**Selected problems from the textbook**

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**(Problems in Mathematics I)**

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# I. LINEAR ALGEBRA

## I.1. Vectors, vector spaces

Given the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and real numbers $\alpha$, $\beta$, $\gamma$. Calculate vector $\mathbf{a}$, which is equal to the linear combination $\alpha\,\mathbf{u} + \beta\,\mathbf{v} + \gamma\,\mathbf{w}$.

**2.** $\mathbf{u} = (4, 2, 0)$, $\mathbf{v} = (5, 3, 2)$, $\mathbf{w} = (-1, 0, -1)$, $\alpha = 2$, $\beta = 2$, $\gamma = 1$

Find vector $\mathbf{x}$, which satisfies the given equation.

**8.** $7\,\mathbf{x} + \mathbf{u} = 3\mathbf{u} + 6\mathbf{v} - \mathbf{x}$, where $\mathbf{u} = (-1, 0, 3, 1)$, $\mathbf{v} = (-1, 0, 3, 5)$

Calculate the scalar product of given vectors. (`Hint:` *Use the formula* $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots + u_n v_n$.)

**15.** $\mathbf{u} = (3, 3, 1)$, $\mathbf{v} = (4, 3, 0)$

Calculate the angle between given vectors. (`Hint:` *Use the formula* $\cos\vartheta = \mathbf{u} \cdot \mathbf{v} / (|\mathbf{u}|\,|\mathbf{v}|)$.)

**22.** $\mathbf{u} = (-1, 3)$, $\mathbf{v} = (2, 2)$

For which value of parameter $\alpha$ are given vectors orthogonal? (`Hint:` *Two vectors are orthogonal if their scalar product equals zero.*)

**25.** $\mathbf{u} = (-2, \alpha + 3)$, $\mathbf{v} = (0, -1 + 2\alpha)$

Given vectors from $\mathbf{V}(\mathbb{E}_2)$, $\mathbf{V}(\mathbb{E}_3)$ or $\mathbf{V}(\mathbb{E}_4)$. a) Are the vectors linearly dependent or linearly independent? b) What is the dimension of the linear hull of these vectors? c) Which vectors form the basis of the linear hull?

Consider the fact that question b) can also be formulated in this way: What is the rank of a matrix, whose rows (or columns) are the given vectors?

**42.** $\mathbf{u} = (2, 1)$, $\mathbf{v} = (-1, 3)$

**43.** $\mathbf{u} = (-1, 1)$, $\mathbf{v} = (10, -10)$

**44.** $\mathbf{u} = (1, 4, 2)$, $\mathbf{v} = (3, 2, 2)$

**45.** $\mathbf{u} = (-1, 5, 1)$, $\mathbf{v} = (3, -15, -3)$

**50.** $\mathbf{x} = (1, 5)$, $\mathbf{y} = (0, 0)$, $\mathbf{z} = (5, 25)$

**51.** $\mathbf{x} = (2, 3, -2)$, $\mathbf{y} = (3, 0, 1)$, $\mathbf{z} = (0, 9, -8)$

**53.** $\mathbf{x} = (1, 0, 2, -2)$, $\mathbf{y} = (3, -2, 5, 2)$, $\mathbf{z} = (4, -6, 5, 20)$

Given vectors from $\mathbf{V}(\mathbb{E}_2)$, $\mathbf{V}(\mathbb{E}_3)$ or $\mathbf{V}(\mathbb{E}_4)$ with parameters. a) For which values of the parameters are the vectors linearly dependent or linearly independent? b) What is the dimension of the linear hull of these vectors? (`Hint:` *Consider the matrix whose rows are the given vectors. Determine the rank of the matrix in dependence on the parameters.*)

**68.** $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (0, 2, 0)$, $\mathbf{c} = (\alpha, 0, \alpha + 1)$, $\mathbf{d} = (\alpha - 1, \alpha, 0)$

**69.** $\mathbf{u} = (k, 1, 0)$, $\mathbf{v} = (0, k - 1, 3)$, $\mathbf{w} = (0, 2, k)$

**70.** $\mathbf{u} = (0, 1, a)$, $\mathbf{v} = (2, a, a)$, $\mathbf{w} = (-1, 0, 1)$

Express the vectors $\mathbf{a}$, $\mathbf{b}$ as a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$ (respectively $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$) (if it is possible). Is the expression unique? (`Hint:` *Vector* $\mathbf{a}$ *has the form* $\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v}$. *Write this expression in coordinates. You obtain a system of equations for the unknowns* $\alpha$ *and* $\beta$.)

**73.** $\mathbf{a} = (3, 2, 5)$, $\mathbf{b} = (5, 6, 7)$, $\mathbf{u} = (1, 3, 2)$, $\mathbf{v} = (2, -1, 3)$, $\mathbf{w} = (5, 1, 8)$

a) Do given vectors form a basis of vector space $\mathbf{V}$? b) What is the dimension of the linear hull of the given vectors?

Consider that question b) can also be formulated as follows: What is the rank of a matrix, whose rows (or columns) are the given vectors?

**79.** $\mathbf{V} = \mathbf{V}(\mathbb{E}_3),\ \mathbf{a} = (0,7,3),\ \mathbf{b} = (5,3,2)$

Given a vector space $\mathbf{V}$ and its subset $\mathbf{V}'$. Determine if $\mathbf{V}'$ is a subspace of vector space $\mathbf{V}$.

**85.** $\mathbf{V} = \mathbf{V}(\mathbb{E}_3),\quad \mathbf{V}' = \{(a,b,c);\ a,b,c \in \mathbb{R},\ a+b=0\}$

**92.** $\mathbf{V} = \mathbf{V}(\mathbb{E}_4),\quad \mathbf{V}' = \{(u,v,w,x);\ u,v,w,x \in \mathbb{R},\ u-2v+w \geq 0\}$

In next problems $\mathbf{V}$ is a set of functions defined on interval $I$. The sum $f+g$ of arbitrary two functions $f$ and $g$ from $\mathbf{V}$ is defined: $(f+g)(x) = f(x)+g(x)$ for $x \in I$. The multiple $\lambda \cdot f$ of an arbitrary function $f$ from $\mathbf{V}$ by an arbitrary real number $\lambda$ is defined: $(\lambda \cdot f)(x) = \lambda \cdot f(x)$ for $x \in I$. Decide if set $\mathbf{V}$ (with the two defined operations) is a vector space.

**93.** $I = (-\infty, +\infty)$, $\mathbf{V}$ is a set of all functions of the form $\alpha \cdot \sin x + \beta \cdot \cos x + \gamma$ where $\alpha,\ \beta,\ \gamma \in \mathbb{R}$.

**94.** $I = (-\infty, +\infty)$, $\mathbf{V}$ is the set of all functions of the form $\alpha + \beta \cdot x + \gamma \cdot x^2$ where $\alpha,\ \beta,\ \gamma \in \mathbb{R}$.

## I.2. Matrices, determinants

Calculate the matrix $A \cdot B$.

**109.** $A = \begin{pmatrix} 2, & -1, & 3 \\ 0, & 5, & 2 \end{pmatrix},\ B = \begin{pmatrix} 3, & 1, & 2, & 4 \\ 0, & 3, & 1, & 0 \\ 5, & 2, & 0, & 1 \end{pmatrix}$ **110.** $A = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},\ B = (1,\ \ 2,\ \ 1)$

**111.** $A = \begin{pmatrix} 1, & 3, & 4, & 2 \\ -2, & 3, & -1, & 2 \\ 4, & 1, & 2, & 3 \\ 1, & 2, & 2, & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 3 \\ -2 \\ -1 \end{pmatrix}$ **112.** $A = \begin{pmatrix} 1, & 2, & 3 \\ 3, & 2, & 1 \\ 1, & 3, & 2 \end{pmatrix},\ B = \begin{pmatrix} 1, & 0 \\ 0, & 1 \\ 0, & 0 \end{pmatrix}$

Calculate

**117.** $\begin{pmatrix} 2, & 1 \\ 1, & 3 \end{pmatrix}^3$ **124.** $\begin{pmatrix} 2, & 1, & -1 \\ -1, & 2, & 0 \end{pmatrix} \cdot \begin{pmatrix} 2, & 1, & -1 \\ -1, & 2, & 0 \end{pmatrix}^T$

Calculate the matrix $A \cdot B - B \cdot A$.

**132.** $A = \begin{pmatrix} 1, & 2, & 1 \\ 2, & 1, & 2 \\ 1, & 2, & 3 \end{pmatrix},\ B = \begin{pmatrix} 4, & 1, & 1 \\ -4, & 2, & 0 \\ 1, & 2, & 1 \end{pmatrix}$ **133.** $A = \begin{pmatrix} 2, & 1, & 0 \\ 1, & 1, & 2 \\ -1, & 2, & 1 \end{pmatrix},\ B = \begin{pmatrix} 3, & 1, & 2 \\ 3, & 2, & 4 \\ -3, & 5, & 1 \end{pmatrix}$

Find $x,\ y \in \mathbb{R}$ that satisfy the given equation.

**139.** $\begin{pmatrix} 1, & 3x-2 \\ 3y+6, & 2 \end{pmatrix} = \begin{pmatrix} 1, & 12 \\ 40, & 2 \end{pmatrix}^T$ **140.** $\begin{pmatrix} x+y, & -3 \\ -2, & x-y \end{pmatrix} = \left[ \begin{pmatrix} 2, & -1 \\ -1, & 1 \end{pmatrix} \cdot \begin{pmatrix} 3, & 0 \\ 0, & 2 \end{pmatrix} \right]^T$

Determine the rank of a given matrix. If it is a square matrix, decide if it is a regular matrix or a singular matrix.

**143.** $\begin{pmatrix} 1, & 2, & 3 \\ 2, & -1, & 1 \\ 1, & 7, & 7 \end{pmatrix}$ **148.** $\begin{pmatrix} 1, & -2, & 3 \\ -3, & -6, & -9 \\ 4, & 8, & 12 \end{pmatrix}$ **149.** $\begin{pmatrix} 1, & 1, & 1, & 2 \\ 2, & -4, & 4, & 1 \\ -1, & -19, & 5, & 0 \\ 3, & 15, & -1, & 2 \end{pmatrix}$

Calculate the inverse matrix (if it exists).

**160.** $\begin{pmatrix} 1, & 2 \\ 2, & 2 \end{pmatrix}$ **161.** $\begin{pmatrix} 1, & 0, & 0 \\ 3, & 1, & 0 \\ 0, & 3, & 1 \end{pmatrix}$ **166.** $\begin{pmatrix} 1, & 2, & 3 \\ 2, & 1, & 3 \\ 1, & 4, & 5 \end{pmatrix}$

**168.** $\begin{pmatrix} 2, & 2, & 3 \\ 1, & -1, & 0 \\ -1, & 2, & 1 \end{pmatrix}$ **170.** $\begin{pmatrix} 1, & 2, & -3 \\ 0, & 1, & 2 \\ 0, & 0, & 1 \end{pmatrix}$ **171.** $\begin{pmatrix} \cos x, & -\sin x \\ \sin x, & \cos x \end{pmatrix}$

Find matrix $X$ that satisfies the equation

**174.** $X \cdot \begin{pmatrix} 1, & 1, & -1 \\ 2, & 1, & 0 \\ 1, & -1, & 1 \end{pmatrix} = \begin{pmatrix} 1, & -1, & 3 \\ 4, & 3, & 2 \\ 1, & -2, & 5 \end{pmatrix}$ **175.** $X \cdot \begin{pmatrix} 2, & -1 \\ 0, & 2 \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$

**176.** $A = \begin{pmatrix} 1, & 2 \\ 0, & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3, & 2 \\ 1, & -1 \end{pmatrix}$. Find matrix $X$, that satisfies $A \cdot X = (A - B)^2$.

**177.** Solve the matrix equation $A \cdot X \cdot B = C$, where $A = \begin{pmatrix} 3, & 1 \\ 5, & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0, & 1 \\ -2, & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2, & 2 \\ 2, & -1 \end{pmatrix}$.

**178.** Given the matrix $A = \begin{pmatrix} x, & 1+x^2, & 1 \\ y, & 1+y^2, & 1 \\ z, & 1+z^2, & 1 \end{pmatrix}$.

For which $x, y, z \in \mathbf{R}$ is matrix $A$ regular? Calculate $A^{-1}$ for $x = 0, y = 1, z = 2$.

Calculate the determinants

**180.** $\begin{vmatrix} \cos x, & \sin x \\ \sin x, & \cos x \end{vmatrix}$ **185.** $\begin{vmatrix} 2, & 5, & 0 \\ -1, & 7, & 1 \\ 4, & 1, & -4 \end{vmatrix}$ **190.** $\begin{vmatrix} a, & a, & a \\ -a, & a, & x \\ -a, & -a, & x \end{vmatrix}$ **200.** $\begin{vmatrix} 1, & 0, & -1, & -1 \\ 0, & -1, & -1, & 1 \\ a, & b, & 0, & 0 \\ -1, & -1, & 1, & 0 \end{vmatrix}$

## I.4. Eigenvalues and eigenvectors of square matrices

Find eigenvalues and eigenvectors of the matrices.

**235.** $\begin{pmatrix} 2, & 1 \\ 1, & 2 \end{pmatrix}$ **236.** $\begin{pmatrix} 3, & 4 \\ 5, & 2 \end{pmatrix}$ **237.** $\begin{pmatrix} 0, & a \\ -a, & 0 \end{pmatrix}$ **238.** $\begin{pmatrix} 5, & 6, & -3 \\ -1, & 0, & 1 \\ 1, & 2, & 1 \end{pmatrix}$

**241.** $\begin{pmatrix} 0, & 0, & 1 \\ 0, & 1, & 0 \\ 1, & 0, & 0 \end{pmatrix}$ **243.** $\begin{pmatrix} 3, & 1, & 0 \\ -4, & -1, & 0 \\ 4, & -8, & -2 \end{pmatrix}$ **244.** $\begin{pmatrix} 2, & 5, & -6 \\ 4, & 6, & -9 \\ 3, & 6, & -8 \end{pmatrix}$

In next problems we assume that $A$ is a square matrix of the type $3 \times 3$, whose entries are real numbers. Decide if matrix $A$ can have given eigenvalues and eigenvectors.

