneous half circle with radius a relative to axis of symmetry.

Example 495: Compute moment of inertia of the curve $c = \{\, [x,y] \in \mathbb{E}_2 : x = a\cos^3 t,$ $y = a\sin^3 t,\ t \in \langle 0, \pi/2 \rangle \,\}$ relative to x-axis, if $\rho = 1$.

- Find center of mass T of a curve c with longitudinal mass density ρ.

Example 496: $c = \{\, [x,y,z] \in \mathbb{E}_3 : x = a\cos t,\ y = a\sin t,\ z = at, a > 0,\ t \in \langle 0, 2\pi \rangle \,\}$, $\rho = \dfrac{z^2}{x^2+y^2}$

Example 497: $c = c_1 \cup c_2; \quad c_1 = \{\, [x,y] \in \mathbb{E}_2 : y = 6\sqrt{x},\ x \in \langle 1, 6 \rangle \,\},$
$c_2 = \{\, [x,y] \in \mathbb{E}_2 : y = -6\sqrt{x},\ x \in \langle 1, 6 \rangle \,\}, \quad \rho = 1$

IV.4. Line integral of a vector function

Consider a smooth single curve c with parametrization $P(t),\ t \in \langle a, b \rangle$. Then

$$\int_c \vec{f}(X)\,\mathrm{d}\vec{s} = \pm \int_a^b \vec{f}\,(P(t)) \cdot \dot{P}(t)\,\mathrm{d}t$$

The sign plus or minus depends on accordance of curve's orientation and orientation induced by parametrization $P(t)$.

Comment: Line integral of vector function $\vec{f} = (U, V, W)$ can be written using differentials in the form

$$\int_c \vec{f}\,\mathrm{d}\vec{s} = \int_c (U, V, W) \cdot (\,\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z) = \int_c (U\,\mathrm{d}x + V\,\mathrm{d}y + W\,\mathrm{d}z).$$

Analogously for $c \subset \mathbb{E}_2$.

- Compute given line integrals on oriented curve c with initial point A.

Example 498: $\int_c (x, -y^2)\,\mathrm{d}\vec{s}$, c is the line segment from $A = [1, -2]$ into $B = [3, 2]$.

Example 499: $\int_c (x^2 - y^2, 1)\,\mathrm{d}\vec{s}$, $\quad c = \{[x, y] \in \mathbb{E}_2 :\ y = x^3\}$, from $A = [0, 0]$ into $B = [3, 27]$.

Example 500: $\int_c (-y, x)\,\mathrm{d}\vec{s}$, $\quad c = \{[x, y] \in \mathbb{E}_2 :\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ x \ge 0,\ y \ge 0\}$, $\ A = [a, 0]$.

Example 501: $\int_c (y, -x, z)\,\mathrm{d}\vec{s}$, $\quad c = \{[x, y, z] \in \mathbb{E}_3 :\ x = R\cos t,\ y = R\sin t,\ z = \dfrac{at}{2\pi}\ R > 0\}$, from intersection with the plane $z = 0$ into intersection with the plane $z = a,\ a > 0$.

Example 502: $\int_c -x\cos y\,\mathrm{d}x + y\sin x\,\mathrm{d}y$, $\quad c$ is the line segment from $A = [0, 0]$ into $B = [\pi, 2\pi]$.

Example 503: $\oint_c x\,\mathrm{d}y$, $\quad c$ is a perimeter of the triangle between lines $x = 0,\ y = 0$ and $2x + 7y = 14$. The curve c is oriented positively.

Example 504: $\int_c (x\cos y, 0)\,\mathrm{d}\vec{s}$, $\quad c$ is oriented line segment from $A = [0, 1]$ into $B = [1, 2]$.

Example 505: $\int_c (x^2 + y^2, x^2 - y^2)\,\mathrm{d}\vec{s}$, $\quad c$ is oriented curve $y = 1 - |1 - x|,\ x \in \langle 0, 2\rangle$, initial point is $A = [0, 0]$.

Example 506: $\int_c (x^2 - 2xy)\,\mathrm{d}x + (y^2 - 2xy)\,\mathrm{d}y$, $\quad c$ is arc of parabola $y = x^2$ from $A = [-1, 1]$ into $B = [1, 1]$.

Example 507: $\int_c (y, x)\,\mathrm{d}\vec{s}$, $\quad c = \{[x, y] \in \mathbb{E}_2 :\ x^2 + y^2 = a^2,\ x \ge 0,\ y \ge 0,\ a > 0\}$, initial point is $A = [a, 0]$.

