Mathematics I – Exam 1

---

1. a) Define the notion of the *linear dependence and independence* of a group of vectors $\vec{u_1}, ..., \vec{u_n}$.

b) For which values of the paremeter $\lambda \in \mathbb{R}$ form the vectors $\vec{u} = (0; 1; -2\lambda)$, $\vec{v} = (2; 1; \lambda)$ and $\vec{w} = (\lambda + 2; 1; 0)$ basis of the space $V(\mathbb{E}_3)$?

c) For $\lambda = 1$ express the vector $\vec{x} = (4; 0; 3)$ using this basis.

2. a) Compute the limit of the function $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\ln(1 - x^2)}$.

(In the case you would like to use the L'Hospital's rule, verify its assumptions.)

b) Find the limit of the sequence $\displaystyle\lim_{n\to+\infty} n\left(\sqrt{n^2 + 3} - n\right)$.

3. Given function $f(x) = \dfrac{x + 3}{\sqrt{x^2 - 1}}$.

a) Compute the derivative $f'(x)$. Find the domains $D(f)$, $D(f')$. Compute the value of the derivative $f'(-2)$ and describe the behavior of the given function f in the neighborhood of the point $x_0 = -2$, i.e. if the function is increasing/decreasing and how fast (inclination of the tangent).

b) Write the equations of the tangent and normal lines to the graph of the given function at the point $[x_0, f(x_0)]$ for $x_0 = -2$.

c) Justify the existence of absolute extrema of the given function on the interval $\langle -7, -2\rangle$. Find those extrema, i.e. find their position, type and value.

4. Describe the behavior of the function $f(x) = \dfrac{\ln x}{x}$.

a) Find the domain $D(f)$ of the function f, find the intersections of the graph of f with the x and y axes, find the limits of f in the boundary points of $D(f)$, find all asymptotes.

b) Determine the intervals of monotonicity and local extrema of f (i.e. find their position, type, value)

c) Find out, where is the function convex (concave-up) and concave (concave-down), find the inflection points. Sketch the graph of the given function f.

5. Find the following integrals and intervals of their existence.

a) $\displaystyle\int (3x - 4)\cos 5x \,\mathrm{d}x$ b) $\displaystyle\int \sin^2 x\,(1 + \cos^3 x)\,\mathrm{d}x$

6. Given function $f(x) = (x^2 - x)\mathrm{e}^x$.

a) Compute the integral $\int f(x)\mathrm{d}x$. Verify the result (from the definition).

b) Compute the (integral) mean value of the function f on the interval $\langle 0; 1\rangle$.