**245.** eigenvalues: 2, 1, $2 + \mathrm{i}$

**246.** eigenvalues: 2, $1 + \mathrm{i}$, $1 - \mathrm{i}$, eigenvectors: $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ -1 + \mathrm{i} \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \\ -\mathrm{i} \end{pmatrix}$

Calculate the inverse matrix to matrix $A$ and find eigenvalues and eigenvectors of the inverse matrix $A^{-1}$. Compare the results with the eigenvalues and eigenvectors of matrix $A$.

**250.** $A = \begin{pmatrix} 2, & 1 \\ 1, & 2 \end{pmatrix}$ **251.** $A = \begin{pmatrix} 3, & 4 \\ 5, & 2 \end{pmatrix}$ **252.** $A = \begin{pmatrix} 0, & 5 \\ -5, & 0 \end{pmatrix}$ **253.** $A = \begin{pmatrix} 5, & 6, & -3 \\ -1, & 0, & 1 \\ 1, & 2, & -1 \end{pmatrix}$

To given square matrix $A$ calculate the matrix $A^2$ and find eigenvalues and eigenvectors of the matrix $A^2$. Compare the results with the eigenvalues and the eigenvectors of matrix $A$.

**255.** $A = \begin{pmatrix} 2, & 1 \\ 1, & 2 \end{pmatrix}$ **256.** $A = \begin{pmatrix} 1, & 0 \\ 3, & 2 \end{pmatrix}$ **257.** $A = \begin{pmatrix} 0, & 2 \\ -2, & 0 \end{pmatrix}$ **258.** $A = \begin{pmatrix} 2, & -1, & 2 \\ 5, & -3, & 3 \\ -1, & 0, & -2 \end{pmatrix}$

## I.5. Systems of linear algebraic equations

Applying Frobenius' theorem decide about the solvability of the system of equations and about the number of solutions.

**273.** $$\begin{aligned} x - 2y &= -3 \\ 2x - y &= 0 \\ 4x - 5y &= -6 \end{aligned}$$

**275.** $$\begin{aligned} x - 2y + 2z &= -9 \\ 3x + 5y + 4z &= 10 \\ 5x + 12y + 6z &= 29 \end{aligned}$$

**276.** $$\begin{aligned} 3x + 2y &= 12 \\ 5x + 4y + z &= 27 \\ x + 2y + 5z &= 33 \end{aligned}$$

Applying Frobenius' theorem decide about the solvability of the system of equations and about the number of solutions in dependence on parameters.

**278.** $$\begin{aligned} ax + y + z &= 1 \\ x + ay + z &= 1 \\ x + y + az &= 1 \end{aligned}$$

**280.** $$\begin{aligned} ax - 3y &= 1 \\ ax - 2y &= 2 \end{aligned}$$

**283.** $$\begin{aligned} 2x - y + z + u &= 1 \\ x + 2y - z + 4u &= 2 \\ x + 7y - 4z + 11u &= \lambda \end{aligned}$$

Can the next systems be solved by means of Cramer's rule? If ,,yes", find the solution applying this rule. (`Hint:` *Cramer's rule can be applied if the matrix of the system is regular.*)

**287.** $$\begin{aligned} 3x - 2y + z &= 11 \\ x + y - 3z &= 7 \\ 11x - 4y - 3z &= 10 \end{aligned}$$

**288.** $$\begin{aligned} 2x - 3y + z &= 0 \\ x + 2y - z &= 3 \\ 2x + y + z &= 12 \end{aligned}$$

**295.** $$\begin{aligned} -5x + y - 2z &= 1 \\ 2x + y + 2z &= 0 \\ -x + 3y + 2z &= 0 \end{aligned}$$

What is the dimension of the vector space of all solutions of the homogeneous system (in dependence on parameter $\lambda$)?

**300.** $$\begin{aligned} 3x + 2y - z &= 0 \\ 2x - y + 3z &= 0 \\ \lambda x + 3y - 4z &= 0 \end{aligned}$$

**301.** $$\begin{aligned} 4x + 2y - 2z &= 0 \\ 2x + y + 3z &= 0 \\ \lambda x + y - \lambda z &= 0 \end{aligned}$$

Solve the homogeneous systems by means of Gaussian elimination.

**308.** $$\begin{aligned} 3x - y + 3z &= 0 \\ x + 2y - 5z &= 0 \\ 3x + y - 2z &= 0 \end{aligned}$$

**309.** $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 4x_1 + 7x_2 + 5x_3 &= 0 \\ x_1 + 6x_2 + 10x_3 &= 0 \\ x_1 + x_2 - 4x_3 &= 0 \end{aligned}$$

**310.** $$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 0 \\ 2x_1 - x_2 + 3x_3 &= 0 \\ 3x_1 - 5x_2 + 4x_3 &= 0 \\ x_1 + 17x_2 + 4x_3 &= 0 \end{aligned}$$

**316.** $$\begin{aligned} 3x_1 + 4x_2 - 5x_3 + 7x_4 &= 0 \\ 2x_1 - 3x_2 + 3x_3 - 2x_4 &= 0 \\ 4x_1 + 11x_2 - 13x_3 + 16x_4 &= 0 \\ 7x_1 - 2x_2 + x_3 + 3x_4 &= 0 \end{aligned}$$

**317.** $$\begin{aligned} x_1 + x_2 - 3x_4 - x_5 &= 0 \\ x_1 - x_2 + 2x_3 - x_4 &= 0 \\ 4x_1 - 2x_2 + 6x_3 + 3x_4 - 4x_5 &= 0 \\ 2x_1 + 4x_2 - 2x_3 + 4x_4 - 7x_5 &= 0 \end{aligned}$$

Solve the inhomogeneous systems by means of Gaussian elimination.

**324.** $$\begin{aligned} x + 2y + 3z &= 4 \\ 2x + y - z &= 3 \\ 3x + 3y + 2z &= 10 \end{aligned}$$

**328.** $$\begin{aligned} x + 2y + 3z &= 5 \\ 2x - y - z &= 1 \\ x + 3y + 4z &= 6 \end{aligned}$$

**332.** $$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 1 \\ 7x_2 - 3x_3 &= -7 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_1 + 4x_2 - 2x_3 &= -6 \end{aligned}$$

**337.** $$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 1 \\ 2x_1 - x_2 \qquad - 3x_4 &= 2 \\ 3x_1 \qquad - x_3 + x_4 &= -3 \\ 2x_1 + 2x_2 - 2x_3 + 5x_4 &= -6 \end{aligned}$$

**338.** $$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 4 \\ x_2 - x_3 + x_4 &= -3 \\ x_1 + 3x_2 \qquad - 3x_4 &= 1 \\ -7x_2 + 3x_3 + x_4 &= -3 \end{aligned}$$

Solve the systems of linear algebraic equations with parameters.

**348.** $$\begin{aligned} ax + y + z &= 4 \\ x + 2y + z &= 3 \\ x + 4y + z &= 4 \end{aligned}$$

**349.** $$\begin{aligned} ax - 2y + z &= 1 \\ x - 2ay + z &= -2 \\ x - 2y + az &= 1 \end{aligned}$$

**350.** $$\begin{aligned} \alpha x + y + z &= 2\alpha + 1 \\ x + \alpha y + z &= 2 \\ x + y + \alpha z &= 1 \end{aligned}$$

Specify the numbers of solutions of the systems of linear algebraic equations in dependence on parameters $a$ and $k$. Solve the system for given values of the parameters. (`Hint:` *Apply Frobenius' theorem.*)

**359.** $$\begin{aligned} (2a-1)x + (a+1)y + z &= 1 - a \\ x - y + z &= 1 \\ x + y + 3z &= 1 \end{aligned}$$ $[\, a = 1\, ]$

**360.** $$\begin{aligned} x + 2y + 3z &= 5 \\ 3x + y + 2z &= k \\ 2x - y - z &= 0 \end{aligned}$$ $[\, k = 5\, ]$

**369.** Find all values of parameter $\lambda$ for which the system $A \cdot X = O$ has a non–zero solution and calculate this solution. Matrix $A$ has the form: $$A = \begin{pmatrix} \lambda, & 4, & 7 \\ 3, & -4, & 5 \\ 1, & \lambda, & 4 \end{pmatrix}$$

(`Hint:` *The system* $A \cdot X = O$ *has a non–zero solution if matrix* $A$ *is singular and its determinant is therefore equal to zero.*)

**370.** Applying Cramer's rule solve the system $$\begin{aligned} 2x + 3y - 3z &= -1 \\ 4x - 4y - z &= 3 \\ 8x \qquad - 9z &= 0. \end{aligned}$$

# III. DIFFERENTIAL CALCULUS

## III.1. Sequences of real numbers

Decide if given sequences are increasing, decreasing, non–increasing, non–decreasing, monotone, strictly monotone, bounded below, bounded above, unbounded. (Assume that $n = 1, 2, 3, \ldots$)

**575.** $\left\{ 2 + 3^n \right\}$ **577.** $\left\{ \dfrac{n}{n+1} \right\}$ **578.** $\left\{ \dfrac{(-1)^n}{n^2+1} \right\}$

**579.** $\left\{ \dfrac{1+(-1)^n}{2} \right\}$ **580.** $\left\{ -\dfrac{n^2}{n+1} \right\}$ **581.** $\left\{ \dfrac{n+5}{n+2} \right\}$

Calculate the limits.

**591.** $\displaystyle\lim_{n\to+\infty} \left(1 + \frac{3}{n}\right)$ **594.** $\displaystyle\lim_{n\to+\infty} \frac{2n^2 - 3n + 5}{3n^2 - 2n + 1}$ **596.** $\displaystyle\lim_{n\to+\infty} \frac{n^2 - n + 3}{n^3 + 2n + 2}$

**599.** $\displaystyle\lim_{n\to+\infty} \frac{5n^3 - 3n^2 + 5n - 1}{4n^2 + n - 2}$ **609.** $\displaystyle\lim_{n\to+\infty} \frac{(3-n)^2 + (3+n)^2}{(3-n)^2 - (3+n)^2}$

**612.** $\displaystyle\lim_{n\to+\infty} \frac{(2n-1)^2 - 4n + 1}{n^2 - (n+5)^2}$ **616.** $\displaystyle\lim_{n\to+\infty} \frac{\sqrt{n+1} - \sqrt[3]{n^3+1}}{\sqrt[4]{n+1} - \sqrt[5]{n^5+1}}$

**619.** $\lim\limits_{n\to+\infty}\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)$

**620.** $\lim\limits_{n\to+\infty}\left(\sqrt{n+2}-\sqrt{n+5}\right)$

**621.** $\lim\limits_{n\to+\infty} n\left(\sqrt{n(n-2)}-\sqrt{n^2-3}\right)$

**622.** $\lim\limits_{n\to+\infty} n\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)$

**623.** $\lim\limits_{n\to+\infty}\left(\sqrt[3]{5+8n^3}-2n\right)$

**629.** $\lim\limits_{n\to+\infty}\dfrac{1+2+3+\ldots+n}{\sqrt{9n^4+1}}$

## III.2. Functions – basic notions and properties

Specify domains of given functions. (`Hint:` *If the domain is not defined explicitly then it is automatically the set of all $x$, for which the right hand side makes sense.*)

**657.** $y=\dfrac{1}{\sqrt{x^2-4x}}$

**660.** $y=\ln(x+3)+\sqrt{5-2x}$

**661.** $y=\arcsin\dfrac{1-2x}{4}$

**664.** $y=\ln(x^2-1)$

**667.** $y=\dfrac{x+\sqrt{x}}{2x^2-7x+6}$

**668.** $y=\sqrt{\ln(x^2-3x+2)}$

Given the functions $f_1$ a $f_2$. Write the composite functions $g=f_1*f_2$ a $h=f_2*f_1$.

**674.** $f_1(x)=x^2,\quad f_2(x)=\sin x$

**675.** $f_1(x)=\ln(x+1),\quad f_2(x)=5x^2+2$

**678.** $f_1(x)=x^2+5x,\quad f_2(x)=\sin(2x+1)$

**679.** $f_1(x)=\cos(x+1),\quad f_2(x)=x+2$

Which of these functions are odd or even?

**687.** $y=x^3+x\cdot\cos x$

**694.** $y=\dfrac{x^2-1}{x+x^3}$

**695.** $y=\cos x+\cos(2x)$

Which of these functions are periodic? What is the period?

**698.** $y=\sin x+\cos(2x)$

**704.** $y=|x+2|$

**707.** $y=|\cos^2(x/2)|$

Using graphs of elementary functions, sketch the graphs of these functions:

**709.** $y=\sin(2x)$

**718.** $y=\arcsin(x-5)$

**723.** $y=\sqrt{x+4}$

**727.** $y=|x|+2$

**733.** $y=\ln|x|$

**734.** $y=\ln|x-5|$

## III.3. Limit and continuity of a function

Given function $f$ and a positive number $\epsilon$. Evaluate the limit $L=\lim_{x\to+\infty}f(x)$ and find a real number $a$ such that $f(x)\in U_\varepsilon(L)$ for all $x\in(a,+\infty)$. (`Hint:` *Use the definition of a limit.*)

**767.** $f(x)=\dfrac{1}{x+1},\ \varepsilon=0.01$

**768.** $f(x)=5+\mathrm{e}^{-x},\ \varepsilon=0.1$

Calculate the limits (if they exist).