IV.5. Work along a curve

- Compute a work W done by a given force $\vec{f}$ along a curve c:

Example 508: $\vec{f} = (x + y, 2x)$, $\quad c = \{[x, y] \in \mathbb{E}_2 :\ x^2 + y^2 = R^2,\ y \ge 0\}$, initial point is $B = [-R, 0]$.

Example 509: $\vec{f} = \left(\dfrac{2y}{x^2 + 4y^2}, \dfrac{-2x}{x^2 + 4y^2}\right)$, $\quad c = \left\{[x, y] \in \mathbb{E}_2 :\ \dfrac{x^2}{4} + y^2 = 1\right\}$, the curve c is oriented positively.

Example 510: $\vec{f} = -y\vec{i} + x\vec{j} + a\vec{k}$, $\quad c = \{[x, y, z] \in \mathbb{E}_3 : \ x^2 + y^2 - 4x + 3 = 0, \ z = 2\}$, the orientation of c is given by tangent vector $\vec{\tau}([2, 1, 2]) = -\vec{i}$.

Example 511: $\vec{f} = (xy, x+y), c = c_1 \cup c_2, c$ is closed curve, where cv_1 is a part of parabola $y = x^2$ and c_2 is a part of straight line $y = x$, c is positively oriented.

- Compute a work done by a given force $\vec{f}$ along a curve c:

Example 512: $\vec{f} = \dfrac{(x - y, x + y)}{x^2 + y^2}$, $\quad c = \{[x, y] \in \mathbb{E}_2 : \ x^2 + y^2 = 4\}$, curve c is negatively oriented.

Example 513: $\vec{f} = \dfrac{2}{x^2 + y^2}(y, -x)$, $\quad c = \{[x, y] \in \mathbb{E}_2 : \ x^2 + y^2 = 16\}$, curve c is positively oriented.

Example 514: $\vec{f} = \left(\dfrac{1}{y}, -\dfrac{1}{x}\right)$, $\quad c$ is a perimeter of $\triangle\, ABC$, where $A = [1, 1], \ B = [2, 1]$, $C = [2, 2]$. Curve c is positively oriented.

Example 515: $\vec{f} = \dfrac{(y^2, -x^2)}{x^2 + y^2}$, $\quad c = \{[x, y] \in \mathbb{E}_2 : \ x^2 + y^2 = a^2, \ a > 0, \ y \geq 0\}$ from the point $[a, 0]$ into the point $[-a, 0]$.

Example 516: $\vec{f} = (y, 2)$, $\quad c$ is a closed curve formed by the positive parts of the half-axes and quarter of an ellipse $x = 2\cos t, \ y = \sin t$ in the first quadrant. Orientation is negative.

Example 517: $\vec{f} = (x + y, 2x)$, $\quad c = \{[x, y] \in \mathbb{E}_2 : \ x = a\cos t, \ y = a\sin t, \ t \in \langle 0, 2\pi\rangle\}$, orientation is positive.

Example 518: $\vec{f} = (y, z, x)$, $\quad c$ is a line segment with initial point $[a, 0, 0]$ and terminal point $[a, a, a]$.

Example 519: $\vec{f} = (y, z, x)$, $\quad c$ is an intersection of surfaces $z = xy, \ x^2 + y^2 = 1$ from the point $[1, 0, 0]$ into the point $[0, 1, 0]$.

Example 520: $\vec{f} = (yz, z\sqrt{R^2 - y^2}, xy)$, $\ c = \{X \in \mathbb{E}_3 : \ X = P(t), \ P(t) = \left(R\cos t, R\sin t, \dfrac{at}{2\pi}\right)$, $a > 0, \ R > 0, \ t \in \langle 0, 2\pi\rangle\}$, curve's orientation is in accordance with parametrization.

Example 521: $\vec{f} = (x, y, xz - y)$, $\quad c = \{X \in \mathbb{E}_3 : \ X = P(t), \ P(t) = (t^2, 2t, 4t^3), \ t \in \langle 0, 1\rangle\}$, curve's orientation is in accordance with parametrization.

Example 522: $\vec{f} = (x, y, z)$, $\quad c$ is a quarter of ellipse $c = \{[x, y] \in \mathbb{E}_2 : \ x^2 + y^2 = 4, \ x + z = 2\}$ from $[2, 0, 0]$ into $[0, 2, 2]$.

