Mathematics I A – Exam 1

1. a) Define the notion of the *linear dependence and independence* of a group of vectors $\vec{u_1}, \ldots, \vec{u_n}$.

b) For which values of the paremeter $\lambda \in \mathbb{R}$ form the vectors $\vec{u} = (0; 1; -2\lambda)$, $\vec{v} = (2; 1; \lambda)$ and $\vec{w} = (\lambda + 2; 1; 0)$ basis of the space $V(\mathbb{E}_3)$?

c) For $\lambda = 1$ express the vector $\vec{x} = (4; 0; 3)$ using this basis.

2. a) Compute the limit of the function $\displaystyle \lim_{x \to 0} \frac{\cos x - 1}{\ln(1 - x^2)}$.

(In the case you would like to use the L'Hospital's rule, verify its assumptions.)

b) Find the limit of the sequence $\displaystyle \lim_{n \to +\infty} n\left(\sqrt{n^2 + 3} - n\right)$.

3. Given function $f(x) = \dfrac{x + 3}{\sqrt{x^2 - 1}}$.

a) Compute the derivative $f'(x)$. Find the domains $D(f)$, $D(f')$. Compute the value of the derivative $f'(-2)$ and describe the behavior of the given function f in the neighborhood of the point $x_0 = -2$, i.e. if the function is increasing/decreasing and how fast (inclination of the tangent).

b) Write the equations of the tangent and normal lines to the graph of the given function at the point $[x_0, f(x_0)]$ for $x_0 = -2$.

c) Justify the existence of absolute extrema of the given function on the interval $\langle -7, -2 \rangle$. Find those extrema, i.e. find their position, type and value.

4. Describe the behavior of the function $f(x) = \dfrac{\ln x}{x}$.

a) Find the domain $D(f)$ of the function f, find the intersections of the graph of f with the x and y axes, find the limits of f in the boundary points of $D(f)$, find all asymptotes.

b) Determine the intervals of monotonicity and local extrema of f (i.e. find their position, type, value)

c) Find out, where is the function convex (concave-up) and concave (concave-down), find the inflection points. Sketch the graph of the given function f.

5. Find the following integrals and intervals of their existence.

a) $\displaystyle \int (3x - 4) \cos 5x \, \mathrm{d}x$ b) $\displaystyle \int \sin^2 x \, (1 + \cos^3 x) \, \mathrm{d}x$

6. Given function $f(x) = (x^2 - x)\mathrm{e}^x$.

a) Compute the integral $\int f(x)\mathrm{d}x$. Verify the result (from the definition).

b) Compute the (integral) mean value of the function f on the interval $\langle 0; 1 \rangle$.

Mathematics I B – Exam 1

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1. a) Compute the determinant of the matrix of the system:

$$\begin{aligned} x - 3y + 2z &= 5 \\ 2x + y + z &= 9 \\ 6x + 3y - 2z &= 2. \end{aligned}$$

b) Explain the use of the Cramer's rule for the solution of the systems of linear equations $\boldsymbol{A}\vec{x} = \vec{b}$ with square matrix $\boldsymbol{A}$.
Find out if it's possible to use the Cramer's rule for the solution of the above given system.

c) Using the Cramer's rule compute the value of the unknown z.

2. a) Compute the limit of the function $\lim\limits_{x \to 0} \dfrac{x^2 + \sin 2x}{1 - \mathrm{e}^{3x}}$.
(In the case you would like to use the L'Hospital's rule, verify its assumptions.)

b) Find the limit of the sequence $\lim\limits_{n \to +\infty} \dfrac{2n - 1}{n^2 - (n+2)^2}$.

3. a) Find the domain of the function $f(x) = \dfrac{x^2 + 1}{2 - 3x}$. Compute the derivative $f'(x)$ and find the points where it is equal to zero.

b) Write the equation of the tangent to the graph of f at point $[0, f(0)]$. Sketch the tangent and describe the behavior of the given function f in the neighborhood of the point $x_0 = 0$, i.e. if the function is increasing/decreasing and how fast (inclination of the tangent).

c) Write the Taylor's polynomial $T_2(x)$ (of the second order) of the given function f centered at point $x_0 = 0$. Use the result, i.e. use the $T_2(x)$, to compute the approximate value of $f(1/2)$. What is the exact value $f(1/2)$?

4. Given function $f(x) = (x - 2)\,\mathrm{e}^x$.

a) Determine the intervals of monotonicity and local extrema of f (i.e. find their position, type, value)

b) Find out, where is the function f convex (concave-up) and concave (concave-down), find the inflection points.

c) Find the intersections of the graph of f with the x and y axes. Sketch the graph of the given function f at the interval $\langle -5, 5 \rangle$.

5. Find the following integrals and intervals of their existence.

a) $\displaystyle\int (x - 2)\mathrm{e}^x \,\mathrm{d}x$ b) $\displaystyle\int x\sqrt{x^2 - 3}\,\mathrm{d}x$

6. a) Compute the integral $\displaystyle\int_0^{\pi/2} \sin^2 x \,\mathrm{d}x$.

b) Sketch the domain bounded by the x axis, by the graph of the function $y = \sin x$ and by the lines $x = 0$ and $x = \pi/2$. Compute the surface area of this domain.

c) Find the volume of the body that arises by rotation of the above described domain around the x axis.