**791.** $\lim\limits_{x\to0}\dfrac{x^3-3x+1}{x-4}$

**792.** $\lim\limits_{x\to2}\dfrac{x^2-4x+1}{2x+1}$

**796.** $\lim\limits_{x\to+\infty}\left(\dfrac{5x^2}{1-x^2}+2^{1/x}\right)$

**797.** $\lim\limits_{x\to+\infty}\dfrac{3x-1}{x^2+1}$

**807.** $\lim\limits_{x\to+\infty}x\left(\sqrt{x^2+1}-x\right)$

**816.** $\lim\limits_{x\to1}\dfrac{x^2-2x+1}{2x^2-x-1}$

**837.** $\lim\limits_{x\to0}\dfrac{\mathrm{tg}(5x)}{3x}$

**839.** $\lim\limits_{x\to0}\dfrac{1-\cos^2x}{x^2}$

**840.** $\lim\limits_{x\to0}\dfrac{\mathrm{arctg}\,x}{x}$

**859.** $\lim\limits_{x\to-1}\dfrac{x^3+1}{\sin(x+1)}$

**863.** $\lim\limits_{x\to+\infty}\sin\dfrac{1}{x}$

**864.** $\lim\limits_{x\to0}\mathrm{arctg}\dfrac{1}{x^2}$

**865.** $\lim\limits_{x\to+\infty}\arcsin\dfrac{x}{x+1}$

**866.** $\lim\limits_{x\to+\infty}\dfrac{\mathrm{e}^x}{x^2}$

**869.** $\lim\limits_{x\to-\infty}x\cdot\mathrm{e}^x$

**882.** $\lim\limits_{x\to 0} \dfrac{e^{2x} - e^{x}}{\sin(2x) - \sin x}$ **887.** $\lim\limits_{x\to +\infty} \left(\dfrac{x+1}{x-2}\right)^{2x-1}$ **891.** $\lim\limits_{x\to 0} \dfrac{\ln(\cos x)}{x^2}$

Calculate the limits.

**902.** $\lim\limits_{x\to 1+} \dfrac{x+2}{x-1}$ **904.** $\lim\limits_{x\to 0+} x \cdot \ln x$ **909.** $\lim\limits_{x\to 0+} \dfrac{5+x}{x(x-1)}$

The next three limits do not exist. Give reasons why.

**921.** $\lim\limits_{x\to 0} \dfrac{1}{x}$ **924.** $\lim\limits_{x\to +\infty} \sin x$ **926.** $\lim\limits_{x\to 2} \dfrac{x+1}{x-2}$

Find maximal intervals where given functions are continuous.

**929.** $y = \dfrac{x}{(1+x)^2}$ **931.** $y = \dfrac{1+x}{1+x^3}$ **932.** $y = \dfrac{x^2-1}{x^3-3x+2}$

**937.** $y = e^{x+1/x}$ **939.** $y = \dfrac{1}{\ln x}$ **943.** $y = \dfrac{1}{e^x - 1}$

## III.4. Derivative of a function, its geometrical and physical sense

Calculate derivatives of given functions. Also specify for which $x$ the derivative exists.

a) Polynomials, rational functions, powers and roots.

**960.** $y = 5x^2 + 7x - 2$ **968.** $y = \sqrt{2x^2 - x + 5}$ **972.** $y = (x+6)\sqrt{x-1}$

**976.** $y = \dfrac{x^2+3}{x+5}$ **979.** $y = \dfrac{x+2}{\sqrt{5-x}}$ **980.** $y = \dfrac{\sqrt{x^2+1}}{x+1}$

b) Trigonometric functions associated composite functions.

**990.** $y = \sin(3x)$ **991.** $y = \cos x^2$ **993.** $y = \sin^2(6x)$

**999.** $y = \sin(x^2 + 2x + 2)$ **1001.** $y = x^2 \cdot \operatorname{tg} x$ **1003.** $y = \sqrt{1 + x + \sin x}$

c) Inverse trigonometric functions and associated composite functions.

**1008.** $y = \arcsin\sqrt{x+1}$ **1009.** $y = \arccos\sqrt{\dfrac{x}{x+1}}$ **1016.** $y = \operatorname{arctg}\sqrt{x}$

d) Exponential and logarithmic functions and associated composite functions.

**1017.** $y = e^{2x}$ **1018.** $y = e^{5x^2-2x+1}$ **1020.** $y = \sqrt{e^x}$

**1021.** $y = (e^{5x}+1)^2$ **1027.** $y = \ln(x^2 + 3x + 4)$ **1028.** $y = \ln\left(x + \sqrt{1+x^2}\right)$

e) Various further functions.

**1047.** $y = \ln\left(7x + \sqrt{49x^2+1}\right)$ **1049.** $y = e^{2x} \cdot (x^2+1)^2$ **1051.** $y = \arcsin\dfrac{1-x^2}{1+x^2}$

Calculate the derivative of a given function $f$ and sketch the graph of function $f$ and its derivative $f'$. (`Hint:` *Consider the formulas* $|x| = x$ *for* $x > 0$, $|x| = -x$ *for* $x < 0$. *Function* $|x|$ *is not does not have a derivative at the point* $x = 0$.)

**1055.** $f(x) = |x|$ **1056.** $f(x) = \ln|x|$ **1057.** $f(x) = x \cdot |x|$

Calculate $f'_+(x_0)$ a $f'_-(x_0)$. (`Hint:` *If the right limit* $\lim_{x\to x_0+} f'(x)$ *exists then the right derivative* $f_+(x_0)$ *also exists and equals the limit. The same assertion also holds on the left derivative.*)

**1058.** $f(x) = |x|,\quad x_0 = 0$ **1062.** $f(x) = |4x - x^2|,\quad x_0 = 4$

Calculate second derivatives of given functions. For which $x$ does the second derivative exist? (`Hint:` *$f''$ is equal to the derivative of function $f'$.*)

**1066.** $y = \sqrt{1+x^2}$ **1068.** $y = \operatorname{cotg} x$ **1071.** $y = \dfrac{1+x}{1-x}$

Write equations of the tangent line and the normal line to the graph of function $f$ at the point $[x_0, f(x_0)]$. (`Hint:` *Equation of the tangent line: $y - y_0 = k \cdot (x - x_0)$, where $y_0 = f(x_0)$ a $k = f'(x_0)$. Equation of the normal line: $y - y_0 = -(1/k)(x - x_0)$.*)

**1109.** $f(x) = \dfrac{e^x}{x+1}$, $x_0 = 0$ **1110.** $f(x) = x \cdot \sin x$, $x_0 = \pi$

**1118.** At which point does the parabola $y = x^2 + 4x$ have a tangent line parallel with the $x$–axis? (`Hint:` *Find point $x_0$ where the derivative of the function $x^2 + 4x$ equals zero. Calculate $y_0$ from the equation of the parabola.*)

**1119.** At which point does the parabola $y = x^2 - 2x + 5$ have a tangent line perpendicular to the axis of the first quadrant? (`Hint:` *The axis of the first quadrant is the straight line $y = x$, whose slope is $k_1 = 1$. Find point $x_0$ where the function $x^2 - 2x + 5$ has the derivative $k_2 = -1$. Calculate $y_0$ from the equation of the parabola.*)

**1125.** For which $x \in D(f)$ does the function $f(x) = \ln \dfrac{1+\sqrt{x^2-1}}{x}$ have a tangent line at the point $[x, f(x)]$? Is some of the tangent lines parallel with the $x$–axis? Write the equation of the tangent line at the point $[x_0, f_(x_0)]$ for $x_0 = \sqrt{5}$.

## III.5. Application of derivative, behaviour of a function

Find intervals, where given function is monotone. Specify if the function is increasing or decreasing. (`Hint:` *Use the sign of the first derivative.*)

**1144.** $f(x) = 2x^3 + 3x^2 - 36x + 4$ **1145.** $f(x) = x + \dfrac{x}{x^2-1}$ **1146.** $f(x) = x^2 \cdot e^x$

**1148.** $f(x) = 3x - x^3$ **1151.** $f(x) = \dfrac{2x}{1+x^2}$ **1156.** $f(x) = x^2 - \ln x^2$

Determine the domain $D(f)$ of function $f$, evaluate one–sided limits at the end points of intervals that form $D(f)$ and find intervals of monotonicity of function $f$. Specify where $f$ is increasing and where it is decreasing. Sketch the graph of function $f$.

**1160.** $f(x) = \dfrac{e^x}{x^2-1}$ **1161.** $f(x) = x^{1/x}$

Decide about the existence of the global extremes (maximum and minimum) on interval $I$. If the extremes exist, find their values and positions (i.e. find points where function $f$ takes their values).

**1164.** $f(x) = x^4 - 2x^2 + 5$, $I = \langle -2, 2 \rangle$ **1169.** $f(x) = \sqrt{5-4x}$, $I = \langle -1, 1 \rangle$

**1174.** $f(x) = x^2 + \dfrac{16}{x} - 16$, $I = \langle 1, 4 \rangle$ **1175.** $f(x) = 4 - x - \dfrac{4}{x^2}$, $I = \langle 1, 4 \rangle$

**1177.** $f(x) = x + 3\sqrt[3]{x^2}$, $I = \langle -1, 1 \rangle$ **1179.** $f(x) = -\frac{2}{3} \ln x - \frac{1}{6} x^2 + x$, $I = (0, 6\rangle$

Find local and global extremes of given functions (on their domains).

**1204.** $y = x - 2 \ln x$ **1209.** $y = \dfrac{\ln x}{x}$ **1211.** $y = x + \dfrac{1}{x}$

**1212.** $y = \dfrac{(4-x)^3}{9(2-x)}$ **1217.** $y = \dfrac{1+x+x^2}{1-x+x^2}$ **1218.** $y = x + 2\sqrt{-x}$

**1241.** Given the function $f(x) = (1 + x^2)\, e^x$ . Specify its domain, calculate one–sided limits at the end points of the domain of $f$ and find the local extremes.

On which maximal intervals are given functions concave up and concave down? Find points of inflection. (`Hint:` *Use the sign of the second derivative.*)

**1255.** $y = 3x^5 - 40x^3 + x - 2$ **1256.** $y = \dfrac{x}{1+x^2}$ **1259.** $y = \sqrt{1+x^2}$

**1260.** $y = \ln(1 + x^2)$ **1262.** $y = \ln \dfrac{x-1}{x+2}$ **1264.** $y = x \cdot \operatorname{arctg} x$

Find asymptotes of given functions (if they exist).

**1267.** $y = \dfrac{x^3}{4-x^2}$ **1268.** $y = 2x - \dfrac{1}{x-2}$ **1270.** $y = 2x + \operatorname{arctg} \dfrac{x}{2}$

**1272.** $y = \dfrac{1-2x}{2+x}$ **1274.** $y = x + \dfrac{1}{x}$ **1275.** $y = x + \dfrac{\ln x}{x}$

Examine the behaviour of given functions.

**1277.** $y = \dfrac{1}{4-x^2}$ **1278.** $y = \sqrt[3]{x^2} - x$ **1279.** $y = e^{-x^2}$

**1281.** $y = 1 + x^2 - \frac{1}{2}\, x^4$ **1282.** $y = x^4 - 2x^2$ **1292.** $y = (x + 2)\, \mathrm{e}^{1/x}$

**1293.** $y = (x - 3)\sqrt{x}$ **1295.** $y = \dfrac{x-2}{\sqrt{x^2+1}}$ **1308.** $y = \mathrm{e}^{2x-x^2}$

**1317.** $y = \arccos \dfrac{1-x^2}{1+x^2}$ **1319.** $y = \ln(4 - x^2)$ **1321.** $y = x + \operatorname{arccotg} x$

## III.6. Taylor's theorem

Write Taylor's polynomial of degree $n$ of function $f$ at point $x_0$. Write the formula for the remainder after the $n$–th term.

**1330.** $f(x) = \mathrm{e}^x, \quad n = 5, \quad x_0 = 0$ **1331.** $f(x) = \mathrm{e}^x, \quad n = 5, \quad x_0 = 1$

**1333.** $f(x) = \mathrm{e}^{3x}, \quad n = 4, \quad x_0 = 0$ **1337.** $f(x) = \sin x, \quad n = 7, \quad x_0 = 0$

**1346.** $f(x) = \ln(x + 1), \quad n = 7, \quad x_0 = 0$ **1351.** $f(x) = \sqrt{x}, \quad n = 4, \quad x_0 = 1$

**1352.** $f(x) = \sqrt{x+3}, \quad n = 3, \quad x_0 = 0$ **1354.** $f(x) = \dfrac{1}{x}, \quad n = 4, \quad x_0 = 1$

Calculate approximately with the accuracy $\varepsilon$ (i.e. with an error less than or equal to $\varepsilon$) given function values. (`Hint:` *We have to evaluate approximately the function value of $f$ at point $x$ which is ,,close" to another point $x_0$, where the function value $f(x_0)$ is known. Find $n$ so large that the remainder $R_{n+1}(x)$ is less than or equal to $\varepsilon$ on some interval containing $x$. Then calculate approximately $f(x)$ by means of Taylor's polynomial $T_n(x)$.*)

**1362.** $1/\mathrm{e}, \quad \varepsilon = 10^{-3}$ **1363.** $\cos 5^o, \quad \varepsilon = 10^{-3}$ **1365.** $\ln 1.2, \quad \varepsilon = 10^{-3}$

**1376.** Write Taylor's polynomial $T_2$ of the function $f(x) = \sqrt[3]{1+x}$ at the point $x_0 = 0$. Using Lagrange's form of the remainder find an upper bound of $|f(\frac{1}{2}) - T_2(\frac{1}{2})|$.