Example 523: $\vec{f} = (y^2, z^2, x^2)$, $\quad c = \{X \in \mathbb{E}_3 : \ X = P(t), \ P(t) = (5, 2 + 4\sin t, -3 + 4\cos t)$, $t \in \langle 0, 2\pi\rangle\}$, curve's orientation is in accordance with parametrization.

- Consider a vector field $\vec{f}$ and an oriented curve c.
  a) Sketch the given curve $c \in \mathbb{E}_2$.
  b) Suggest its parametrization $P(t)$ and justify, if the curve c is oriented in accordance with your parametrization.
  c) Compute a work, which is done by the force $\vec{f}$ along the oriented curve c.

Example 524: $\vec{f} = (y, z, x)$, c is a line segment with initial point $[a, 0, 0]$ and terminal point $[a, a, a]$.

Example 525: $\vec{f} = (xy, y - 1)$, c is a segment of the curve $y = x^2$ with initial point $A = [0, 0]$ and terminal point $B = [2, 4]$.

Example 526: $\vec{f} = (\sqrt{x} + y, x + \sqrt{y})$, c is a segment of the curve $x = y^2$ from the point $A = [8, 2]$ into the point $B = [2, 1]$.

Example 527: $\vec{f} = \left(x^3, \frac{1}{y} \ln y\right)$, c is given by $y = \mathrm{e}^x$, where $|x| \leq 1$ and the initial point is at $x = -1$.

Example 528: $\vec{f} = (2, xy)$, $c = \{[x, y] \in \mathbb{E}_2 : \ y = \ln x + 1, \ x \in \langle 1, \mathrm{e} \rangle \}$

Example 529: $\vec{f} = (0, x)$, $c = \{[x, y] \in \mathbb{E}_2 : \ xy = 1, \ x \in \langle 2, 1 \rangle \}$

Example 530: $\vec{f} = \left(\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2}\right)$, $c = \{[x, y] \in \mathbb{E}_2 : \ x^2 + y^2 = 4, \ x \geq 0 \}$ is oriented from $[0, 2]$ to $[0, -2]$.

Example 531: $\vec{f} = \left(\frac{2y}{x^2 + 4y^2}, -\frac{2x}{x^2 + 4y^2}\right)$, $c = \{[x, y] \in \mathbb{E}_2 : \ \frac{x^2}{4} + y^2 = 1 \}$, with negative orientation.

- Let us have a line segment AB, a vector function $\vec{f}$ and a scalar function ρ.
  a) Suggest a parametrization $P(t)$ of the line segment and compute tangent vector $\dot{P}(t)$.
  b) Using a line integral of the vector function compute a work, which is done by force $\vec{f}$ along the line segment AB from A to B.
  c) Compute a mass of the curve k, if a longitudinal mass density is $\rho(x, y)$.

Example 532: $A = [1, 0]$, $B = [2, 3]$, $\vec{f}(x, y) = (x^2, xy)$, $\rho(x, y) = x^2 + y^2$

Example 533: $A = [0, 1]$, $B = [1, 2]$, $\vec{f}(x, y) = (x\sqrt{y^2 - 2x}, 0)$, $\rho(x, y) = xy$

IV.6. Green's theorem

The line integral of vector field along a closed curve we call *circulation of vector field $\vec{f}$ around the circle c* and write $\oint_c \vec{f} \cdot \mathrm{d}\vec{s}$.

Let 1) the vector function $\vec{f} = (U(x,y), V(x,y))$ has continuous partial derivatives in a region $G \subset \mathbb{E}_2$,

2) curve $c \subset G$ is positively oriented, closed, simple and smooth by parts,

3) $\mathrm{int} c \subset G$.

Then

$$\oint_c \vec{f} \cdot \mathrm{d}\vec{s} = \iint_{\mathrm{int} c} \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y.$$

Comment: If the curve is oriented negatively, the integral has negative sign.
Comment: In differentials, the Green's theorem has the following form:

$$\oint_c U \, \mathrm{d}x + V \, \mathrm{d}y = \iint_{\mathrm{int} c} \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y.$$

Example 534: Using Green's theorem, compute circulation of the vector field $\vec{f} = (2x + 3y, 5x - y - 4)$ along a perimeter of triangle ABC in direction $A \to B \to C$, where $A = [1,0],\ B = [1,-3],\ C = [-3,0]$.

