MATHEMATICS I

selected problems from the exam tests in previous years

I. LINEAR ALGEBRA

1. a) Define the notions *dimension* and *basis* of a vector space V.
   b) Decide whether the vectors $\vec{x} = (4, 2, 0)$, $\vec{y} = (1, 2, -1)$ a $\vec{z} = (7, 8, 1)$ form a basis in the vector space $V(\mathbb{E}_3)$.
   c) If the vector $\vec{u} = (21, 18, 3)$ can be expressed as a linear combination of the vectors $\vec{x}$, $\vec{y}$, $\vec{z}$, find the coefficients in this combination.

2. a) Define what it means that vectors $\vec{u}_1, \ldots, \vec{u}_n$ are *linearly dependent*, respectively *linearly independent*.
   b) For which values of parameter $a \in \mathbb{R}$ are the vectors $\vec{u} = (-1, 0, 1)$, $\vec{v} = (0, 1, a)$, $\vec{w} = (2, a, a)$ linearly dependent?
   c) What is, in this case, dimension of the vector space generated by these vectors?

3. a) Define the notions *rank of a matrix* and *regular matrix*.
   b) For which values of parameter $\alpha \in \mathbb{R}$ is the rank of the matrix $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & \alpha \\ -3 & 4 & 1 \end{pmatrix}$ equal to 3 and for which α is the rank equal to 2?
   c) Is matrix A regular for $\alpha = 1$? (Give reasons for your answer.)

4. a) Define the notion of an *inverse matrix* to a square matrix A.
   b) Decide about the existence of the inverse matrix to the matrix $A = \begin{pmatrix} 1, & 0, & 0 \\ 3, & 1, & 0 \\ 0, & 3, & 1 \end{pmatrix}$.
   c) If the inverse matrix exists, calculate it. Verify the result by computing the product $A \cdot A^{-1}$.

5. $A = \begin{pmatrix} 3, & 1 \\ 5, & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0, & 1 \\ -2, & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2, & 2 \\ 2, & -1 \end{pmatrix}$
   a) Find the matrix A^{-1}. b) Find the matrix B^{-1}.
   c) Calculate matrix X such that $A \cdot X \cdot B = C$.

6. Given the matrix $A = \begin{pmatrix} 1, & 1, & 0 \\ 0, & 0, & 3 \\ 0, & 2, & 1 \end{pmatrix}$.
   a) Calculate the matrix $B = A^2$ $(= A \cdot A)$.
   b) Define the notion of a *regular matrix*. Decide if the given matrix A is regular.
   c) If the inverse matrix to A exists, calculate it.

7. Given the matrix with parameters $a, b \in \mathbb{R}$: $A = \begin{pmatrix} 1, & 0, & -1, & -1 \\ 0, & -1, & -1, & 1 \\ a, & b, & 0, & 0 \\ -1, & -1, & 1, & 0 \end{pmatrix}$.
   a) Calculate the determinant of matrix A.
   b) Define the notions of a *regular matrix* and a *singular matrix*.
   c) For which values of parameters a, b does the homogeneous system of linear algebraic equations $A \cdot X = O$ have a non–zero solution?

8. a) Explain principles of the Cramer rule. Under which conditions it can be applied?

b) Verify the assumptions for the system

$$\begin{aligned} x + y + 2z &= 1 \\ 2x + y &= -4 \\ 5x + y - 3z &= -13 \end{aligned}$$

c) Applying Cramer's rule, calculate the value of unknown y.

9. a) Calculate the determinant of the system with parameter $a \in \mathbb{R}$:

$$\begin{aligned} x + 2y + az &= 0 \\ -x + 3y + az &= -8 \\ 3x - y + 2z &= 13. \end{aligned}$$

b) Explain principles of the Cramer rule. Is it possible to apply Cramer's rule to the system given above if $a = 1$? (Give reasons for your answer.)

c) Assuming that $a = 1$, calculate the value of unknown z.

10. a) Write the Frobenius theorem.

b) What is the number of solutions of the given system in dependence on parameter $a \in \mathbb{R}$:

$$\begin{aligned} x - y + z &= 1 \\ x + y + 3z &= 1 \\ (2a-1)x + (a+1)y + z &= 1 - a \end{aligned}$$

c) Solve the system for $a = 1$.

