CZECH TECHNICAL UNIVERSITY IN PRAGUE

FACULTY OF MECHANICAL ENGINEERING

MATHEMATICS I

Problems from previous examinations

November 8, 2026

Selected problems from examination tests in previous years

CHAPTER I.

LINEAR ALGEBRA

1. a) Define the notions *dimension* and *basis* of a vector space V.

b) Decide whether the vectors $\vec{x} = (4; 2; 0)$, $\vec{y} = (1; 2; -1)$ a $\vec{z} = (7; 8; 1)$ form a basis in the vector space $V(\mathbb{E}_3)$.

c) If the vector $\vec{u} = (21; 18; 3)$ can be expressed as a linear combination of the vectors $\vec{x}$, $\vec{y}$, $\vec{z}$, find the coefficients in this combination.

2. a) Define what it means that vectors $\vec{u}_1, \ldots, \vec{u}_n$ are *linearly dependent*, respectively *linearly independent*.

b) For which values of parameter $a \in \mathbb{R}$ are the vectors $\vec{u} = (-1; 0; 1)$, $\vec{v} = (0; 1; a)$, $\vec{w} = (2; a; a)$ linearly dependent?

c) What is, in this case, dimension of the vector space generated by these vectors?

3. a) Define the notions *rank of a matrix* and *regular matrix*.

b) For which values of parameter $\alpha \in \mathbb{R}$ is the rank of the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & \alpha \\ -3 & 4 & 1 \end{pmatrix}$ equal to 3 and for which α is the rank equal to 2?

c) Is matrix $\boldsymbol{A}$ regular for $\alpha = 1$? (Give reasons for your answer.)

4. a) Define the notion of an *inverse matrix* to a square matrix $\boldsymbol{A}$.

b) Decide about the existence of the inverse matrix to the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$.

c) If the inverse matrix exists, calculate it. Verify the result by computing the product $\boldsymbol{A} \cdot \boldsymbol{A}^{-1}$.

5. $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, $\boldsymbol{C} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$

a) Find the matrix $\boldsymbol{A}^{-1}$. b) Find the matrix $\boldsymbol{B}^{-1}$.

c) Calculate matrix $\boldsymbol{X}$ such that $\boldsymbol{A} \cdot \boldsymbol{X} \cdot \boldsymbol{B} = \boldsymbol{C}$.

6. Given the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix}$.

a) Calculate the matrix $\boldsymbol{B} = \boldsymbol{A}^2$ $(= \boldsymbol{A} \cdot \boldsymbol{A})$.

b) Define the notion of a *regular matrix*. Decide if the given matrix $\boldsymbol{A}$ is regular.

c) If the inverse matrix to $\boldsymbol{A}$ exists, calculate it.

7. Given the matrix with parameters $a, b \in \mathbb{R}$: $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ a & b & 0 & 0 \\ -1 & -1 & 1 & 0 \end{pmatrix}$.

a) Calculate the determinant of matrix $\boldsymbol{A}$.

b) Define the notions of a *regular matrix* and a *singular matrix.*

c) For which values of parameters a, b does the homogeneous system of linear algebraic equations $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{O}$ have a non–zero solution?

8. a) Explain principles of the Cramer rule. Under which conditions it can be applied?

b) Verify the assumptions for the system

$$\begin{aligned} x + y + 2z &= 1 \\ 2x + y &= -4 \\ 5x + y - 3z &= -13 \end{aligned}$$

c) Applying Cramer's rule, calculate the value of unknown y.

9. a) Calculate the determinant of the system with parameter $a \in \mathbb{R}$:

$$\begin{aligned} x + 2y + az &= 0 \\ -x + 3y + az &= -8 \\ 3x - y + 2z &= 13 \end{aligned}$$

b) Explain principles of the Cramer rule. Is it possible to apply Cramer's rule to the system given above if $a = 1$? (Give reasons for your answer.)

c) Assuming that $a = 1$, calculate the value of unknown z.

10. a) Write the Frobenius theorem.

b) What is the number of solutions of the given system in dependence on parameter $a \in \mathbb{R}$:

$$\begin{aligned} x - y + z &= 1 \\ x + y + 3z &= 1 \\ (2a-1)x + (a+1)y + z &= 1 - a \end{aligned}$$

c) Solve the system for $a = 1$.

