Mathematics I A – Exam 3

1. a) Write *Frobenius theorem* (including all assumptions).

b) Find the number of solutions depending on the value of the parameter $a \in \mathbb{R}$:

$$\begin{aligned} -x \qquad\quad + z &= 0 \\ 2x + ay + az &= 8 \\ y + az &= 4 \end{aligned}$$

c) Using the Gauss algorithm or Cramer's rule find the solution of the system for $a = 3$.

2. a) Define the notion of *eigenvalue* and *eigenvector* of a square matrix. Write down and explain the property of a matrix that will guarantee the existence of zero eigenvalue.

b) Find eigenvalues of the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & -5 \\ 2 & 1 & 3 \end{pmatrix}$

c) Choose one of the eigenvalues, construct the system of equations for computing the eigenvectors and find that eigenvector.

3. Given function $f(x) = \sqrt{4x-3} + \dfrac{x^2}{3}$

a) Compute 1^{st} a 2^{nd} derivative of this function. Find domain $D(f)$ and $D(f')$.

b) Find equation of tangent line to the graph of function f at the point $x_0 = 1$.

c) Write Taylor's polynomial $T_2(x)$ of 2^{nd} degree with the center at $x_0 = 1$ of function f. Using the $T_2(x)$ find the approximate value of $f(x)$ for $x = 2$.

d) Write Lagrange's form of the remainder $R_3(x)$. Use it to estimate the error of the approximation of the value of function f at the point $x = 2$ by $T_2(x)$ from part c).

4. For the function $f(x) = 4 \arctan\, x - 2x$

a) Find intervals of monotonicity and local extrema of the given function f.

b) Find intervals of convexity or concavity of the function f. Find inflection points.

c) Find the asymptote of the function f for $x \to +\infty$.
Sketch the graph of the function f on interval $\langle -1; +\infty)$.

5. Compute integrals a) $\displaystyle\int \sin^3\varphi \, \cos^3\varphi \, \mathrm{d}\varphi$, b) $\displaystyle\int 2x \, \arctan x \, \mathrm{d}x$.
Find intervals of existence of these integrals.

6. a) Compute integral $\displaystyle\int \frac{1}{x^2 - x - 6} \, \mathrm{d}x$, find intervals of its existence.

b) Compute area of the surface, which is for $x \in \langle\, 0, 2 \,\rangle$ bounded by axis x and by the curve $y = \dfrac{1}{x^2 - x - 6}$. Simplify the result.

c) Decide (by computation) about the convergence of the improper integral $\displaystyle\int_0^3 \frac{1}{x^2 - x - 6} \, \mathrm{d}x$.

Mathematics I B – Exam 3

---

1. Is given a system of equations

$$\begin{aligned} x + y + 2z &= 2 \\ 3x - y - z &= 7 \\ -x + 2y + z &= -5 \end{aligned}$$

a) Use the Gauss algorithm or Cramer's rule to find the solution of the given system of equations. Verify the result.

b) Find the rank of the matrix of this system and rank of the extended matrix of this system.

2. Given matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

a) Compute the eigenvalues of matrix $\boldsymbol{A}$.

b) For one of the eigenvalues write the system of equations for the computation of the eigenvector, compute this eigenvector.

c) Decide and explain if there exists an inverse matrix $\boldsymbol{A}^{-1}$ to matrix $\boldsymbol{A}$.
Find $\boldsymbol{A}^{-1}$ and check the result.

3. For the function $f(x) = \mathrm{e}^{2x-4}$

a) Compute derivatives $f'(x)$, $f''(x)$.

b) Write equation of a tangent line to the graph of function f at point $[x_0, f(x_0)]$, if $x_0 = 2$. Use the equation of the tangent to compute the approximate value of function f at point $x = 2.2$.

c) Sketch the tangent line from part b). Describe the behavior of the given function f in the neighborhood of the point $x_0 = 2$, i.e. if it's increasing/decreasing, how fast (the inclination of the tangent in degrees).

d) Write Taylor's polynomial of the 2^{nd} order $T_2(x)$ of function f with center at point $x_0 = 2$.

4. Is given a function $f(x) = 3x - x^3$

a) Find its domain $D(f)$, compute its derivative and find $D(f')$.

b) Find intervals where the function f is increasing and where it's decreasing.

c) Find local extrema of the function f.

d) Compute limits of f at the endpoints of its domain $D(f)$. Sketch the graph of f.

5. Compute integrals and find intervals of their existence:

a) $\displaystyle\int \sin^3\varphi \, \cos\varphi \, \mathrm{d}\varphi$, b) $\displaystyle\int \frac{2x+1}{(x-1)(x+2)} \, \mathrm{d}x$.

6. Given function $f(x) = \dfrac{1}{x^2} + x^{3/2}$.

a) Compute the indefinite integral of f, find intervals of its existence.

b) Compute area of the region bounded for $x \in \langle 1, 2 \rangle$ by the x-axis and by the graph of the given function f.