Example 535: Prove the existence of integral $\oint_c \left(\ln(x^2 + y^2), -2\mathrm{argtg}\frac{y}{x} \right) \cdot \mathrm{d}\vec{s}$ and justify the possibility of use the Green's theorem in this case, if $c \subset \mathbb{E}_2$ is positively oriented curve defined by one of the following equations:

a) $x^2 + y^2 = 1$, b) $(x+1)^2 + y^2 = 1$, c) $(x.2)^2 + y^2 = 1$,

d) c is a perimeter of square with vertices $A = [1,0],\ B = [0,1],\ C = [-1,0],\ D = [0,-1]$.

If the integral exists, compute them using Green's theorem.

Example 536: Compute circulation of the vector field $\vec{f} = (-y, x)$ along the negatively oriented curve $c = c_1 \cup c_2$, where $c_1 = \{[x,y] \in \mathbb{E}_2 :\ x^2 - 2x + y^2 = 0,\ y \geq 0\}$ and $c_2 = \{[x,y] \in \mathbb{E}_2 :\ y = 0,\ x \in \langle 0, 2 \rangle\}$

a) directly as line integral, b) using Green's theorem.

Example 537*: Using line integral, compute area of interior of the curve $c = \{[x,y] \in \mathbb{E}_2 :\ x^{2/3} + y^{2/3} = a^{2/3},\ a > 0\}$.

- There is given a set D and vector field $\vec{f}$.
  a) Write Green's theorem (both assumptions and statement).
  b) Sketch the set D and highlit a curve c, which forms a positively oriented boundary of D.
  c) Compute a circulation of vector field $\vec{f}$ along curve c.

Example 538: $D = \{[x, y] \in \mathbb{E}_2 : \; x^2 + y^2 \leq 4, \; x \geq 0, \; y \geq 0\}, \quad \vec{f} = (xy, x^2 + 2x)$

Example 539: $D = \{[x, y] \in \mathbb{E}_2 : \; 0 \leq x \leq 2, \; 0 \leq y \leq 2x\}, \quad \vec{f} = (y^2 - 3y, xy)$

Example 540: $D = \{[x, y] \in \mathbb{E}_2 : \; 0 \leq x \leq 3, \; 0 \leq y \leq 1\}, \quad \vec{f} = (\frac{1}{3}y^3, x^2 + y^2)$

Example 541: $D = \{[x, y] \in \mathbb{E}_2 : \; 0 \leq x \leq 4, \; 4 - 4x \leq y \leq 4\}, \quad \vec{f} = (y^2, (x + y)^2)$

Example 539: Examine an existence of integral $\oint_c \left(-\frac{-1}{x^2}, 2x \right) \, \mathrm{d}\vec{s}$ and decide if Green's theorem can be used, if $c \subset \mathbb{E}_2$ is negatively oriented curve:
a) $c = \{[x, y] \in \mathbb{E}_2 : \; x^2 + y^2 = 1\}$,
b) $c = \{[x, y] \in \mathbb{E}_2 : \; x^2 + (y - 2)^2 = 1\}$,
c) $c = \{[x, y] \in \mathbb{E}_2 : \; (x - 2)^2 + y^2 = 1\}$.
If yes, compute the integral using Green's theorem.

- There are given a vector field $\vec{f}$ and a curve c.
  a) Write Green's theorem and verify if its assumptions needed for computation the integral $\oint \vec{f} \, \mathrm{d}\vec{s}$ are met.
  b) Compute the integral using Green's theorem.
  c) Compute the same integral directly (without Green's theorem).

Example 543: $\vec{f} = (-y, x), \; c = \{[x, y] \in \mathbb{E}_2 : \; x^2 + y^2 = 16\}$, orientation is negative.

Example 544: $\vec{f} = (1 - x^2, x(1 + y^2)), \; c = \partial D$ is a boundary of the set $D = \langle 0, 2\rangle \times \langle 0, 2\rangle$, orientation is negative.

Example 545*: $\vec{f} = \left(\frac{2y^3}{3} + g(x), xy^2 + h(y) \right)$, where g, h are arbitrary functions of one variable having continuous derivatives in $\mathbb{R}$, c is negatively oriented boundary of the set $D = \{[x, y] \in \mathbb{E}_2 : \; y \geq 0, \; y \leq 2 - x, \; x \geq y^2\}$.