## IV. INDEFINITE INTEGRAL

### IV.1. Fundamental properties of indefinite integrals, table of basic integrals

Using the table of basic integrals, calculate:

**1448.** $\int 3x^7 \, \mathrm{d}x$ **1450.** $\int (3-x^2)^3 \, \mathrm{d}x$ **1452.** $\int \sqrt[3]{x} \, \mathrm{d}x$

**1454.** $\int \frac{1}{x^2} \, \mathrm{d}x$ **1455.** $\int \frac{1}{2\sqrt{x}} \, \mathrm{d}x$ **1458.** $\int (1-2u) \, \mathrm{d}u$

**1459.** $\int (\sqrt{x}+1)(x-\sqrt{x}+1) \, \mathrm{d}x$ **1460.** $\int (2x^{-1{,}2} + 3x^{-0{,}8} - 5x^{0{,}38}) \, \mathrm{d}x$

**1461.** $\int \left(\frac{1-x}{x}\right)^2 \mathrm{d}x$ **1464.** $\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}} \, \mathrm{d}x$ **1467.** $\int \left(\frac{3^3}{x^3} + \frac{3^2}{x^2} + \frac{3}{x}\right) \mathrm{d}x$

**1468.** $\int 10^x \, \mathrm{d}x$ **1470.** $\int \frac{3 \cdot 2^x - 2 \cdot 3^x}{2^x} \, \mathrm{d}x$ **1473.** $\int \frac{1+\cos^2 x}{1+\cos 2x} \, \mathrm{d}x$

**1474.** $\int \frac{\cos 2x}{\cos^2 x \cdot \sin^2 x} \, \mathrm{d}x$ **1475.** $\int 2\sin^2 \frac{x}{2} \, \mathrm{d}x$

### IV.2. Integration by parts

Applying the integration by parts, calculate:

**1481.** $\int x\, e^x \, \mathrm{d}x$ **1482.** $\int x \, \ln x \, \mathrm{d}x$ **1483.** $\int x \, \sin x \, \mathrm{d}x$

**1484.** $\int x \, \mathrm{arctg}\, x \, \mathrm{d}x$ **1485.** $\int x \, \cos x \, \mathrm{d}x\,;$ **1486.** $\int x^2 e^x \, \mathrm{d}x$

**1489.** $\int x^n \ln x \, \mathrm{d}x, \ n \neq -1$ **1494.** $\int \mathrm{arctg}\, x \, \mathrm{d}x$ **1502.** $\int \arcsin^2 t \, \mathrm{d}t$

**1504.** $\int e^x \sin x \, \mathrm{d}x$ **1506.** $\int e^{7x} \cos 5x \, \mathrm{d}x$ **1510.** $\int (x^2-3x+2)\, \mathrm{e}^x \, \mathrm{d}x$

### IV.3. Integration by substitution

Using an appropriate substitution, calculate the integrals:

**1514.** $\int \frac{1}{1-x} \, \mathrm{d}x$ **1515.** $\int \frac{e^{2x}}{2+e^{2x}} \, \mathrm{d}x$ **1516.** $\int \mathrm{cotg}\, x \, \mathrm{d}x$

**1518.** $\int \frac{2x}{\sqrt{x^2+1}} \, \mathrm{d}x$ **1519.** $\int (1+x)^{15} \, \mathrm{d}x$ **1532.** $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}} \, \mathrm{d}x$

**1533.** $\int \cos^3 x \, \sin 2x \, \mathrm{d}x$ **1534.** $\int \frac{\sqrt{\ln x}}{x} \, \mathrm{d}x$ **1542.** $\int \cos(1-2x) \, \mathrm{d}x$

**1546.** $\int \frac{2x-3}{x^2-3x+8} \, \mathrm{d}x$ **1555.** $\int e^{-x^3} x^2 \, \mathrm{d}x$ **1572.** $\int \frac{x}{2x+1} \, \mathrm{d}x$

**1579.** $\int \frac{x^4}{1-x} \, \mathrm{d}x$ **1580.** $\int \frac{x^4}{x^2+1} \, \mathrm{d}x$ **1594.** $\int \frac{\sqrt{x}}{\sqrt[3]{x^2}-\sqrt[4]{x}} \, \mathrm{d}x$

**1615.** $\int \frac{\arcsin x}{\sqrt{(1-x^2)^3}} \, \mathrm{d}x$ **1620.** $\int \frac{\sin x}{e^{\cos x}} \, \mathrm{d}x$ **1625.** $\int \frac{\sqrt{x}}{1+x^{\frac{3}{2}}} \, \mathrm{d}x$

**1628.** $\int \frac{2x+3}{\sqrt{1-x^2}} \, \mathrm{d}x$ **1666.** $\int \frac{\ln \cos x}{\cos^2 x} \, \mathrm{d}x$ **1688.** $\int x^5 e^{-x^2} \, \mathrm{d}x$

### IV.4. Integration of rational functions

Calculate the integrals:

**1720.** $\displaystyle\int \frac{8}{3x-1}\,\mathrm{d}x$ **1722.** $\displaystyle\int \frac{x^4}{x^2-2}\,\mathrm{d}x$ **1724.** $\displaystyle\int \frac{1}{(x-1)^3}\,\mathrm{d}x$

Using the decomposition to partial fractions, calculate the integrals:

**1731.** $\displaystyle\int \frac{u-1}{u^2+u}\,\mathrm{d}u$ **1733.** $\displaystyle\int \frac{5x-4}{x^2-8x+12}\,\mathrm{d}x$ **1734.** $\displaystyle\int \frac{x^2}{x^2-5x+4}\,\mathrm{d}x$

**1739.** $\displaystyle\int \frac{x^3}{x^2+3x+2}\,\mathrm{d}x$ **1748.** $\displaystyle\int \frac{x}{(x+1)(x+3)(x+5)}\,\mathrm{d}x$ **1749.** $\displaystyle\int \frac{x-8}{x^3-4x^2+4x}\,\mathrm{d}x$

**1751.** $\displaystyle\int \frac{3x-2}{x(x^2+1)}\,\mathrm{d}x$ **1754.** $\displaystyle\int \frac{1}{x(x^2+1)}\,\mathrm{d}x$ **1755.** $\displaystyle\int \frac{1}{x^2-x+1}\,\mathrm{d}x$

**1761.** $\displaystyle\int \frac{3x-1}{x^2-x+1}\,\mathrm{d}x$ **1793.** $\displaystyle\int \frac{x^2}{x^3+5x^2+8x+4}\,\mathrm{d}x$ **1798.** $\displaystyle\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}\,\mathrm{d}x$

### IV.5. Integration of trigonometric functions and their powers

Calculate the integrals:

**1814.** $\displaystyle\int \sin^7 x\,\mathrm{d}x$ **1815.** $\displaystyle\int \sin^3 x\,\mathrm{d}x$ **1822.** $\displaystyle\int \sin^3 x\,\cos^5 x\,\mathrm{d}x$

**1823.** $\displaystyle\int \sin x\,\cos^5 x\,\mathrm{d}x$ **1828.** $\displaystyle\int \sin^2 x\,\mathrm{d}x$ **1832.** $\displaystyle\int \sin^2 x\,\cos^2 x\,\mathrm{d}x$

**1833.** $\displaystyle\int \sin^4 5x\,\mathrm{d}x$ **1858.** $\displaystyle\int \frac{1}{5+4\sin x}\,\mathrm{d}x$ **1864.** $\displaystyle\int \frac{1+\cos x}{\sin^3 x}\,\mathrm{d}x$

**1865.** $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x$ **1871.** $\displaystyle\int \cos 2x\,\cos 3x\,\mathrm{d}x$ **1873.** $\displaystyle\int \cos 2x\,\sin 4x\,\mathrm{d}x$

### IV.6. Integrals of the type $\displaystyle\int R\left(x,\ \sqrt[s]{\frac{ax+b}{cx+d}}\right)\,\mathrm{d}x\,.$

Calculate the integrals.

**1892.** $\displaystyle\int \frac{\sqrt[4]{x}}{1+\sqrt{x}}\,\mathrm{d}x$ **1895.** $\displaystyle\int \frac{(\sqrt{x^3}-\sqrt[3]{x})}{6\sqrt[4]{x}}\,\mathrm{d}x$ **1896.** $\displaystyle\int \frac{\sqrt{1-x}}{\sqrt{1+x}}\,\frac{1}{x^2}\,\mathrm{d}x$

**1898.** $\displaystyle\int \sqrt{\frac{2+3x}{x-3}}\,\mathrm{d}x$ **1899.** $\displaystyle\int \frac{x}{x+\sqrt{x+2}}\,\mathrm{d}x$

## V. DEFINITE (RIEMANN'S) INTEGRAL

### V.1. Basic properties of definite integrals, Newton–Leibniz' formula

Applying the table of basic indefinite integrals and the Newton–Leibniz formula, calculate the definite integrals:

**1985.** $\displaystyle\int_1^2 \left(x^3+3x^2-5\right)\,\mathrm{d}x$ **1986.** $\displaystyle\int_0^a \left(a^2x-x^3\right)\,\mathrm{d}x$ **1989.** $\displaystyle\int_1^2 \left(x^2+\frac{1}{x^4}\right)\,\mathrm{d}x$

**1991.** $\displaystyle\int_0^{\pi/4} \frac{1}{\cos^2 x}\,\mathrm{d}x$ **1992.** $\displaystyle\int_{-1}^8 \sqrt[3]{x}\,\mathrm{d}x$ **1993.** $\displaystyle\int_0^\pi \sin x\,\mathrm{d}x$

**1996.** $\displaystyle\int_1^{\mathrm{e}} \frac{1}{x}\,\mathrm{d}x$ **2000.** $\displaystyle\int_0^1 x^2\left(1-x^2\right)\,\mathrm{d}x$ **2002.** $\displaystyle\int_1^4 \left(1-\sqrt{x}\right)^2\,\mathrm{d}x$

## V.2. Integration by parts and by substitution in the definite integral

Applying integration by parts and integration by substitution, calculate the definite integrals:

**2010.** $\displaystyle\int_2^3 \frac{x}{x^2+1}\,\mathrm{d}x$ **2011.** $\displaystyle\int_0^\pi x\,\sin x\,\mathrm{d}x$ **2012.** $\displaystyle\int_0^{\mathrm{e}-1} \ln(x+1)\,\mathrm{d}x$

**2015.** $\displaystyle\int_0^1 \frac{\sqrt{x}}{1+x}\,\mathrm{d}x$ **2017.** $\displaystyle\int_1^{\mathrm{e}^3} \frac{1}{x\cdot\sqrt{1+\ln x}}\,\mathrm{d}x$ **2020.** $\displaystyle\int_5^1 \frac{t}{\sqrt{5+4t}}\,\mathrm{d}t$

**2024.** $\displaystyle\int_0^{\pi/2} \sin^3 t\sqrt{\cos t}\,\mathrm{d}t$ **2030.** $\displaystyle\int_0^1 \frac{\operatorname{arctg} z}{1+z^2}\,\mathrm{d}z$ **2031.** $\displaystyle\int_0^{\ln 2} x\,\mathrm{e}^{-x}\,\mathrm{d}x$

**2032.** $\displaystyle\int_1^2 \frac{1}{x^2+x}\,\mathrm{d}x$ **2035.** $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x$ **2036.** $\displaystyle\int_{-1}^1 x\,\operatorname{arctg} x\,\mathrm{d}x$

**2038.** $\displaystyle\int_1^{\mathrm{e}} \frac{\ln^2 x}{x}\,\mathrm{d}x$ **2039.** $\displaystyle\int_1^2 x\,\ln x\,\mathrm{d}x$ **2040.** $\displaystyle\int_0^{\pi/2} \sin^3 x\,\mathrm{d}x$

**2043.** $\displaystyle\int_0^1 \arcsin x\,\mathrm{d}x$ **2044.** $\displaystyle\int_1^2 \frac{\mathrm{e}^{1/x}}{x^2}\,\mathrm{d}x$

## V.3. Improper Riemann's integral

Do these integrals converge? If „yes", find their values.

**2050.** $\displaystyle\int_1^{\mathrm{e}} \frac{1}{x\,\ln x}\,\mathrm{d}x$ **2051.** $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ **2054.** $\displaystyle\int_2^3 \frac{x}{\sqrt[4]{x^2-4}}\,\mathrm{d}x$

**2056.** $\displaystyle\int_2^{+\infty} \frac{1}{x^3}\,\mathrm{d}x$ **2057.** $\displaystyle\int_1^{+\infty} \frac{1}{x^2+2x}\,\mathrm{d}x$ **2058.** $\displaystyle\int_1^{+\infty} \frac{\ln x}{x^2}\,\mathrm{d}x$

**2060.** $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2+2x+2}\,\mathrm{d}x$ **2063.** $\displaystyle\int_0^{\pi/2} \operatorname{tg} x\,\mathrm{d}x$

## V.4. Some geometrical applications of the definite integral

Calculate areas $P$ of curvilinear trapezoids bounded by the $x$–axis and the curves with the equations:

**2067.** $y = x\sqrt{1-x^2},\ x=0,\ x=1$ **2068.** $y = x^2-4,\ x=0,\ x=6$

Calculate areas of planar regions bounded by given curves:

**2069.** $y = 3-2x-x^2,\ y=0$ **2070.** $y=x^3,\ y=x$

**2074.** Calculate the volume of the circular ellipsoid that arises by the revolution of the ellipse $4x^2+9y^2-36=0$ a) about the $x$–axis, b) about the $y$–axis.

**2075.** Calculate the volume of the circular body that arises by the revolution of the planar region, bounded by the curves $y^2=8x$, $y=x^2$ a) about the $x$–axis, b) about the $y$–axis.

Calculate the length of the graph of a given function.
(`Hint:` *Apply the formula* $l=\int_a^b\sqrt{[f'(x)]^2+1}\,\mathrm{d}x$.)

**2077.** $y=\frac{3}{2}\sqrt[3]{x^2}$ pro $x\in\langle 1,8\rangle$

## V.5. Further problems

Sketch the region bounded by given curves and evaluate its area.

**1.** $y = x^2,\ y = \sqrt{x}$

**2.** $y = 3 - x,\ y = 2/x$

**3.** $y = e^x,\ y = e^{-x},\ y = e$

**4.** $y = \dfrac{2}{1+x^2},\ y = x^2$

**5.** Find coordinates of intersections of the graphs of given functions $f$ and $g$, resp. $f$ and $h$, resp. $g$ and $h$: $f(x) = x/8,\ g(x) = 8/x,\ h(x) = 8/x^2$. Sketch the picture and calculate the area of the region between the graphs of these three functions.

Evaluate the volume of the circular body that arises by rotation of a given curve about the $x$–axis. Sketch the region in the $x$, $y$–plane, bounded by the curve and the $x$–axis.

**6.** $y = \sin x,\ x \in \langle 0, \pi/2\rangle$

**7.** $y = 3 - x,\ x \in \langle 0, 3\rangle$

**8.** $y = e^x,\ x \in \langle 0, 1\rangle$

**9.** $y = \dfrac{1}{\cos x},\ x \in \langle 0, \pi/4\rangle$

**10.** $y = \sin x + \cos x,\ x \in \langle 0, \pi/2\rangle$

**11.** Calculate the volume of the body that arises by rotation of the planar region, bounded by the curves $y = \sqrt{x},\ y = x/2$ about the $x$–axis.