11. a) Define the notions *eigenvalue* and *eigenvector* of a square matrix.

b) For which value of parameter $c \in \mathbb{R}$ does matrix A have the eigenvalue $\lambda = 0$?

$$A = \begin{pmatrix} c, & c-5 \\ 6, & -3 \end{pmatrix}$$

c) Calculate the eigenvalues and the associated eigenvectors of matrix A if $c = 4$.

12. a) Define the notions *eigenvalue* and *eigenvector* of a square matrix.

b) Find all eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3, & 1, & 0 \\ -4, & -1, & 0 \\ 4, & -8, & -2 \end{pmatrix}$.

II. DIFFERENTIAL CALCULUS

1. a) Evaluate $$\lim_{n \to +\infty} (\sqrt{n^2+1} - \sqrt{n^2-1}).$$

b) Define what it means that the sequence $\{a_n\}$ is increasing.

c) Create an increasing sequence whose limit is 3.

2. a) Evaluate $$\lim_{n \to +\infty} \frac{n + \cos(n^2)}{2n+1}.$$

b) Define what it means that the sequence $\{a_n\}$ is decreasing.

c) Create a decreasing sequence whose limit is 3.

3. a) Evaluate $$\lim_{n \to +\infty} \frac{(2n-1)^2 - 4n^2 + 1}{n^2 - (n+5)^2}.$$

b)Write the theorem on a limit of a subsequence.

c) Create a sequence that has no limit. (Give reasons why your sequence has no limit.)

4. a) Using the definition, decide about the monotonicity of the sequence $\{\frac{n+1}{2n+1}\}$.

b) Evaluate the limit of the sequence $\lim_{n \to +\infty} n(\sqrt{n^2+1} - n)$.

c) Evaluate the limit of the function $\lim_{x\to 0} \frac{\cos x - 1}{x \sin x}$. If you apply l'Hospital's rule, verify its assumptions.

5. a) Given the function $f : f(x) = \frac{x}{\sqrt{x^2 - 4x}}$. Find $D(f)$.

   b) Calculate the limits of $f(x)$ for $x \to +\infty$ and $x \to 0$ (if the limits exist).

   c) Is the given function f odd, even or periodic? (Give reasons for your answer.)

6. a) Given the function $f(x) = \arccos(x^2 - 1)$. Specify $D(f)$.

   b) Write the equation of the tangent line to the graph of function f at the point $[x_0, f(x_0)]$, if $x_0 = 1$.

   c) Is function f even or odd? (Give reasons for your answer.)

7. a) Given the function $f : f(x) = \ln(x^2 + 4x + 3)$. Specify $D(f)$.

   b) Write the equation of the tangent line to the graph of function f at the point $[x_0, f(x_0)]$, if $x_0 = 1$.

   c) Using the linear approximation of f (i.e. the result of part b)), calculate an approximate value of function f at the point $x = 0,9$.

   Other variants of problem 7 with different functions f and points x_0:

   b) $f(x) = x + \sqrt{1 - x^2}$ c) $x_0 = 0$

   b) $f(x) = \frac{x + 2}{\sqrt{5 - x}}$ c) $x_0 = 1$

   b) $f(x) = \arcsin \sqrt{x + 1}$ c) $x_0 = -\frac{1}{2}$

8. a) Define the notion *derivative of function f at point x_0*.

   b) Calculate the derivative of the function $f : f(x) = \sqrt[3]{e^{-x} + 1}$. Specify the domain of the derivative.

   c) Write the equation of the tangent line and the normal line to the graph of function f at the point $[x_0, f(x_0)]$, if $x_0 = 0$.

In problems 9–11:

a) define absolute extremes of function f on the interval I,

b) give reasons for the existence of the absolute extremes,

c) find the absolute extremes (i.e. find their positions and values).

9. $f(x) = 3 - x - \frac{4}{(x + 2)^2}$, $I = \langle -1, 2 \rangle$

10. $f(x) = x + 3\sqrt[3]{x^2}$, $I = \langle -1, 1 \rangle$

11. $f(x) = 2\sqrt{x - 1} - x + 2$, $I = \langle 1, 5 \rangle$

In problems 12–14:

a) find intervals of monotonicity and local extremes of the given function f,

b) find points of inflection and intervals where function f is concave up or down.

c) Sketch the graph.