11. a) Define the notions *eigenvalue* and *eigenvector* of a square matrix.

b) For which value of parameter $c \in \mathbb{R}$ does matrix $\boldsymbol{A}$ have the eigenvalue $\lambda = 0$?

$$\boldsymbol{A} = \begin{pmatrix} c & c-5 \\ 6 & -3 \end{pmatrix}$$

c) Calculate the eigenvalues and the associated eigenvectors of matrix $\boldsymbol{A}$ if $c = 4$.

12. a) Define the notions *eigenvalue* and *eigenvector* of a square matrix.

b) Find all eigenvalues and eigenvectors of the matrix $\boldsymbol{A} = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}$.

CHAPTER II.

DIFFERENTIAL CALCULUS

1. a) Evaluate $\lim\limits_{n\to+\infty}(\sqrt{n^2+1}-\sqrt{n^2-1})$.

b) Define what it means that the sequence $\{a_n\}$ is increasing.

c) Create an increasing sequence whose limit is 3.

2. a) Evaluate $\lim\limits_{n\to+\infty}\dfrac{n+\cos(n^2)}{2n+1}$.

b) Define what it means that the sequence $\{a_n\}$ is decreasing.

c) Create a decreasing sequence whose limit is 3.

3. a) Evaluate $\lim\limits_{n\to+\infty}\dfrac{(2n-1)^2-4n^2+1}{n^2-(n+5)^2}$.

b)Write the theorem on a limit of a subsequence.

c) Create a sequence that has no limit. (Give reasons why your sequence has no limit.)

4. a) Using the definition, decide about the monotonicity of the sequence $\left\{\frac{n+1}{2n+1}\right\}$.

b) Evaluate the limit of the sequence $\lim\limits_{n\to+\infty} n(\sqrt{n^2+1}-n)$.

c) Evaluate the limit of the function $\lim\limits_{x\to 0}\dfrac{\cos x-1}{x\sin x}$. If you apply l'Hospital's rule, verify its assumptions.

5. a) Given the function $f: f(x)=\dfrac{x}{\sqrt{x^2-4x}}$. Find $D(f)$.

b) Calculate the limits of $f(x)$ for $x\to+\infty$ and $x\to 0$ (if the limits exist).

c) Is the given function f odd, even or periodic? (Give reasons for your answer.)

6. a) Given the function $f(x)=\arccos(x^2-1)$. Specify $D(f)$.

b) Write the equation of the tangent line to the graph of function f at the point $[x_0; f(x_0)]$, if $x_0=1$.

c) Is function f even or odd? (Give reasons for your answer.)

7. a) Given the function $f: f(x)=\ln(x^2+4x+3)$. Specify $D(f)$.

b) Write the equation of the tangent line to the graph of function f at the point $[x_0; f(x_0)]$, if $x_0=1$.

c) Using the linear approximation of f (i.e. the result of part b)), calculate an approximate value of function f at the point $x=0.9$.

Other variants of problem 7 with different functions f and points x_0:

b) $f(x)=x+\sqrt{1-x^2}$
c) $x_0=0$

b) $f(x)=\dfrac{x+2}{\sqrt{5-x}}$
c) $x_0=1$

b) $f(x)=\arcsin\sqrt{x+1}$
c) $x_0=-\frac{1}{2}$

8. a) Define the notion *derivative of function f at point x_0*.

b) Calculate the derivative of the function $f : f(x) = \sqrt[3]{\mathrm{e}^{-x}+1}$. Specify the domain of the derivative.

c) Write the equation of the tangent line and the normal line to the graph of function f at the point $[x_0; f(x_0)]$, if $x_0 = 0$.

In problems 9–11:

a) define absolute extremes of function f on the interval I,

b) give reasons for the existence of the absolute extremes,

c) find the absolute extremes (i.e. find their positions and values).

9. $f(x) = 3 - x - \dfrac{4}{(x+2)^2}, \qquad I = \langle -1; 2\rangle$

10. $f(x) = x + 3\sqrt[3]{x^2}, \qquad I = \langle -1; 1\rangle$

11. $f(x) = 2\sqrt{x-1} - x + 2, \qquad I = \langle 1; 5\rangle$

In problems 12–14:

a) find intervals of monotonicity and local extremes of the given function f,

b) find points of inflection and intervals where function f is concave up or down.

c) Sketch the graph.

12. $f(x) = 1 + x^2 - \frac{1}{2}x^4$ **13.** $f(x) = (x-3)\sqrt{x}$ **14.** $f(x) = \mathrm{e}^{2x-x^2}$

15. Given the function $f(x) = \dfrac{1}{9-x^2}$.

a) Specify $D(f)$. Is the given function even or odd? (Give reasons for your answer.)

b) Find intervals of monotonicity and local extremes.

c) Calculate the limits of f for $x \to +\infty$, $x \to 3+$ and $x \to 3-$. Sketch the graph.