Calculate lengths of the graphs of given functions.

**12.** $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$ for $x \in \langle 2,\ 4\rangle$

**13.** $y = \sqrt{x^3}$ for $x \in \langle 0, 4\rangle$

Calculate the mean value of function $f$ on given interval. (`Hint:` *Apply the formula*
$\mu(f) = \dfrac{1}{(b-a)} \displaystyle\int_a^b f(x)\,\mathrm{d}x$.)

**14.** $f(x) = \sin^2 x,\ x \in \langle 0, \pi\rangle$

**15.** $f(x) = x\,\sin x,\ x \in \langle 0, \pi\rangle$

## RESULTS

### I.1. Vectors, vector spaces

2. $\mathbf{a} = (17, 10, 3)$ 8. $\mathbf{x} = (-1, 0, 3, 4)$ 15. 21 22. $63.4^{\circ}$

25. $\alpha = \frac{1}{2}, -3$ 42. LN, 2; $\mathbf{u}$, $\mathbf{v}$ 43. LZ, 1; e.g. $\mathbf{u}$ 44. LN, 2; $\mathbf{u}$, $\mathbf{v}$

45. LZ, 1; e.g. $\mathbf{u}$ 50. LZ, 1; e.g. $\mathbf{x}$ 51. LZ, 2; e.g. $\mathbf{x}$, $\mathbf{y}$ 53. LN, 3; $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$

68. for all $\alpha$; dim = 2 for $\alpha = -1$, dim = 3 for $\alpha \neq -1$

69. $k = 0,\ 3,\ -2$; dim = 2 70. $a = 2,\ -1$; dim = 2

73. e.g. $\mathbf{a} = \mathbf{u} + \mathbf{v}$, the expression is not unique, $\mathbf{b}$ cannot be expressed

79. no, 2 85. yes 92. no 93. yes 94. yes

### I.2. Matrices, determinants

109. $\begin{pmatrix} 21, 5, 3, 11 \\ 10, 19, 5, 2 \end{pmatrix}$ 110. $\begin{pmatrix} 3, 6, 3 \\ 2, 4, 2 \\ 1, 2, 1 \end{pmatrix}$ 111. $\begin{pmatrix} 0 \\ 7 \\ 0 \\ 2 \end{pmatrix}$ 112. $\begin{pmatrix} 1, 2 \\ 3, 2 \\ 1, 3 \end{pmatrix}$ 117. $\begin{pmatrix} 15, 20 \\ 20, 35 \end{pmatrix}$

124. $\begin{pmatrix} 6, 0 \\ 0, 5 \end{pmatrix}$ 132. $\begin{pmatrix} -10, -4, -7 \\ 6, 14, 4 \\ -5, 5, -4 \end{pmatrix}$ 133. $\begin{pmatrix} 4, -4, 4 \\ -4, -1, 0 \\ 2, 4, -4 \end{pmatrix}$ 139. $x = 14,\ y = 2$ 140. $x = 4,\ y = 2$

143. 3, regular 148. 2, singular 149. 3, singular 160. $\begin{pmatrix} -1, 1 \\ 1, -0.5 \end{pmatrix}$ 161. $\begin{pmatrix} 1, 0, 0 \\ -3, 1, 0 \\ 9, -3, 1 \end{pmatrix}$

166. does not exist 168. $\begin{pmatrix} 1, -4, -3 \\ 1, -5, -3 \\ -1, 6, 4 \end{pmatrix}$ 170. $\begin{pmatrix} 1, -2, 7 \\ 0, 1, -2 \\ 0, 0, 1 \end{pmatrix}$ 171. $\begin{pmatrix} 2, -1, 0, 0 \\ -3, 2, 0, 0 \\ 31, -19, 3, -4 \\ -23, 14, -2, 3 \end{pmatrix}$

174. $\begin{pmatrix} -3, 2, 0 \\ -4, 5, -2 \\ -5, 3, 0 \end{pmatrix}$ 175. $\begin{pmatrix} 0.5, 0.25 \\ 0, 0.5 \end{pmatrix}$ 176. $\begin{pmatrix} 5, -9 \\ -0.5, 4.5 \end{pmatrix}$ 177. $\begin{pmatrix} 8, -1 \\ -19, 2 \end{pmatrix}$

178. $x \neq y,\ x \neq z,\ y \neq z,\ \begin{pmatrix} -1.5, 2, -0.5 \\ 0.5, -1, 0.5 \\ 0.5, 1, -0.5 \end{pmatrix}$

180. $\cos 2x$ 185. $-58$ 190. $2a^2(a + x)$ 200. $3a - b$

### I.4. Eigenvalues and eigenvectors of square matrices

235. $\lambda_1 = 3,\ X_1 = \alpha \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad \lambda_2 = 1,\ X_2 = \beta \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},\quad \alpha, \beta \in \mathbf{C},\ \alpha, \beta \neq 0$

236. $\lambda_1 = 7,\ X_1 = \alpha \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad \lambda_2 = -2,\ X_2 = \beta \cdot \begin{pmatrix} 4 \\ -5 \end{pmatrix},\quad \alpha, \beta \in \mathbf{C},\ \alpha, \beta \neq 0$

237. $\lambda_1 = ai,\ X_1 = \alpha \cdot \begin{pmatrix} 1 \\ i \end{pmatrix},\quad \lambda_2 = -ai,\ X_2 = \beta \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix},\quad \alpha, \beta \in \mathbf{C},\ \alpha, \beta \neq 0$

238. $\lambda = 2,\ X = \alpha \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\quad \alpha, \beta \in \mathbf{C},\ |\alpha| + |\beta| \neq 0$

241. $\lambda_1 = 1,\ X_1 = \begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix},\quad \lambda_2 = -1,\ X = \begin{pmatrix} -\gamma \\ 0 \\ \gamma \end{pmatrix},\quad \alpha, \beta, \gamma \in \mathbf{C},\ |\alpha| + |\beta| \neq 0,\ \gamma \neq 0$

243. $\lambda_1 = -2,\ X_1 = \alpha \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\quad \lambda_2 = 1,\ X_2 = \beta \cdot \begin{pmatrix} 3 \\ -6 \\ 20 \end{pmatrix},\quad \alpha, \beta \in \mathbf{C},\quad \alpha, \beta \neq 0$

244. $\lambda = 1,\ X = \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\quad \alpha \in \mathbf{C},\ \alpha \neq 0$ 245. no 246. no

250. $A$: eigenvalues $\begin{matrix} \lambda_1 = 3, \\ \lambda_2 = 1, \end{matrix}$, eigenvectors $X_1 = \begin{pmatrix} p \\ p \end{pmatrix},\ X_2 = \begin{pmatrix} q \\ -q \end{pmatrix},\ p, q \in \mathbf{C},\ p, q \neq 0$

$A^{-1} = \begin{pmatrix} 2/3, -1/3 \\ -1/3, 2/3 \end{pmatrix}$, eigenvalues $\begin{matrix} \eta_1 = 1/\lambda_1 = 1/3, \\ \eta_2 = 1/\lambda_2 = 1, \end{matrix}$, eigenvectors $X_1,\ X_2$

251. $A$: eigenvalues $\begin{matrix} \lambda_1 = 7, \\ \lambda_2 = -2, \end{matrix}$ eigenvectors $X_1 = \begin{pmatrix} p \\ p \end{pmatrix},\ X_2 = \begin{pmatrix} -4q \\ 5q \end{pmatrix},\ p, q \in \mathbf{C},\ p, q \neq 0$

$A^{-1} = \begin{pmatrix} -2/14, 4/14 \\ 5/14, -3/14 \end{pmatrix}$, eigenvalues $\begin{matrix} \eta_1 = 1/\lambda_1 = 1/7, \\ \eta_2 = 1/\lambda_2 = -1/2, \end{matrix}$ eigenvectors $X_1,\ X_2$

252. $A$: eigenvalues $\begin{matrix} \lambda_1 = i\sqrt{5}, \\ \lambda_2 = -i\sqrt{5}, \end{matrix}$ eigenvectors $X_1 = \begin{pmatrix} p \\ pi \end{pmatrix},\ X_2 = \begin{pmatrix} q \\ -qi \end{pmatrix},\ p, q \in \mathbf{C},\ p, q \neq 0$

$A^{-1} = \begin{pmatrix} 0, -1/5 \\ 1/5, 0 \end{pmatrix}$, eigenvalues $\begin{matrix} \eta_1 = 1/\lambda_1 = -i/5, \\ \eta_2 = 1/\lambda_2 = i/5, \end{matrix}$ eigenvectors $X_1,\ X_2$

253. $A$: $\begin{matrix} \lambda_1 = 2, \\ \lambda_{2,3} = 1 \pm \sqrt{3}, \end{matrix}$ eigenvectors $X_1 = \begin{pmatrix} 2p \\ -p \\ 0 \end{pmatrix},\ X_{2,3} = \begin{pmatrix} 3(2 \pm \sqrt{3})q \\ -2(2 \pm \sqrt{3})q \\ q \end{pmatrix},\ p, q \neq 0$

$A^{-1} = \begin{pmatrix} 1/2, 0, -3/2 \\ 0, 1/2, 1/2 \\ 1/2, 1, -1/2 \end{pmatrix}$, eigenvalues $\begin{matrix} \eta_1 = 1/\lambda_1 = 1/2 \\ \eta_{2,3} = 1/\lambda_{2,3} = 1/(1 \pm \sqrt{3}), \end{matrix}$, eigenvectors $X_1,\ X_{2,3}$

255. $A$: eigenvalues $\begin{matrix} \lambda_1 = 3, \\ \lambda_2 = 1, \end{matrix}$, eigenvectors $X_1 = \begin{pmatrix} p \\ p \end{pmatrix},\ X_2 = \begin{pmatrix} q \\ -q \end{pmatrix},\ p, q \in \mathbf{C},\ p, q \neq 0$

$A^2 = \begin{pmatrix} 5, 4 \\ 4, 5 \end{pmatrix}$, eigenvalues $\begin{matrix} \eta_1 = \lambda_1^2 = 9, \\ \eta_2 = \lambda_2^2 = 1, \end{matrix}$, eigenvectors $X_1,\ X_2$

256. $A$: eigenvalues $\begin{matrix} \lambda_1 = 1, \\ \lambda_2 = 2, \end{matrix}$, eigenvectors $X_1 = \begin{pmatrix} -p \\ 3p \end{pmatrix},\ X_2 = \begin{pmatrix} 0 \\ q \end{pmatrix},\ p, q \in \mathbf{C},\ p, q \neq 0$

$A^2 = \begin{pmatrix} 1, 0 \\ 9, 4 \end{pmatrix}$, eigenvalues $\begin{matrix} \eta_1 = \lambda_1^2 = 1, \\ \eta_2 = \lambda_2^2 = 4, \end{matrix}$, eigenvectors $X_1,\ X_2$

257. $A$: eigenvalues $\begin{matrix} \lambda_1 = 2i, \\ \lambda_2 = -2i, \end{matrix}$, eigenvectors $X_1 = \begin{pmatrix} -p \\ pi \end{pmatrix},\ X_2 = \begin{pmatrix} -q \\ -qi \end{pmatrix},\ p, q \in \mathbf{C},\ p, q \neq 0$

$A^2 = \begin{pmatrix} -4, 0 \\ 0, -4 \end{pmatrix}$, vl. číslo $\eta = \lambda_1^2 = \lambda_2^2 = -4$, eigenvectors $X_1,\ X_2$

258. $A$: eigenvalue $\lambda = -1$, eigenvectors $X = \begin{pmatrix} -p \\ -p \\ p \end{pmatrix},\ p \in \mathbf{C},\ p \neq 0$

$A^2 = \begin{pmatrix} -3, 1, -3 \\ -8, 4, -5 \\ 0, 1, 2 \end{pmatrix}$, eigenvalue $\eta = \lambda^2 = 1$, eigenvectors $X$

### I.5. Systems of linear algebraic equations

273. yes 1 275. yes, $\infty$ 276. yes, 1

278. $a \neq 1, -2$ ... 1 solution, $a = 1$ ... $\infty$ solution, $a = -2$ ... 0 solution

280. $a \neq 0$ ... 1 solution, $a = 0$ ... 0 solution, 283. $\lambda \neq 5$ ... 0 solution, $\lambda = 5$ ... $\infty$ solution

287. no 288. yes, $x = 2, y = 3, z = 5$ 295. no

300. 1 for $\lambda = 1$, 0 for $\lambda \neq 1$ 301. 1 for $\lambda = 2$, 0 for $\lambda \neq 2$ 308. $x = 0, y = 0, z = 0$

309. $x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0$

310. $x_1 = -11p,\ x_2 = -p,\ x_3 = 7p,\ p \in \mathbb{R}$

316. $x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0$

317. $x_1 = 5p + q,\ x_2 = q,\ x_3 = 2p,\ x_4 = 6p,\ p, q \in \mathbb{R}$

324. solution does not exist

328. $x = 1,\ y = -1,\ z = 2$

332. $x_1 = p - 1,\ x_2 = 3p - 1,\ x_3 = 7p,\ p \in \mathbb{R}$

337. $x = 0,\ y = 2,\ z = \frac{5}{3},\ v = -\frac{4}{3}$

338. $x_1 = -8,\ x_2 = 6,\ x_3 = 12,\ x_4 = 3$

348. solution does not exist for $a = 1$, $x = \dfrac{3}{2(a-1)},\quad y = \dfrac{1}{2},\quad z = \dfrac{4a-7}{2(a-1)}$ for $a \neq 1$

349. solution does not exist for $a = 1$, $x = -6p + 4,\ y = -p + \frac{3}{2},\ z = 2p,\ p \in \mathbb{R}$, for $a = 2$
$x = y = z = 1/(a-1)$ for $a \neq 1,\ a \neq 2$