Example 546: Using line integral of vector function compute the work, which id done by the force $\vec{f} = (x + 1, 2y)$ along a given oriented curve:
a) c_1 is line segment AB with initial point $A = [-1, 0]$ and terminal point $B = [1, 0]$.
b) $c_2 = \{[x, y] \in \mathbb{E}_2 : \; x^2 + y^2 = 1, \; y \geq 0\}$ with initial poinr $B = [1, 0]$.
c) c is closed curve $c_1 \cup c_2$ (Use Green's theorem).

Example 547: Compute circulation of $\vec{f} = \dfrac{2(y,-x)}{x^2+y^2}$ along a positively oriented curve $c = \{[x,y] \in \mathbb{E}_2 :\ x^2 + y^2 = 16\,\}$. Can be used Green's theorem? Justify your answer.

Example 548: Using Green's theorem compute circulation of $\vec{f} = \big(y, (x-y)^2\big)$ along the negatively oriented curve $c = \{[x,y] \in \mathbb{E}_2 :\ (x-1)^2 + y^2 = 1\,\}$.

Example 549*: Using line integral derive a formula for area of the region, which is bounded by ellipse $c = \Big\{[x,y] \in \mathbb{E}_2 :\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,\Big\}$.

Example 550*: Using line integral compute area of the region, which is bounded by arc of cykloide $x = a(t - \sin t),\ y = a(1 - \cos t),\ t \in \langle 0, 2\pi\rangle$ and the line segment from point $[0,0]$ into $[2\pi a, 0]$.

Example 551*: Let c_1 be a line segment from $[0,0]$ to $[1,1]$, c_2 is a part of parabola $y = x^2$, also from $[0,0]$ to $[1,1]$. Using Green's theorem compute $I_1 - I_2$, where $I_1 = \displaystyle\int_{c_1} (x+y)^2\,\mathrm{d}x - (x-y)^2\,\mathrm{d}y, \quad I_2 = \int_{c_2} (x+y)^2\,\mathrm{d}x - (x-y)^2\,\mathrm{d}y.$

Example 552*: Using Green's theorem compute the integral $\displaystyle\oint_c \left(x\mathrm{e}^{-y^2}, -x^2 y\mathrm{e}^{-y^2} + \frac{1}{x^2+y^2}\right)\mathrm{d}\vec{s}$, where c is positively oriented perimeter of the square with vertices $[1,0],\ [2,0],\ [2,1],\ [1,1]$.

Example 553: Using Green's theorem compute the integral $\displaystyle\oint_c \big(x^2\mathrm{e}^x - y^3, 2y\mathrm{e}^x - 3\big)\,\mathrm{d}\vec{s}$, where $c = c_1 \cup c_2;\ \ c_1 = \{[x,y] \in \mathbb{E}_2 :\ x = 0,\ y \in \langle -2, 2\rangle\,\},\ \ c_2 = \{[x,y] \in \mathbb{E}_2 :\ 4x^2 + y^2 = 4,\ x \geq 0\,\}$ and $[0,2]$ is the initial point of c_2.

IV.7. Conservative vector field

The vector field $\vec{f} = (U, V, W)$ we call **conservative** or **potential in region** $G \subset \mathbb{E}_3$, if there exists a scalar function φ, such that $\vec{f} = \mathrm{grad}\varphi$ in region G. Similarly for $\vec{f} = (U, V)$ in region $G \subset \mathbb{E}_2$.

The scalar function φ we call **potential function of vector field $\vec{f}$ in** G.

Theorem: Let $\vec{f}$ be conservative continuous vector field with potential function φ in a domain $G \subset \mathbb{E}_3$ ($\mathbb{E}_2$ respectively). Let c be a oriented curve in G with initial point A and with final point B. Then

$$\int_c \vec{f} \cdot \mathrm{d}\vec{s} = \varphi(B) - \varphi(A).$$

Remark: Because in conservative vector field the line integral does not depend on the curve c but on both initial and final points only, we can write

$$\int_c \vec{f}\cdot \mathrm{d}\vec{s} = \int_A^B \vec{f}\cdot \mathrm{d}\vec{s}$$

Theorem: Let $\vec{f}$ be conservative continuous vector field with potential function φ in a domain $G \subset \mathbb{E}_3$ ($\mathbb{E}_2$ respectively). Then the following three assertions are equivalent:

a) $\vec{f}$ is conservative (or potential) vector field in G.

b) Line integral $\displaystyle\int_c \vec{f}\,\mathrm{d}\vec{s}$ doesn't depend on integration path in G.

c) The circulation of the field $\vec{f}$ around any closed curce in G is null.