16. Given the function $f(x) = (x+2)\,\mathrm{e}^{1/x}$with the restricted domain $D(f) = (0; +\infty)$.

a) Find intervals of monotonicity and local extremes.

b) Calculate the limits of function f for $x \to +\infty$ and $x \to 0+$.

c) Sketch the graph.

In problems 17–20:

a) find intervals of monotonicity and local extremes for the given function f,

b) find intervals where function f is concave up or concave down and find points of inflection,

c) evaluate the limits at the end points of $D(f)$ and sketch the graph.

17. $f(x) = 3 - x - \dfrac{4}{(x+2)^2}$ with the restricted domain $D(f) = (-2; +\infty)$

18. $f(x) = x\,\ln x$ **19.** $f(x) = (x-2)\,\mathrm{e}^x$ **20.** $f(x) = x^2 + 2\ln(x+2)$.

In problems 21 and 22:

a) find intervals of monotonicity and local extremes for the given function f,

b) find asymptotes of function f, c) sketch the graph.

21. $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$ **22.** $f(x) = \dfrac{x-1}{x^2+3}$

23. $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$ a) Specify $D(f)$. Calculate the limits for $x \to +\infty$ and $x \to -\infty$.

b) Find $f'(x)$ (including the domain).

c) Find intervals of monotonicity and local extremes.

24. a) Evaluate the coefficients and write Taylor's polynomial of the 5–th degree of the function $f(x) = \mathrm{e}^x$ with the center at the point $x_0 = 0$. Write the form of the remainder.

b) Applying Taylor's polynomial, calculate the value of $\mathrm{e}^{-1/3}$ with the error less than 0.001.

25. a) Evaluate the coefficients and write Taylor's polynomial of the 6–th degree of the function $f(x) = \cos\, x$ with the center at the point $x_0 = 0$. Write the form of the remainder.

b) Evaluate the coefficients and write Taylor's polynomial of the 5–th degree of the same function f with the center at the point $x_0 = \pi/2$.

26. a) Evaluate the coefficients and write Taylor's polynomial T_2 of the 2nd degree of the function $f(x) = x + \sqrt{x+1}$ withy the center $x_0 = 0$. Write the form of the remeinder.

b) Estimate the error when using polynomial T_2 for the approximate evaluation of the function value of f at the point $x = 1/2$.

CHAPTER III.

INTEGRAL CALCULUS

In problems 1–6:

a) write the theorem on the integration by parts (including the assumptions),
b) evaluate the integral $\int f(x)\,\mathrm{d}x$, where function f has the concrete form

1. $f(x) = x\,\arctan\,x$

2. $f(x) = x^2 \ln x$

3. $f(x) = (x^2 + x + 2)\,\mathrm{e}^x$

4. $f(x) = \ln^2 x$

5. $f(x) = (3x - 5)\,\sin x$

6. $f(x) = (2x + 3)\,\mathrm{e}^{3x}$

On which intervals do the integrals exist?

In problems 7–20:

a) write the theorem on integration by substitution (including the assumptions),
b) evaluate the integral $\int f(x)\,\mathrm{d}x$, where function f has the concrete form

7. $f(x) = \cos(1 - 2x)$

8. $f(x) = \dfrac{x-2}{x^2 - 4x + 8}$

9. $f(x) = \dfrac{\mathrm{e}^{2x}}{2 + \mathrm{e}^{2x}}$

10. $f(x) = \dfrac{x^3}{\sqrt{x^4 + 7}}$

11. $f(x) = \dfrac{1}{1 + \sqrt{x}}$

12. $f(x) = \dfrac{\mathrm{e}^{1/x}}{x^2}$

13. $f(x) = x\sqrt{1 - x^2}$

14. $f(x) = \dfrac{\cos x}{\sqrt[3]{\sin^2 x}}$

15. $f(x) = \left(\dfrac{1}{1 + \ln^2 x} + \dfrac{1}{\sqrt{\ln\,x}}\right)\dfrac{1}{x}$

16. $f(x) = \sin^2 x \cos^3 x$

17. $f(x) = \cos^2 x + \cos^3 x$

18. $f(x) = \cos^7 x$

19. $f(x) = x^3\,\mathrm{e}^{-x^2}$

20. $f(x) = \dfrac{\sqrt{x-2}}{x-1}$

On which intervals do the integrals exist?