350. solution does not exist for $\alpha = 1$, $x = p + \frac{4}{3},\ y = p - \frac{1}{3},\ z = p,\ p \in \mathbb{R}$ for $\alpha = -2$
$x = (1-2\alpha)/(1-\alpha),\ y = 0,\ z = 1/(1-\alpha)$ for $\alpha \neq 1,\ \alpha \neq -2$

359. solution does not exist for $a = \frac{2}{5}$, a unique solution for $a \neq \frac{2}{5}$,
$x = \frac{1}{3},\ y = -\frac{1}{3},\ z = \frac{1}{3}$ for $a = 1$

360. solution does not exist for $k \neq 5$,
for $k = 5$ there is infinitely many solutions: $x = -p + 1,\ y = -7p + 2,\ z = 5p,\ p \in \mathbb{R}$

369. $\lambda = 0\ \ldots \begin{pmatrix} -16p \\ -7p \\ 4p \end{pmatrix},\quad \lambda = 1\ \ldots \begin{pmatrix} -3q \\ -q \\ q \end{pmatrix},\quad \begin{matrix} p \neq 0 \\ q \neq 0 \end{matrix}$

370. $x = \frac{45}{60},\ y = -\frac{10}{60},\ z - \frac{40}{60}$

**III.1. Sequences of real numbers**

| | incr. | decr. | non –incr. | non –decr. | mon. | strictly mon. | lower bound. | upper bound. | bound. | unbound. |
|---|---|---|---|---|---|---|---|---|---|---|
| 575. | + | − | − | + | + | + | + | − | − | + |
| 577. | + | − | − | + | + | + | + | + | + | − |
| 578. | − | − | − | − | − | − | + | + | + | − |
| 579. | − | − | − | − | − | − | + | + | + | − |
| 580. | − | + | + | − | + | + | − | + | − | + |
| 581. | − | + | + | − | + | + | + | + | + | − |

591. $\mathbf{e}^3$ 594. $\frac{2}{3}$ 596. 0 599. $+\infty$ 609. $-\infty$ 612. $-\infty$ 616. $-1$
619. 0 620. 0 621. $-\infty$ 622. 1 623. 0 629. $\frac{1}{6}$

**III.2. Functions – basic notions and properties**

657. $(-\infty, 0) \cup (4, +\infty)$ 660. $(-3, \frac{5}{2}\rangle$ 661. $\langle -\frac{3}{2}, \frac{5}{2}\rangle$ 664. $(-\infty, -1) \cup (1, +\infty)$

667. $\langle 0, \frac{3}{2}) \cup (\frac{3}{2}, 2) \cup (2, +\infty)$ 668. $(-\infty, (3-\sqrt{5})/2\rangle \cup \langle (3+\sqrt{5})/2, +\infty)$

674. $g(x) = (\sin x)^2,\ h(x) = \sin x^2$ 675. $g(x) = \ln(5x^2+3),\ h(x) = 5\ \ln^2(x+1) + 2$

678. $g(x) = \sin^2(2x+1) + 5\ \sin(2x+1),\ h(x) = \sin(2x^2 + 10x + 1)$

679. $g(x) = \cos(x+3),\ h(x) = \cos, x + 2$

687. odd 694. odd 695. even 698. yes, $2\pi$ 704. no 707. yes, $2\pi$

| | supremum | infimum | maximum | minimum | upper bound. | lower bound. | bound. |
|---|---|---|---|---|---|---|---|
| 709. | 1 | −1 | 1 | −1 | yes | yes | yes |
| 718. | $\pi/2$ | $-\pi/2$ | $\pi/2$ | $-\pi/2$ | yes | yes | yes |
| 723. | $+\infty$ | 0 | does not exist | 0 | no | yes | no |
| 727. | $+\infty$ | 2 | does not exist | 2 | no | yes | no |
| 733. | $+\infty$ | $-\infty$ | does not exist | does not exist | no | no | no |
| 734. | $+\infty$ | $-\infty$ | does not exist | does not exist | no | no | no |

## III.3. Limit and continuity of a function

767. $L = 0$, e.g. $a = 100$ 768. $L = 0$, e.g. $a = \ln 10$ 791. $-\frac{1}{4}$
792. $-\frac{3}{5}$ 796. $-4$ 797. $0$ 807. $\frac{1}{2}$ 816. $0$ 837. $\frac{5}{3}$
839. $1$ 840. $1$ 859. $3$ 863. $0$ 864. $\pi/2$ 865. $\pi/2$
866. $+\infty$ 869. $0$ 882. $1$ 887. $\mathbf{e}^6$ 891. $-\frac{1}{2}$ 902. $+\infty$
904. $0$ 909. $-\infty$ 921. the left limit $(-\infty) \neq$ the right limit $(+\infty)$
924. If we choose e.g. $x_n = \pi/2 + \pi n$ then $x_n \to +\infty$, but $\lim_{n\to+\infty} \sin x_n$ does not exist, because $\sin x_n = (-1)^n$.
926. the left limit $(-\infty) \neq$ the right limit $(+\infty)$
929. $(-\infty, -1),\ (-1, +\infty)$ 931. $(-\infty, -1),\ (-1, +\infty)$ 932. $(-\infty, 1),\ (1, 2),\ (2, +\infty)$
937. $(-\infty, 0),\ (0, +\infty)$ 939. $(0, 1),\ (1, +\infty)$ 943. $(-\infty, 0),\ (0, +\infty)$

## III.4. Derivative of a function, its geometrical and physical sense

960. $10x + 7,\ x \in (-\infty, +\infty)$ 968. $\dfrac{4x-1}{2\sqrt{2x^2 - x + 5}},\ x \in (-\infty, +\infty)$
972. $\sqrt{x-1} + \dfrac{x+6}{2\sqrt{x-1}},\ x \in (1, +\infty)$ 976. $\dfrac{x^2 + 10x - 3}{(x+5)^2},\ x \in (-\infty, -5) \cup (-5, +\infty)$
979. $\dfrac{1}{\sqrt{5-x}} + \dfrac{x+2}{2\,(5-x)^{3/2}},\ x \in (-\infty, 5)$
980. $\dfrac{x}{(x+1)\sqrt{x^2+1}} - \dfrac{\sqrt{x^2+1}}{(x+1)^2},\ x \in (-\infty, -1) \cup (-1, +\infty)$
990. $3\cos(3x),\ x \in (-\infty, +\infty)$ 991. $-\sin x^2 \cdot 2x,\ x \in (-\infty, +\infty)$
993. $12\sin(6x)\cos(6x),\ x \in (-\infty, +\infty)$
999. $\cos(x^2 + 2x + 2) \cdot (2x + 2),\ x \in (-\infty, +\infty)$
1001. $2x \cdot \operatorname{tg} x + \dfrac{x^2}{\cos^2 x},\ x \in (-\pi/2 + k\pi, +\pi/2 + k\pi)$, $k$ integer
1003. $\dfrac{1 + \cos x}{2\sqrt{1 + x + \sin x}},\ x \in (x_0, +\infty)$, where $x_0$ is a solution of the equation $1 + x + \sin x = 0$
1008. $\dfrac{1}{2\sqrt{-x(x+1)}},\ x \in (-1, 0)$ 1009. $-\dfrac{1}{2\sqrt{x}\,(x+1)},\ x \in (0, +\infty)$
1016. $\dfrac{1}{2\sqrt{x}\,(1+x)},\ x \in (0, +\infty)$ 1017. $2\,\mathbf{e}^{2x},\ x \in (-\infty, +\infty)$
1018. $(10x - 2) \cdot \mathbf{e}^{5x^2 - 2x + 1},\ x \in (-\infty, +\infty)$ 1020. $\frac{1}{2}\,\mathbf{e}^{x/2},\ x \in (-\infty, +\infty)$
1021. $10\,(\mathbf{e}^{5x} + 1)\,\mathbf{e}^{5x},\ x \in (-\infty, +\infty)$ 1027. $\dfrac{2x+3}{x^2 + 3x + 4},\ x \in (-\infty, +\infty)$
1028. $\dfrac{1}{\sqrt{1+x^2}},\ x \in (-\infty, +\infty)$ 1047. $\dfrac{7\sqrt{49x^2+1} + 49x}{(7x + \sqrt{49x^2+1})\sqrt{49x^2+1}},\ x \in (-\infty, +\infty)$
1049. $2\,\mathrm{e}^{2x} \cdot (x^2+1)^2 + 4x\,\mathrm{e}^{2x} \cdot (x^2+1),\ x \in (-\infty, +\infty)$
1051. $-\dfrac{2\operatorname{sgn} x}{1+x^2},\ x \in (-\infty, 0) \cup (0, +\infty)$
1055. $f'(x) = \operatorname{sgn} x,\ x \neq 0$ 1056. $f'(x) = 1/x,\ x \neq 0$
1057. $f'(x) = 2x \cdot \operatorname{sgn} x,\ x \in (-\infty, +\infty)$ 1058. $f'_+(0) = 1,\ \ f'_-(0) = -1$
1062. $f'_+(4) = 4,\ \ f'_-(4) = -4$ 1066. $(1 + x^2)^{-3/2},\ x \in (-\infty, +\infty)$
1068. $\dfrac{2\cos x}{\sin^3 x},\ x \neq k\pi$, $k$ integer 1071. $-4\,(x-1)^{-3},\ x \neq 1$
1109. $y - 1 = 0,\ \ x = 0$ 1110. $y = -\pi \cdot (x - \pi),\ \ y = (x - \pi)/\pi$ 1118. $[-2, -4]$ 1119. $[\,\frac{1}{2},\ \frac{17}{4}\,]$
1125. tangent line exists for $x > 0$, parallel with the $x$–axis at the point $[\sqrt{2},\ f(\sqrt{2})]$,
tangent line at the point $[\sqrt{5},\ f(\sqrt{5})]$ is $y - \ln(3/\sqrt{5}) = -(x - \sqrt{5})/(6\sqrt{5})$

**III.5. Application of derivative, behaviour of a function**

1144. $f$ increasing on $(-\infty, -3\rangle$ and on $\langle 2, +\infty)$, decreasing on $\langle -3, 2\rangle$
1145. $f$ increasing on $(-\infty, -\sqrt{3}\rangle$ and on $\langle \sqrt{3}, +\infty)$, decreasing on $\langle -\sqrt{3}, -1)$, $(-1, 1)$, $(1, \sqrt{3}\rangle$
1146. $f$ increasing on $(-\infty, -2\rangle$ and on $\langle 0, +\infty)$, decreasing on $\langle -2, 0\rangle$
1148. $f$ increasing on $\langle -1, 1\rangle$, decreasing on $(-\infty, -1\rangle$ and on $\langle 1, +\infty)$
1151. $f$ decreasing on $(-\infty, -1\rangle$ and on $\langle 1, +\infty)$, increasing on $\langle -1, 1\rangle$
1156. $f$ increasing on $\langle -1, 0)$ and on $\langle 1, +\infty)$, decreasing on $(-\infty, -1\rangle$ and on $(0, 1\rangle$
1160. $D(f) = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to-1-} f(x) = +\infty$
$\lim_{x\to-1+} f(x) = -\infty$, $\lim_{x\to1-} f(x) = -\infty$, $\lim_{x\to1+} f(x) = +\infty$, $\lim_{x\to+\infty} f(x) = +\infty$
$f$ increasing on $(-\infty, -1)$, $(-1, 1-\sqrt{2}\rangle$ and on $\langle 1+\sqrt{2}, +\infty)$,
decreasing on $\langle 1-\sqrt{2}, 1)$ and on $(1, 1+\sqrt{2}\rangle$
1161. $D(f) = (0, +\infty)$, $\lim_{x\to0+} f(x) = 0$, $\lim_{x\to+\infty} f(x) = 1$
$f$ increasing on $(0, \mathrm{e}\rangle$, decreasing on $\langle \mathrm{e}, +\infty)$
1164. $\max_I f = f(-2) = f(2) = 13$, $\min_I f = f(-1) = f(1) = 4$
1169. $\max_I f = f(-1) = 3$, $\min_I f = f(1) = 1$
1174. $\max_I f = f(4) = 4$, $\min_I f = f(2) = -4$
1175. $\max_I f = f(2) = 1$, $\min_I f = f(1) = -1$
1177. $\max_I f = f(1) = 4$, $\min_I f = f(0) = 0$
1179. $\max_I f$ does not exist, $\min_I f = f(6) = -\frac{2}{3} \ln 6$
1204. absolute minimum $y = 2 - 2 \ln 2$ at the point $x = 2$
1209. absolute maximum $y = 1/\mathrm{e}$ at the point $x = \mathrm{e}$
1211. local minimum $y = 2$ at the point $x = 1$, local maximum $y = -2$ at the point $x = -1$
1212. local minimum $y = -2$ at the point $x = 1$, local maximum $y = 2$ at the point $x = -1$
1217. absolute minimum $y = \frac{1}{3}$ at the point $x = -1$, absolute maximum $y = 3$ at the point $x = 1$
1218. absolute minimum $y = 1$ at the point $x = -1$
1241. $D(f) = (-\infty, +\infty)$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to+\infty} f(x) = +\infty$, $f$ has no local extremes
1255. concave down on $(-\infty, -2\rangle$ and on $\langle 0, 2\rangle$, concave up on $\langle -2, 0\rangle$ and on $\langle 2, +\infty)$, points of inflection $-2$, $0$, $2$
1256. concave down on $(-\infty, -\sqrt{3}\rangle$ and on $\langle 0, \sqrt{3}\rangle$, concave up on $\langle -\sqrt{3}, 0\rangle$ and on $\langle \sqrt{3}, +\infty)$, points of inflection $-\sqrt{3}$, $0$, $\sqrt{3}$
1259. concave up on $(-\infty, +\infty)$
1260. concave up on $\langle -1, 1\rangle$, concave down on $(-\infty, -1\rangle$ and on $\langle 1, +\infty)$, points of inflection $\pm 1$
1262. concave up on $(-\infty, -2)$, concave down on $(1, +\infty)$
1264. concave up on $(-\infty, +\infty)$
1267. slanted asymptote $y = -x$ (for $x \to -\infty$ and for $x \to +\infty$), vertical asymptotes $x = -2$, $x = 2$
1268. slanted asymptote $y = 2x$ (for $x \to -\infty$ and for $x \to +\infty$), vertical asymptote $x = 2$
1270. slanted asymptotes $y = 2x - \pi/2$ (for $x \to -\infty$) a $y = 2x + \pi/2$ (for $x \to +\infty$)
1272. slanted asymptote $y = -2$ (for $x \to -\infty$ and for $x \to +\infty$), vertical asymptote $x = -2$
1274. slanted asymptote $y = x$ (for $x \to -\infty$ and for $x \to +\infty$), vertical asymptote $x = 0$
1275. slanted asymptote $y = x$ (for $x \to +\infty$), vertical asymptote $x = 0$
1277. $D(f) = (-\infty, -2) \cup (-2, 2) \cup (2, +\infty)$, $f$ continuous on $(-\infty, -2)$, $(-2, 2)$ and on $(2, +\infty)$, $f$ even,
$\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to-2-} f(x) = -\infty$, $\lim_{x\to-2+} f(x) = +\infty$,
$\lim_{x\to+2-} f(x) = +\infty$, $\lim_{x\to+2+} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = 0$,

$f'(x) = \dfrac{2x}{(4-x^2)^2}$ for $x \in D(f)$,
$f$ decreasing on $(-\infty,-2)$ and on $(-2,0\rangle$, increasing on $\langle 0,2)$ and on $(2,+\infty)$,
$f$ has local minimum $y_0 = \frac{1}{4}$ at the point $x_0 = 0$,
$f''(x) = \dfrac{8+6x^2}{(4-x^2)^3}$ for $x \in D(f)$,
$f$ concave down on $(-\infty,-2)$ and on $(2,+\infty)$, concave up on $-2,2)$,
asymptote $y=0$ for $x \to -\infty$ and for $x \to +\infty$ and vertical asymptotes $x=-2$ a $x=2$.