Theorem: Let

a) G be a simply connected domain in $\mathbb{E}_2$ and

b) $\vec{f} = (U, V)$ be a vector field and $U(x,y),\ V(x,y)$ have continuous derivatives in G and

c) $\dfrac{\partial V}{\partial x} = \dfrac{\partial U}{\partial y}$ holds in G.

Then $\vec{f}$ is conservative vector field in G.

Example 554: Find the biggest domain (or domains) in which the given vector field is conservative.

a) $\vec{f} = \left(\frac{y^2}{x^2} - \sin y,\ -\frac{2y}{x} + y^2\right)$,

b) $\vec{f} = \left(\frac{y^2}{x^2},\ -\frac{2y}{x} + y^2\right)$.

Example 555:

a) Verify that the vector field $\vec{f} = (3 + 2xy,\ x^2 - 3y^2)$ is conservative in $\mathbb{E}_2$.

b) Find a potential function of the given vector field.

c) Compute the line integral $\displaystyle\int_c \vec{f}\,\mathrm{d}\vec{s}$, where c is a line segment from $A = [1,3]$ into $B = [2,1]$.

Example 556:

a) Verify that there are valid sufficient conditions for vector field $\vec{f} = (3x^2y - 3y^2,\ x^3 - 6xy)$ to be conservative in $G = \mathbb{E}_2$.

b) Find a potential function of the given vector field.

c) Compute the line integral $\displaystyle\int_A^B \vec{f}\,\mathrm{d}\vec{s}$, where $A = [1,3],\ B = [2,1]$.

Example 557: Consider the integral $\displaystyle\int_c \vec{f}\,\mathrm{d}\vec{s}$ where $\vec{f} = (3x^2y,\ x^3 + \sqrt{y})$. Write sufficient conditions to be this integral independent of integration path in a domain $D \subset \mathbb{E}_2$. Find a potential function φ and use this for evaluation of the given integral when:

a) c is a line segment from $A = [2,4]$ to $B = [1,1]$.

c) c is negatively oriented boundary of $M = \{[x,y] \in \mathbb{E}_2 :\ \ x^2 + (y-3)^2 \le 8,\ x \ge 0\}$.

Example 558: Consider the vector field $\vec{f} = (2x\cos y,\ -x^2\sin y)$.

a) Verify that $\vec{f}$ in conservative in $\mathbb{E}_2$.

b) Find its potential function φ.

c) Compute a line integral of $\vec{f}$ along a curve c with initial point $A = [2, 0]$ and terminal point $B = [4, \pi/2]$.

Example 559: Find domains $G \subset \mathbb{E}_2$, where the field $\vec{f} = \left(\frac{1}{y} - \frac{y}{x^2} + 2y - 5,\ \frac{1}{x} - \frac{x}{y^2} + 2x + 11\right)$ is conservative and compute its potential function $\varphi(x, y)$ satisfying the condition $\varphi(-2, 2) = 0$.

Example 560: Find the largest domain $G \subset \mathbb{E}_2$, where the vector function $\vec{f} = \left(\frac{y^2}{\sqrt{x}},\ 4y\sqrt{x}\right)$ is continuous and determine, if the integral $\int_c \vec{f} \cdot \mathrm{d}\vec{s}$ depends on an integration path. If this region exists, then compute $\int_{[1,2]}^{[4,-2]} \vec{f} \cdot \mathrm{d}\vec{s}$.

Example 561: There is given the function $\varphi(x, y) = x^3y + x^2y^2$. Find

a) a force field $\vec{f}$, which has the potential function φ;

b) the work, which is done by the force $\vec{f}$ when moving from $M = [1, 1]$ to $N = [-2, 3]$;

c) the work, which is done by the force $\vec{f}$ when moving along the positively oriented curve $c = \{[x, y] \in \mathbb{E}_2;\ x^2 + 4y^2 = 4$.

Example 562: We have the vector field $\vec{f} = \dfrac{(x - y,\ x + y)}{x^2 + y^2}$ in $G = \mathbb{E}_2 - \{[0, 0]\}$

a) Verify that $\dfrac{\partial U}{\partial y} = \dfrac{\partial V}{\partial x}$ in G.

b) Compute the integral $\oint_c \vec{f} \cdot \mathrm{d}\vec{s}$, where c is a negatively oriented circle with center $S = [0, 0],\ r = 2$. This result should imply that the field $\vec{f}$ is not conservative. Isn't it?

