Mathematics I – Exam 3

1. a) Write *Frobenius theorem* (including all assumptions).

b) Find the number of solutions depending on the value of the parameter $a \in \mathbb{R}$:

$$\begin{aligned} -x \quad\quad + z &= 0 \\ 2x + ay + az &= 8 \\ y + az &= 4 \end{aligned}$$

c) Using the Gauss algorithm or Cramer's rule find the solution of the system for $a = 3$.

2. a) Define the notion of *eigenvalue* and *eigenvector* of a square matrix. Write down and explain the property of a matrix that will guarantee the existence of zero eigenvalue.

b) Find eigenvalues of the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & -5 \\ 2 & 1 & 3 \end{pmatrix}$

c) Choose one of the eigenvalues, construct the system of equations for computing the eigenvectors and find that eigenvector.

3. Given function $f(x) = \sqrt{4x - 3} + \dfrac{x^2}{3}$

a) Compute 1^{st} a 2^{nd} derivative of this function. Find domain $D(f)$ and $D(f')$.

b) Find equation of tangent line to the graph of function f at the point $x_0 = 1$.

c) Write Taylor's polynomial $T_2(x)$ of 2^{nd} degree with the center at $x_0 = 1$ of function f. Using the $T_2(x)$ find the approximate value of $f(x)$ for $x = 2$.

d) Write Lagrange's form of the remainder $R_3(x)$. Use it to estimate the error of the approximation of the value of function f at the point $x = 2$ by $T_2(x)$ from part c).

4. For the function $f(x) = 4\,\operatorname{arctan}\, x - 2x$

a) Find intervals of monotonicity and local extrema of the given function f.

b) Find intervals of convexity or concavity of the function f. Find inflection points.

c) Find the asymptote of the function f for $x \to +\infty$.
Sketch the graph of the function f on interval $\langle -1; +\infty)$.

5. Compute integrals a) $\displaystyle\int \sin^3\varphi \cos^3\varphi \,\mathrm{d}\varphi$, b) $\displaystyle\int 2x \operatorname{arctan} x \,\mathrm{d}x$.
Find intervals of existence of these integrals.

6. a) Compute integral $\displaystyle\int \frac{1}{x^2 - x - 6}\,\mathrm{d}x$, find intervals of its existence.

b) Compute area of the surface, which is for $x \in \langle 0, 2 \rangle$ bounded by axis x and by the curve $y = \dfrac{1}{x^2 - x - 6}$. Simplify the result.

c) Decide (by computation) about the convergence of the improper integral $\displaystyle\int_0^3 \frac{1}{x^2 - x - 6}\,\mathrm{d}x$.