1278. $D(f) = (-\infty,+\infty)$, $f$ continuous on $(-\infty,+\infty)$,
$\lim_{x\to-\infty} f(x) = +\infty$, $\lim_{x\to+\infty} f(x) = -\infty$,
$f'(x) = \dfrac{3}{2\sqrt[3]{x}} - 1$ for $x \in (-\infty,0) \cup (0,+\infty)$,
$f$ decreasing on $(-\infty,0\rangle$ and on $\langle \frac{8}{27},+\infty)$, increasing on $\langle 0,\frac{8}{27}\rangle$,
$f$ has local minimum $y=0$ at the point $x=0$ a local maximum $y=\frac{4}{27}$ at the point $x=\frac{8}{27}$,
$f''(x) = -\dfrac{2}{9\sqrt[3]{x^4}}$ for $x \in (-\infty,0) \cup (0,+\infty)$,
$f$ concave down on $(-\infty,0\rangle$ and on $\langle 0,+\infty)$, $f$ has no asymptotes

1279. $D(f) = (-\infty,+\infty)$, $f$ continuous on $(-\infty,+\infty)$, $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$,
$f'(x) = -2x\,\mathrm{e}^{-x^2}$ for $x \in D(f)$, $f$ increasing on $(-\infty,0\rangle$ and decreasing on $\langle 0,+\infty)$,
$f$ has absolute maximum $y=1$ at the point $x=0$,
$f''(x) = (-2+4x^2)\,\mathrm{e}^{-x^2}$ for $x \in D(f)$,
$f$ concave up on $(-\infty,-\sqrt{2}/2\rangle$ and on $\langle\sqrt{2}/2,+\infty)$, concave down on $\langle -\sqrt{2}/2,\sqrt{2}/2\rangle$,
asymptote $y=0$ for $x \to -\infty$ and for $x \to +\infty$

1281. $D(f) = (-\infty,+\infty)$, $f$ continuous on $(-\infty,+\infty)$, sudá,
$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = -\infty$, $f'(x) = 2x - 2x^3$ for $x \in D(f)$,
$f$ increasing on $(-\infty,-1\rangle$ and on $\langle 0,1\rangle$, decreasing on $\langle -1,0\rangle$ and on $\langle 1,+\infty)$,
absolute maximum $y=1.5$ at the points $x=\pm 1$, local minimum $y=1$ at the point $x=0$,
the graph intersects the $x$–axis at the points $\pm\sqrt{1+\sqrt{3}}$,
$f''(x) = 2-6x^2$ for $x \in D(f)$, $f$ concave down on $(-\infty,-\sqrt{3}/3\rangle$ and on $\langle\sqrt{3}/3,+\infty)$,
concave up on $\langle -\sqrt{3}/3,\sqrt{3}/3\rangle$, $f$ has no asymptotes

1282. $D(f) = (-\infty,+\infty)$, $f$ continuous on $(-\infty,+\infty)$, sudá,
$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = +\infty$, $f'(x) = 4x^3 - 4x$ for $x \in D(f)$,
$f$ decreasing on $(-\infty,-1\rangle$ and on $\langle 0,1\rangle$, increasing on $\langle -1,0\rangle$ and on $\langle 1,+\infty)$,
absolute minimum $y=-1$ at the points $x=\pm 1$, local maximum $y=0$ at the point $x=0$,
the graph intersects the $x$–axis at the points $\pm\sqrt{2}$ and touches it at the point $x=0$,
$f''(x) = 12x^2-4$ for $x \in D(f)$, $f$ concave up on $(-\infty,-\sqrt{3}/3\rangle$ and on $\langle\sqrt{3}/3,+\infty)$,
concave down on $\langle -\sqrt{3}/3,\sqrt{3}/3\rangle$, $f$ has no asymptotes

1292. $D(f) = (-\infty,0) \cup (0,+\infty)$, $f$ continuous on $(-\infty,0)$ and on $(0,+\infty)$,
$\lim\limits_{x\to-\infty} f(x) = -\infty$, $\lim\limits_{x\to 0-} f(x) = 0$, $\lim\limits_{x\to 0+} f(x) = +\infty$, $\lim\limits_{x\to+\infty} f(x) = +\infty$,
$f'(x) = \mathrm{e}^{1/x}\,\dfrac{(x-2)(x+1)}{x^2}$ for $x \in (-\infty,0) \cup (0,+\infty)$,
$f$ increasing on $(-\infty,-1\rangle$ and on $\langle 2,+\infty)$, decreasing on $\langle -1,0)$ and on $(0,2\rangle$,
$f$ has local maximum $y=\mathrm{e}^{-1}$ at the point $x=-1$, local minimum $y=4\sqrt{\mathrm{e}}$ at the point $x=2$
$f''(x) = \mathrm{e}^{1/x}\,\dfrac{5x+2}{x^4}$ for $x \in (-\infty,0) \cup (0,+\infty)$,
$f$ concave down on $(-\infty,-\frac{2}{5}\rangle$, concave up on $\langle -\frac{2}{5},0)$ and on $(0,+\infty)$,
vertical asymptote $x=0$, slanted asymptote $y=x+3$ for $x \to -\infty$ and for $x \to +\infty$

1293. $D(f) = \langle 0,+\infty)$, $f$ continuous on $\langle 0,+\infty)$, $f(0)=0$, $\lim_{x\to+\infty} f(x) = +\infty$,

$f'(x) = \dfrac{3\,(x-1)}{2\sqrt{x}}$ for $x \in (0, +\infty)$, $f$ increasing on $\langle 1, +\infty)$, decreasing on $\langle 0, 1\rangle$,
absolute minimum $y = -2$ at the point $x = 1$,
$f''(x) = \dfrac{3\,(x+1)}{4x\,\sqrt{x}}$ for $x \in (0, +\infty)$, $f$ concave up on $\langle 0, +\infty)$, $f$ has no asymptotes

1295. $D(f) = (-\infty, +\infty)$, $f$ continuous on $(-\infty, +\infty)$, $\lim_{x\to-\infty} f(x) = -1$, $\lim_{x\to+\infty} f(x) = 1$,
$f'(x) = \dfrac{1+2x}{(x^2+1)\sqrt{x^2+1}}$ for $x \in (-\infty, +\infty)$,
$f$ increasing on $\langle -0.5, +\infty)$, decreasing on $(-\infty, -0.5\rangle$,
absolute minimum $y = -5/\sqrt{5}$ at the point $x = -0.5$,
$f''(x) = -\dfrac{4x^2+3x-2}{(x^2+1)^{5/2}}$ for $x \in (-\infty, +\infty)$,
$f$ concave down on $(-\infty, -(3+\sqrt{41})/8\rangle$ and on $\langle(-3+\sqrt{41})/8, +\infty)$,
concave up on $\langle -(3+\sqrt{41})/8, (-3+\sqrt{41})/8\rangle$,
asymptote $y = -1$ for $x \to -\infty$ and $y = 1$ for $x \to +\infty$

1308. $D(f) = (-\infty, +\infty)$, $f$ continuous on $(-\infty, +\infty)$,
$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$,
$f'(x) = \mathrm{e}^{2x-x^2}\, 2\,(1-x)$ for $x \in (-\infty, +\infty)$,
$f$ increasing on $(-\infty, 1\rangle$, decreasing on $\langle 1, +\infty)$,
absolute maximum $y = \mathrm{e}$ at the point $x = 1$,
$f''(x) = 2\,\mathrm{e}^{2x-x^2}\,(2x^2-4x+1)$ for $x \in (-\infty, +\infty)$,
$f$ concave down on $\langle 1-\sqrt{2}/2, 1+\sqrt{2}/2\rangle$, concave up on $(-\infty, 1-\sqrt{2}/2\rangle$, $\langle 1+\sqrt{2}/2, +\infty)$,
asymptote $y = 0$ for $x \to -\infty$ and for $x \to +\infty$

1317. $D(f) = (-\infty, +\infty)$, $f$ continuous on $(-\infty, +\infty)$, $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = \pi$,
$f'(x) = \dfrac{2\,\mathrm{sgn}\,x}{(1+x^2)^2}$ for $x \in (-\infty, 0) \cup (0, +\infty)$,
$f$ decreasing on $(-\infty, 0\rangle$, increasing on $\langle 0, +\infty)$,
absolute minimum $y = 0$ at the point $x = 0$,
$f''(x) = -\dfrac{4x\,\mathrm{sgn}\,x}{(1+x^2)^2}$ for $x \in (-\infty, 0) \cup (0, +\infty)$,
$f$ concave down on $(-\infty, 0\rangle$, concave up on $\langle 0, +\infty)$,
asymptote $y = \pi$ for $x \to -\infty$ and for $x \to +\infty$

1319. $D(f) = (-2, 2)$, $f$ continuous on $(-1, 1)$, sudá, $\lim\limits_{x\to-2+} f(x) = \lim\limits_{x\to+2-} f(x) = -\infty$,
$f'(x) = -\dfrac{2x}{4-x^2}$ for $x \in (-2, 2)$,
$f$ increasing on $(-2, 0\rangle)$, decreasing on $\langle 0, 2)$,
absolute maximum $y = \ln 4$ at the point $x = 0$,
$f''(x) = -\dfrac{8}{(4-x^2)^2}$ for $x \in (-2, 2)$,
$f$ is concave down on $(-2, 2)$, $f$ has vertical asymptotes $x = -2$ a $x = 2$

1321. $D(f) = (-\infty, +\infty)$, $f$ continuous on $(-\infty, +\infty)$, lichá, $\lim\limits_{x\to-\infty} f(x) = -\infty$,
$\lim\limits_{x\to+\infty} f(x) = +\infty$, $f'(x) = \dfrac{x^2}{1+x^2}$ for $x \in (-\infty, +\infty)$, $f$ increasing on $(-\infty, +\infty)$,
$f''(x) = \dfrac{2x}{(1+x^2)^2}$ for $x \in (-\infty, +\infty)$,
$f$ concave down on $(-\infty, 0\rangle$, concave up on $\langle 0, +\infty)$,
asymptote $y = x + \pi/2$ for $x \to -\infty$ a $y = x$ for $x \to +\infty$

## III.6. Taylor's theorem

1330. $T_5(x) = 1 + \dfrac{x}{1\,!} + \ldots + \dfrac{x^5}{5\,!}$, $R_6(x) = \dfrac{\mathrm{e}^{\xi} x^6}{6\,!}$

1331. $T_5(x) = \mathrm{e} + \dfrac{\mathrm{e}}{1\,!}\,(x-1) + \ldots + \dfrac{\mathrm{e}}{5\,!}\,(x-1)^5, \quad R_6(x) = \dfrac{\mathrm{e}^{\xi}}{6\,!}\,(x-1)^6$

1333. $T_4(x) = 1 + 3x + \dfrac{(3x)^2}{2\,!} + \dfrac{(3x)^3}{3\,!} + \dfrac{(3x)^4}{4\,!}, \quad R_5 x = \dfrac{(\ln\,2)^5}{5\,!}\,2^{\xi}\,x^5$

1337. $T_7(x) = x - \dfrac{x^3}{3\,!} + \dfrac{x^5}{5\,!} - \dfrac{x^7}{7\,!}, \quad R_8(x) = \dfrac{\sin\,\xi}{8\,!}\,x^8$

1346. $T_7(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dfrac{x^6}{6} + \dfrac{x^7}{7}, \quad R_8(x) = -\dfrac{x^8}{8}\,(\xi+1)^8$

1351. $T_4(x) = 1 + \frac{1}{2}\,(x-1) - \frac{1}{8}\,(x-1)^2 + \frac{1}{16}\,(x-1)^3 - \frac{5}{128}\,(x-1)^4, \quad R_5(x) = \frac{7}{256}\,\xi^{-9/2}(x-1)^5$

1352. $T_3(x) = \sqrt{3} + \dfrac{x}{2\sqrt{3}} - \dfrac{x^2}{24\sqrt{3}} + \dfrac{x^3}{144\sqrt{3}}, \quad R_4(x) = -\dfrac{5\,x^4}{128\,(\xi+3)^{7/2}}$

1354. $T_4(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4, \quad R_5(x) = -(x-1)^5/\xi^6$

1362. $\dfrac{1}{\mathrm{e}} \doteq \left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!}\right)\Big|_{x=-1} = \dfrac{265}{720} = 0.3681$