12. $f(x) = 1 + x^2 - \frac{1}{2}x^4$ 13. $f(x) = (x - 3)\sqrt{x}$ 14. $f(x) = e^{2x - x^2}$

15. Given the function $f(x) = \frac{1}{9 - x^2}$.

    a) Specify $D(f)$. Is the given function even or odd? (Give reasons for your answer.)

    b) Find intervals of monotonicity and local extremes.

c) Calculate the limits of f for $x \to +\infty$, $x \to 3+$ and $x \to 3-$. Sketch the graph.

16. Given the function $f(x) = (x+2)\,\mathrm{e}^{1/x}$ with the restricted domain $D(f) = (0, +\infty)$.
    a) Find intervals of monotonicity and local extremes.
    b) Calculate the limits of function f for $x \to +\infty$ and $x \to 0+$.
    c) Sketch the graph.

In problems 17–20:

a) find intervals of monotonicity and local extremes for the given function f,

b) find intervals where function f is concave up or concave down and find points of inflection,

c) evaluate the limits at the end points of $D(f)$ and sketch the graph.

17. $f(x) = 3 - x - \dfrac{4}{(x+2)^2}$ with the restricted domain $D(f) = (-2, +\infty)$

18. $f(x) = x\,\ln x$ 19. $f(x) = (x-2)\,e^x$ 20. $f(x) = x^2 + 2\ln(x+2)$.

In problems 21 and 22:

a) find intervals of monotonicity and local extremes for the given function f,

b) find asymptotes of function f, c) sketch the graph.

21. $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$ 22. $f(x) = \dfrac{x-1}{x^2+3}$

23. $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$ a) Specify $D(f)$. Calculate the limits for $x \to +\infty$ and $x \to -\infty$.
    b) Find $f'(x)$ (including the domain).
    c) Find intervals of monotonicity and local extremes.

24. a) Evaluate the coefficients and write Taylor's polynomial of the 5–th degree of the function $f(x) = \mathrm{e}^x$ with the center at the point $x_0 = 0$. Write the form of the remainder.
    b) Applying Taylor's polynomial, calculate the value of $\mathrm{e}^{-1/3}$ with the error less than 0.001.

25. a) Evaluate the coefficients and write Taylor's polynomial of the 6–th degree of the function $f(x) = \cos x$ with the center at the point $x_0 = 0$. Write the form of the remainder.
    b) Evaluate the coefficients and write Taylor's polynomial of the 5–th degree of the same function f with the center at the point $x_0 = \pi/2$.

26. a) Evaluate the coefficients and write Taylor's polynomial T_2 of the 2nd degree of the function $f(x) = x + \sqrt{x+1}$ withy the center $x_0 = 0$. Write the form of the remeinder.
    b) Estimate the error when using polynomial T_2 for the approximate evaluation of the function value of f at the point $x = 1/2$.

III. INTEGRAL CALCULUS

In problems 1–6:

a) write the theorem on the integration by parts (including the assumptions),

b) evaluate the integral $\int f(x)\,\mathrm{d}x$, where function f has the concrete form

1. $f(x) = x\,\mathrm{arctg}\,x$
2. $f(x) = x^2 \ln x$
3. $f(x) = (x^2 + x + 2)\,e^x$
4. $f(x) = \ln^2 x$
5. $f(x) = (3x-5)\,\sin x$
6. $f(x) = (2x+3)\,e^{3x}$

On which intervals do the integrals exist?

In problems 7–20:

a) write the theorem on integration by substitution (including the assumptions),
b) evaluate the integral $\int f(x)\,\mathrm{d}x$, where function f has the concrete form

7. $f(x) = \cos(1-2x)$
8. $f(x) = \dfrac{x-2}{x^2-4x+8}$
9. $f(x) = \dfrac{e^{2x}}{2+e^{2x}}$
10. $f(x) = \dfrac{x^3}{\sqrt{x^4+7}}$
11. $f(x) = \dfrac{1}{1+\sqrt{x}}$
12. $f(x) = \dfrac{e^{1/x}}{x^2}$
13. $f(x) = x\sqrt{1-x^2}$
14. $f(x) = \dfrac{\cos x}{\sqrt[3]{\sin^2 x}}$
15. $f(x) = \Big(\dfrac{1}{1+\ln^2 x} + \dfrac{1}{\sqrt{\ln x}}\Big)\dfrac{1}{x}$
16. $f(x) = \sin^2 x \cos^3 x$
17. $f(x) = \cos^2 x + \cos^3 x$
18. $f(x) = \cos^7 x$
19. $f(x) = x^3\, e^{-x^2}$
20. $f(x) = \dfrac{\sqrt{x-2}}{x-1}$

On which intervals do the integrals exist?