Remark: To compute this integral, we can not use Green's theorem, because the point $[0, 0] \in \mathrm{int}\, c = \{[x, y] \in \mathbb{E}_2;\ x^2 + y^2 < 4]$ *does not belong to* $D(\vec{f})$.

Example 563: Compute $\int_M^N \vec{f} \cdot \mathrm{d}\vec{s}$, where $M = [1,\ 0,\ \mathrm{e}],\ N = /[2,\ -1,\ \mathrm{e}^2]$, if you know that the field $\vec{f}$ is conservative in $\mathbb{E}_3$ and its potential is the function $\varphi(x, y, z) = xy^2 \ln z$. Find the biggest domain G, where φ is potential function of the field $\vec{f}$,

- Consider a vector field $\vec{f}$. Verify, that $\vec{f}$ is conservative in $G \subset \mathbb{E}_2$ ($\mathbb{E}_3$ respectively). Find the potential function and compute $\displaystyle\int_A^B \vec{f} \cdot \mathrm{d}\vec{s}$.

Example 564*: $\vec{f} = (3x^2y - z^2 + 2z,\ x^3 + 2yz - 3,\ y^2 - 2xz + 2x + 5),\ A = [0,1,1],\ B = [3,0,2],\ G \subset \mathbb{E}_3$.

Example 565: $\vec{f} = \left(x\mathrm{e}^{2y},\ (x^2+1)\mathrm{e}^{2y}\right),\ A = [1,0],\ B = [3,1]$.

Example 566: $\vec{f} = \left(3x^2y - 2xy^2,\ x^3 - 2x^2y\right),\ A = [1,1],\ B = [2,-1]$.

Example 567: $\vec{f} = (\cos 2y + y + x,\ y - 2x\sin 2y + x),\ A = [0,7],\ B = [1,0]$.

Example 568*: $\vec{f} = \left(x^2 - 2yz,\ y^2 - 2xz,\ z^2 - 2xy\right),\ A = [0,0,3],\ B = [3,3,0]$.

Example 569*: $\vec{f} = \left(2y + z^2,\ 2x + 1,\ 2xz + 2\right),\ A = [0,1,1],\ B = [3,0,2]$.

Example 570*: $\vec{f} = \left(\dfrac{x}{x^2+y^2+1},\ \dfrac{y}{x^2+y^2+1},\ 2z\right),\ A = [1,0,0],\ B = [0,1,1]$.

Example 571: Verify that the field $\vec{f} = \left(y^2,\ 2xy\right)$ is conservative in $\mathbb{E}_2$. Find potential function $\varphi(x,y)$, which fulfill $\varphi(-4,3) = -9$.

Example 572*: Find potential function of the field $\vec{f} = \left(1 - \dfrac{1}{y} + \dfrac{y}{z},\ \dfrac{x}{z} + \dfrac{x}{y^2},\ 2z - \dfrac{xy}{z^2}\right)$ on $G \subset \mathbb{E}_3:\ y > 0,\ z > 0$.

Example 573: Consider the force field $\vec{f}$ with potential function $\varphi(x,y) = \operatorname{arctg}\frac{y}{x}$. Compute the work, which is done by the field $\vec{f}$ when moving
a) from the point $M = [1, \sqrt{3}]$ to the point $N = [\sqrt{2}, \sqrt{2}]$;
b) along the curve $c = \{[x,y] \in \mathbb{E}_2;\ (x-y)^2 + y^2 = 1\}$ in positive direction.

- Compute:

Example 574: $\displaystyle\int_{[2,1]}^{[1,2]} \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2}$

Example 575: $\displaystyle\int_{[\pi/4,2]}^{[\pi/6,1]} 2y\sin 2x\,\mathrm{d}x + (1 - \cos 2x)\,\mathrm{d}y$

Example 576: $\displaystyle\oint_c (2x+y)\,\mathrm{d}x + (x+2y)\,\mathrm{d}y, \quad c:\ x^2 + y^2 = a^2$

- Determine domains G, in which a vector field $\vec{f}$ is conservative and find potential function:

Example 577: $\vec{f} = (x^3y^2 + x,\, y^2 + yx^4)$

Example 578: $\vec{f} = \left(\ln y - \dfrac{\mathrm{e}^y}{x^2},\, \dfrac{\mathrm{e}^y}{x} + \dfrac{x}{y}\right)$

Example 579*: $\vec{f} = \left(\dfrac{z}{x-y},\, \dfrac{z}{y-x},\, \ln(x-y) + \dfrac{1}{\sqrt{z}}\right)$

- Consider a vector field $\vec{f} = (U, V)$ and a curve c.
  a) Using line integral of vector function compute the work, which is done by force $\vec{f}$ along a curve c.
  b) Verify that the vector field $\vec{f} = (U, V)$ is conservative in $\mathbb{E}_2$.
  c) Find potential function φ of this field and use this to verify result of part a).