1363. $\cos\,5^{\mathrm{o}} \doteq \left(1 - \dfrac{x^2}{2!}\right)\Big|_{x=\pi/36} = 1 - \dfrac{\pi^2}{2592} = 0.9961923$

1365. $\ln\,1.2 \;=\; \ln\,(1+x)\,|_{x=0.2} \doteq \left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3}\right)\Big|_{x=0.2} \;=\; 0.1826$

1376. $T_2(x) = 1 + \frac{1}{3}\,x - \frac{1}{9}\,x^2, \quad |f(\frac{1}{2}) - T_2(\frac{1}{2})| \le \frac{5}{81}$

### IV.1. Table of basic integrals, fundamental properties of indefinite integrals

1448. $\frac{3}{8}x^8 + C,\; x \in (-\infty, +\infty)$ 1450. $27x - 9x^3 + \frac{9}{5}x^5 - \frac{1}{7}x^7 + C$ 1452. $\frac{3}{4}x\sqrt[3]{x} + C$

1454. $-x^{-1} + C$ 1455. $\sqrt{x} + C,\; x \in \langle 0, +\infty)$ 1458. $u - u^2 + C$ 1459. $\frac{2}{5}x^2\sqrt{x} + x + C$

1460. $-10x^{-0{,}2} + 15x^{0{,}2} - 3{,}62x^{1{,}38} + C, \;\; x \in (0, +\infty)$ 1461. $x - 2\ln|x| - x^{-1} + C$

1464. $\frac{6}{7}\,\sqrt[6]{x^7} - \frac{4}{3}\,\sqrt[4]{x^3} + C,\; x \in \langle 0, +\infty)$ 1467. $3\ln|x| - \dfrac{9}{x} - \dfrac{27}{2x^2} + C$ 1468. $\dfrac{10^x}{\ln 10} + C$

1470. $3x - 2\dfrac{(1,5)^x}{\ln\,1,5} + C$ 1473. $0.5\,(\mathrm{tg}\,x + x) + C\,,\;\, x \in ((2k-1)\dfrac{\pi}{2}, (2k+1)\dfrac{\pi}{2}),\;\, k$ integer

1474. $C - \mathrm{cotg}\,x - \mathrm{tg}\,x$ 1475. $x - \sin x + C$

### IV.2. Integration by parts

1481. $e^x(x-1) + C$ 1482. $\frac{1}{4}x^2(2\ln x - 1) + C,\; x \in (0, +\infty)$ 1483. $\sin x - x\cos x + C$

1484. $\frac{1}{2}\,[(x^2+1)\mathrm{arctg}\,x - x] + C$ 1485. $x\sin x + \cos x + C$ 1486. $e^x(x^2 - 2x + 2) + C$

1489. $\dfrac{x^{n+1}}{n+1}(\ln x - \dfrac{1}{n+1}) + C, \;\; x \in (0, +\infty)$ 1494. $x\,\mathrm{arctg}\,x - \ln\sqrt{1+x^2} + C$

1502. $t\,\mathrm{arcsin}^2 t + 2\sqrt{1-t^2}\mathrm{arcsin}\,t - 2t + C,\; t \in \langle -1, 1\rangle$ 1504. $\frac{1}{2}\,(e^x\sin x - e^x\cos x) + C$

1506. $\dfrac{e^{7x}(7\cos 5x + 5\sin 5x)}{74} + C,$ choose $u = e^{7x}$ 1510. $e^x(x^2 - 5x + 7) + C$

### IV.3. Integration by substitution

1514. $\ln\left|\dfrac{1}{1-x}\right| + C$ 1515. $\dfrac{\ln(2+e^{2x})}{2} + C$ 1516. $\ln|\sin x| + C\,,\; x \in (k\pi, (k+1)\pi)\,,\; k$ integer

1518. $2\,\sqrt{1+x^2} + C$ 1519. $\dfrac{(x+1)^{16}}{16} + C$ 1532. $3\sqrt[3]{\sin x} + C$ 1533. $C - \frac{2}{5}\cos^5 x$

1534. $\frac{2}{3}\sqrt{\ln^3 x} + C,\; x \in \langle 1, +\infty)$ 1542. $C - \frac{1}{2}\sin(1-2x)$ 1546. $\ln(x^2 - 3x + 8) + C$

1555. $C - \frac{1}{3}e^{-x^3}$ 1572. $\frac{1}{2}\left[x - \frac{1}{2}\ln|2x+1|\right] + C,\; x \in (-\infty, -\frac{1}{2})\,, x \in (-\frac{1}{2}, +\infty)$

1579. $C-\frac{1}{4}x^4-\frac{1}{3}x^3-\frac{1}{2}x^2-x-\ln|1-x|\,,\ x\in(-\infty,1)\,,\ x\in(1,+\infty)$ 1580. $\dfrac{x^3}{3}-x+\operatorname{arctg}x+C$

1594. $\frac{6}{5}\,[\sqrt[6]{x^5}+2\,\sqrt[12]{x^5}+2\ln|\,\sqrt[12]{x^5}-1|\,]+C\,,\ x\in(0,1)\,,\ x\in(1,+\infty)$

1615. $\dfrac{x\arcsin x}{\sqrt{1-x^2}}+\dfrac{1}{2}\ln|1-x^2|+C\,,\ x\in(-1,1)$ 1620. $e^{-\cos x}+C$

1625. $\frac{2}{3}\ln(1+x^{\frac{3}{2}})+C\,,\ x\in\langle 0,+\infty)$ 1628. $2\sqrt{1+x^2}+3\ln(x+\sqrt{1+x^2})+C$

1666. $\operatorname{tg}x\cdot\ln(\cos x)+\operatorname{tg}x-x+C\,,\ x\in(-\dfrac{\pi}{2}+2k\pi,\dfrac{\pi}{2}+2k\pi)\,,$ $k$ integer

1688. $C-\frac{1}{2}e^{-x^2}(x^4+2x^2+2)$

## IV.4. Integration of rational functions

1720. $\frac{8}{3}\ln|3x-1|+C\,,\ x\in(-\infty,\frac{1}{3})\,,\ x\in(\frac{1}{3},+\infty)$ 1722. $\dfrac{x^3}{3}+2x+\sqrt{2}\ln\left|\dfrac{x-\sqrt{2}}{x+\sqrt{2}}\right|+C$

1724. $-\dfrac{1}{2(x-3)^2}+C\,,\ x\in(-\infty,3)\,,\ x\in(3,+\infty)$ 1731. $\ln\dfrac{(u+1)^2}{|u|}+C$

1733. $\dfrac{13\ln|x-6|}{2}-\dfrac{3\ln|x-2|}{2}+C$

1734. $x+\dfrac{16\ln|x-4|}{3}-\dfrac{\ln|x-1|}{3}+C\,,\ x\in(-\infty,1)\,,\ x\in(1,4)\,,\ x\in(4,+\infty)$

1739. $\dfrac{x^2}{2}-3x+\ln\dfrac{(x+2)^8}{|x+1|}+C,\ x\in(-\infty,-2)\,,\ x\in(-2,-1)\,,\ x\in(-1,+\infty)$

1748. $\dfrac{1}{8}\ln\dfrac{(x+3)^6}{|(x+5)^5(x+1)|}+C\,,\ x\in(-\infty,-5)\ x\in(-5,-3)\,,\ x\in(-3,-1)\,,\ x\in(-1,+\infty)$

1749. $\dfrac{3}{x-2}+\ln\dfrac{(x-2)^2}{x^2}+C$

1751. $3\operatorname{arctg}x+\ln(x^2+1)-2\ln|x|+C\,,\ x\in(-\infty,0)\,,\ x\in(0,+\infty)$

1754. $\ln\dfrac{|x|}{\sqrt{x^2+1}}+C\,,\ x\in(-\infty,0)\,,\ x\in(0,+\infty)$ 1755. $\dfrac{2\sqrt{3}}{3}\operatorname{arctg}\dfrac{\sqrt{3}}{3}(2x-1)+C$

1761. $\dfrac{3}{2}\ln(x^2-x+1)+\dfrac{\sqrt{3}}{3}\operatorname{arctg}\dfrac{\sqrt{3}}{3}(2x-1)+C$

1793. $\dfrac{4}{x+2}+\ln|x+1|+C\,,\ x\in(-\infty,-2)\,,\ x\in(-2,-1)\,,\ x\in(-1,+\infty)$

1798. $\ln\dfrac{\sqrt{(x^2-2x+5)^3}}{|x-1|}+\dfrac{1}{2}\operatorname{arctg}\dfrac{x-1}{2}+C\,,\ x\in(-\infty,1)\,,\ x\in(1,+\infty)$

## IV.5. Integration of trigonometric functions and their powers

1814. $\dfrac{\cos^7 x}{7}-\dfrac{3\cos^5 x}{5}+\cos^3 x-\cos x+C$ 1815. $\dfrac{\cos^3 x}{3}-\cos x+C$

1822. $\dfrac{\cos^8 x}{8}-\dfrac{\cos^6 x}{6}+C$ 1823. $-\dfrac{\cos^6 x}{6}+C$ 1828. $\dfrac{2x-\sin 2x}{4}+C$ 1832. $\dfrac{x}{8}-\dfrac{\sin 4x}{32}+C$

1833. $\dfrac{3x}{8}-\dfrac{\sin 10x}{20}+\dfrac{\sin 20x}{160}+C$

1858. $\dfrac{2}{3}\operatorname{arctg}\dfrac{5\operatorname{tg}(x/2)+4}{3}+C\,,\ x\in(-\pi+2k\pi,\pi+2k\pi),$ $k$ is an integer

1864. $\ln\sqrt{\left|\operatorname{tg}\dfrac{x}{2}\right|}-\dfrac{1}{4}\operatorname{cotg}^2\dfrac{x}{2}+C,\ x\in(k\pi,(k+1)\pi),$ $k$ is an integer

1865. $\dfrac{x}{2}-\ln\sqrt{|\sin x+\cos x|}+C\,,\ x\in(-\frac{\pi}{4}+k\pi,\frac{3\pi}{4}+k\pi),$ $k$ is an integer

1871. $\dfrac{\sin 5x}{10}+\dfrac{\sin x}{2}+C$ 1873. $-\dfrac{\cos 6x}{12}-\dfrac{\cos 2x}{4}+C$

### IV.6. Integrals of the type $\int R\left(x,\ \sqrt[s]{\dfrac{ax+b}{cx+d}}\right)\mathbf{d}x$

1892. $\frac{4}{3}\sqrt[4]{x^3}-4\sqrt[4]{x}+4\operatorname{arctg}\sqrt[4]{x}+C\,,\ x\in\langle 0,+\infty)$

1895. $\frac{2}{27}x^2\sqrt[4]{x}-\frac{2}{13}x\sqrt[12]{x}+C,\ x\in(0,+\infty)$

1896. $\ln\left|\dfrac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}-\sqrt{1+x}}\right|-\dfrac{\sqrt{1-x^2}}{x}+C\,,\ x\in(-1,0)\,,\ x\in(0,1\rangle$

1898. $\sqrt{3x^2-7x-6}+\dfrac{11}{2\sqrt{3}}\ln(x-\frac{7}{6}+\sqrt{x^2-\frac{7}{3}x-2})+C$

1899. $x+2-2\sqrt{x+2}+\ln\sqrt[3]{\dfrac{(2+\sqrt{x+2})^8}{(\sqrt{x+2}-1)^2}}+C,\ x\in(-2,-1),\ x\in(-1,+\infty)$

### V.1. Basic properties of definite integrals, Newton–Leibniz' formula

| | | | | | |
|---|---|---|---|---|---|
| 1985. 23/4 | 1986. $a^4/4$ | 1989. 21/8 | 1991. 1 | 1992. 45/4 | 1993. 2 |
| 1996. 1 | 2000. 2/15 | 2002. 7/6 | | | |

### V.2. Integration by parts and by substitution in the definite integral

| | | | | |
|---|---|---|---|---|
| 2010. $\ln 2/2$ | 2011. $\pi$ | 2012. 1 | 2015. $2-\pi/2$ | 2017. 2. |
| 2020. $-17/6$ | 2024. 8/21 | 2030. $\pi^2/32$ | 2031. $(1-\ln 2)/2$ | 2032. $\ln(4/3)$ |
| 2035. $\pi/6$ | 2036. $\pi/2-1$ | 2038. 1/3 | 2039. $2\ln 2-3/4$ | 2040. 2/3 |
| 2043. $\pi/2-1$ | 2044. $\mathrm{e}-\sqrt{\mathrm{e}}$ | | | |

### V.3. Improper Riemann integral

| | | | | |
|---|---|---|---|---|
| 2050. divergent | 2051. 4 | 2054. $2\sqrt[4]{125}/3$ | 2056. 1/8 | 2057. $\ln\sqrt{3}$ |
| 2058. 1 | 2060. $\pi$ | 2063. divergent | | |

### V.4. Some geometrical application of the definite integral

2067. 1/3 2068. the sum of areas of the two parts: 176/3

2069. 32/3 2070. the sum of areas of the two parts: 1/2

2074. a) $V_x=\pi\int_{-3}^{3}(4/9)(9-x^2)\,\mathrm{d}x=16\pi$, b) $V_y=\pi\int_{-2}^{2}(9/4)(4-y^2)\,\mathrm{d}y=24\pi$

2075. a) $V_x=9,6\pi$, b) $V_y=4,8\pi$ 2077. $4\sqrt{5}-\sqrt{2}$

### V.5. Further problems

| | | | | |
|---|---|---|---|---|
| 1. 1/3 | 2. $3/2-2\ln 2$ | 3. 2 | 4. $\pi-2/3$ | 5. $8\ln 8-9$ |
| 6. $\pi^2/4$ | 7. $9\pi$ | 8. $(e^2-1)\pi/2$ | 9. $\pi$ | 10. $\pi+\pi^2/2$ |
| 11. $8\pi/3$ | 12. $6+\ln 2/4$ | 13. $8(10\sqrt{10}-1)/27$ | | 14. $\frac{1}{2}$ |
| 15. 1 | | | | |