In problems 21–26 calculate the integral of the given rational function. On which intervals do the integrals exist?

21. $\displaystyle\int \frac{x^3}{x^2 + 3x + 2}\,\mathrm{d}x$

22. $\displaystyle\int \frac{2x+1}{x^2 + 4x + 4}\,\mathrm{d}x$

23. $\displaystyle\int \frac{x}{(x+1)(x+2)(x+5)}\,\mathrm{d}x$

24. $\displaystyle\int \frac{1}{(x+1)^2(x+2)}\,\mathrm{d}x$

25. $\displaystyle\int \frac{x-8}{x^3 - 4x^2 + 4x}\,\mathrm{d}x$

26. $\displaystyle\int \frac{1}{x^2 - x + 1}\,\mathrm{d}x$

27. a) Calculate the area of the region, which is for $x \in \langle\, 1; 2\,\rangle$ bounded by the x–axis and the curve $y = x^2 + \dfrac{1}{x^2}$.

b) Evaluate the definite integral $\displaystyle\int_0^1 (3x + 1)\,\mathrm{e}^x\,\mathrm{d}x$.

28. a) Find the antiderivative (and the interval of its existence) to the function $f(x) = \dfrac{1}{4+x^2}$.

b) Calculate the area of the region, which is bounded by the x–axis and by the curves $y = \dfrac{1}{4+x^2}$, $x = 0$, $x = 2$.

c) Evaluate the improper integral $\displaystyle\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$.

29. Given the function $f(x) = \dfrac{1}{x^2+x}$.

a) Calculate the integral $\int f(x)\,\mathrm{d}x$. Give the intervals of its existence.

b) Evaluate the definite integral $\int_1^3 f(x)\,\mathrm{d}x$.

c) Evaluate the integrals $\int_0^1 f(x)\,\mathrm{d}x$ a $\int_3^\infty f(x)\,\mathrm{d}x$. Do the integrals converge?

30. a) Find the domain and sketch the graph of the function $y = \sqrt{x-1}$.

b) Sketch the region bounded by the curves $y = \sqrt{x-1}$, $x = 0$, $y = 0$ a $y = 1$ and evaluate its area.

c) Calculate the volume of the body which arises by rotation of the above region about the y–axis.

31. Given the function $f(x) = x^2 \sin x$.

a) Calculate the integral $\int f(x)\,\mathrm{d}x$. Verify the result by differentiation.

b) Find the mean value of the function f on the interval $\langle 0;\pi\rangle$, i.e. the value $\mu = \dfrac{1}{\pi}\displaystyle\int_0^\pi f(x)\,\mathrm{d}x$.

32. a) Sketch a region in the 1st quadrant in $\mathbb{E}_2$, that is bounded by the graph of the function $f(x) = \sin x$ and by the straight line $x = \pi/2$. Calculate the volume of the body, which arises by rotation of this region about the x–axis.

b) Calculate the area of the region, that is for $x \in \langle\, 0;\pi/2\,\rangle$ bounded by the x–axis and the curve $y = \cos^4 x\ \sin x$.

33. a) Calculate the integral and give intervals of its existence: $\displaystyle\int \frac{\sqrt{x}-2}{x}\,\mathrm{d}x$.

b) Evaluate the area of the region, that is for $x \in \langle\, \frac{1}{4};1\,\rangle$ bounded by the x–axis ad the curve $y = \dfrac{\sqrt{x}-2}{x}$.

c) Evaluate the improper integral $\displaystyle\int_0^1 \left(\frac{1}{\sqrt{x}} - \frac{1}{x}\right)\,\mathrm{d}x$. Does the integral converge?

34. Given the function $f(x) = (2x+3)\,\sin 2x$.

a) Calculate the integral $\int f(x)\,\mathrm{d}x$. Verify the result by differentiation.

b) Evaluate the area of the region, which is for $x \in \langle 0;\pi/4\rangle$ bounded by the x–axis and by the curve $y = (2x+3)\,\sin 2x$.

35. a) Give intervals of the existence and calculate the integral $\displaystyle\int \frac{x-8}{x^3-4x^2+4x}\,\mathrm{d}x$.

b) Give reasons for the existence and evaluate the definite integral $\displaystyle\int_3^4 f(x)\,\mathrm{d}x$. Simplify the result.

c) Calculate the improper integral $\displaystyle\int_3^{+\infty} \frac{x-8}{x^3-4x^2+4x}\,\mathrm{d}x$. Does the integral converge?