Example 580: $\vec{f} = (x + 3y,\, 3x),\;$ c is oriented line segment with initial point $A = [0, 1]$ and terminal point $B = [1, 3]$.

Example 581: $\vec{f} = (y^2,\, 2xy),\;$ c is a part of parabola $y = x^2$ with initial point $A = [0, 0]$ and terminal point $B = [2, 4]$.

Example 582: $\vec{f} = (2x - y^2,\, 3 - 2xy),\; c = \{[x, y] \in \mathbb{E}_2;\; x = -y^2 - 1\}$ from the point $[-1, 0]$ to the point $[-5, -2]$.

Example 583: $\vec{f} = (-x,\, y),\; c\{[x, y] \in \mathbb{E}_2;\; x^2 + y^2 = 4\}$ from the point $A = [2, 0]$ to the point $B = [0, 2]$.

Example 584:
a) Explain, what it means when we say that an integral $\displaystyle\int_c \vec{f} \cdot \mathrm{d}\vec{s}$ does not depend on integration path in region $G \subset \mathbb{E}_2$.
b) Justify the independence of $\displaystyle\int_c (y \sin x,\, y - \cos x)\, \mathrm{d}\vec{s}$ on integration path in $\mathbb{E}_2$.
c) If exists, find a potential function of the field $\vec{f} = (y \sin x,\, y - \cos x)$ in $\mathbb{E}_2$ and compute a line integral of this field along a curve c with initial point $A = [0, 0]$ and terminal point $B = [0, \pi]$.

- Consider scalar function $\varphi(x,y,z)$, which creates potential function of some vector field $\vec{f}$.
  a) Find the largest region $G \subset \mathbb{E}_2$ in which φ has continuous partial derivatives and find the field $\vec{f}$ in this region.
  b) If there exists any other function, which can be potential function of $\vec{f}$ in this region, find it.
  c) Compute a line integral $\displaystyle\int_c \vec{f} \cdot \mathrm{d}\vec{s}$ along a given curve c.

Example 585: $\varphi(x,y,z) = \dfrac{1}{2}\ln(x^2+y^2) + z^2$; c is a curve in G with initial point $A = [1,0,0]$ and terminal point $B = [0,1,1]$.

Example 586: $\varphi(x,y,z) = xy + xz + yz$;
$c = \{[x,y,z] \in \mathbb{E}_3;\ x = -1+t,\ y = 2 + \dfrac{t}{2},\ z = -\cos\left(\dfrac{t\pi}{4}\right),\ \text{for } t \in \langle 0,4\rangle\,\}$.

- Consider a vector field $\vec{f} = (U,V)$, region D and a curve c.
  a) Write the sufficient condition for a field $\vec{f} = (U,V)$ to be conservative in the region D.
  b) Verify these conditions for a given vector field $\vec{f}$ in the given region D (if D is not given, find the largest possible).
  c) Find potential function and use this to compute an integral of $\vec{f}$ along a given curve c.

Example 587: $\vec{f} = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right)$, $D = \{[x,y] \in \mathbb{E}_2;\ x > 0,\ y > 0\,\}$, c is line segment with initial point $A = [2,4]$ and terminal point $B = [1,2]$.

Example 588: $\vec{f} = \left(\dfrac{y}{x^2+y^2}, \dfrac{-x}{x^2+y^2}\right)$, $D = \{[x,y] \in \mathbb{E}_2;\ y > 0\,\}$, c is positively oriented circle $(x-1)^2 + (y-1)^2 = \frac{1}{4}$.

Example 589: $\vec{f} = \left(y^2 - \dfrac{x}{\sqrt{y-x^2}}, 2xy + \dfrac{1}{\sqrt{y-x^2}}\right)$, $D = \{[x,y] \in \mathbb{E}_2;\ y > x^2\,\}$, c is a curve with initial point $A = [0,1]$ and terminal point $B = [1,2]$.

Example 590: $\vec{f} = \left(\dfrac{y^2}{x} + x^2, 2y\ln x - \cos 2y\right)$, c is a curve with initial point $A = [1,\pi/4]$ and terminal point $B = [2,0]$.