In problems 21–26 calculate the integral of the given rational function. On which intervals do the integrals exist?

21. $\displaystyle\int \frac{x^3}{x^2+3x+2}\,\mathrm{d}x$
22. $\displaystyle\int \frac{2x+1}{x^2+4x+4}\,\mathrm{d}x$
23. $\displaystyle\int \frac{x}{(x+1)(x+2)(x+5)}\,\mathrm{d}x$
24. $\displaystyle\int \frac{1}{(x+1)^2(x+2)}\,\mathrm{d}x$
25. $\displaystyle\int \frac{x-8}{x^3-4x^2+4x}\,\mathrm{d}x$
26. $\displaystyle\int \frac{1}{x^2-x+1}\,\mathrm{d}x$

27. a) Calculate the area of the region, which is for $x \in \langle 1, 2 \rangle$ bounded by the x–axis and the curve $y = x^2 + \dfrac{1}{x^2}$.

    b) Evaluate the definite integral $\displaystyle\int_0^1 (3x+1)\,\mathrm{e}^x\,\mathrm{d}x$.

28. a) Find the antiderivative (and the interval of its existence) to the function $f(x) = \dfrac{1}{4+x^2}$.

    b) Calculate the area of the region, which is bounded by the x–axis and by the curves $y = \dfrac{1}{4+x^2}$, $x = 0$, $x = 2$.

    c) Evaluate the improper integral $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$.

29. Given the function $f(x) = \dfrac{1}{x^2+x}$.

    a) Calculate the integral $\int f(x)\,\mathrm{d}x$. Give the intervals of its existence.

    b) Evaluate the definite integral $\int_1^3 f(x)\,\mathrm{d}x$.

    c) Evaluate the integrals $\int_0^1 f(x)\,\mathrm{d}x$ a $\int_3^\infty f(x)\,\mathrm{d}x$. Do the integrals converge?

30. a) Find the domain and sketch the graph of the function $y = \sqrt{x-1}$.

b) Sketch the region bounded by the curves $y = \sqrt{x-1}$, $x = 0$, $y = 0$ a $y = 1$ and evaluate its area.

c) Calculate the volume of the body which arises by rotation of the above region about the y–axis.

31. Given the function $f(x) = x^2 \sin x$.

a) Calculate the integral $\int f(x)\,\mathrm{d}x$. Verify the result by differentiation.

b) Find the mean value of the function f on the interval $\langle 0, \pi \rangle$, i.e. the value $\mu = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} f(x)\,\mathrm{d}x$.

32. a) Sketch a region in the 1st quadrant in $\mathbb{E}_2$, that is bounded by the graph of the function $f(x) = \sin x$ and by the straight line $x = \pi/2$. Calculate the volume of the body, which arises by rotation of this region about the x–axis.

b) Calculate the area of the region, that is for $x \in \langle 0, \pi/2 \rangle$ bounded by the x–axis and the curve $y = \cos^4 x\ \sin x$.

33. a) Calculate the integral and give intervals of its existence: $\displaystyle\int \frac{\sqrt{x}-2}{x}\,\mathrm{d}x$.

b) Evaluate the area of the region, that is for $x \in \langle \frac{1}{4}, 1 \rangle$ bounded by the x–axis ad the curve $y = \dfrac{\sqrt{x}-2}{x}$.

c) Evaluate the improper integral $\displaystyle\int_0^1 \left(\frac{1}{\sqrt{x}} - \frac{1}{x}\right)\mathrm{d}x$. Does the integral converge?

34. Given the function$f(x) = (2x+3)\sin 2x$.

a) Calculate the integral $\int f(x)\,\mathrm{d}x$. Verify the result by differentiation.

b) Evaluate the area of the region, which is for $x \in \langle 0, \pi/4 \rangle$ bounded by the x–axis and by the curve $y = (2x+3)\sin 2x$.

35. a) Give intervals of the existence and calculate the integral $\displaystyle\int \frac{x-8}{x^3 - 4x^2 + 4x}\,\mathrm{d}x$.

b) Give reasons for the existence and evaluate the definite integral $\int_3^4 f(x)\,\mathrm{d}x$. Simplify the result.

c) Calculate the improper integral $\displaystyle\int_3^{+\infty} \frac{x-8}{x^3 - 4x^2 + 4x}\,\mathrm{d}x$. Does the integral converge?