Mathematics I – Exam 5

1. a) Given system of equations with parameter $a \in \mathbb{R}$:

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2x + 3y + 4z &= 5 \\ ax + ay + az &= a \end{aligned}$$

a) Write the *Frobenius theorem* (assumptions and statement).

b) Find the number of solutions of the system depending on the value of parameter a.

c) Find the solution of the given system for $a = 0$.

d) Verify that for $a = 1$ the $x = -2$, $y = 3$, $z = 0$ is a solution of the system.

2. a) Find the eigenvalues of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 3 \end{pmatrix} \quad .$$

b) For the largest eigenvalue write down the system of equations for the computation of eigenvectors. Solve the system and find the eigenvectors.

c) Verify (by calculation) that for eigenvalues of the given matrix holds $det\boldsymbol{A} = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$ and moreover $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$. Based on the knowledge of eigenvalues of matrix $\boldsymbol{A}$ decide whether there exists the inverse matrix $\boldsymbol{A}^{-1}$.

d) What is the relation between the eigenvalues of a matrix and eigenvalues of its inverse matrix? Find the eigenvalues of the inverse matrix $\boldsymbol{A}^{-1}$ (if it exists).

3. Given function $f(x) = \ln x^2$.

a) Compute the derivative $f'(x)$ and find the domains $D(f),\ D(f')$.

b) Write the Taylor polynomial $T_2(x)$ of order 2 centered at $x_0 = 1$ of the function $f(x)$.

c) Write the equation of the tangent to the graph of the function f at point $[x_0, f(x_0)]$, where $x_0 = 1$.

d) Based on the results from problems b) and c), decide if the function is in the neighborhood of given point $[x_0, f(x_0)]$ increasing/decreasing, convex/concave.
Sketch the graph of the function $f(x)$ *in the neighborhood of point* $[x_0, f(x_0)]$.

4. Given function $f(x) = x^2\,\mathrm{e}^{-x}$.

a) Compute the derivative $f'(x)$ and find its domain $D(f')$.

b) Find the intervals where the function f is increasing/decreasing. Find local extrema.

c) Compute the limits of given function for $x \to -\infty$ and $x \to +\infty$ and sketch the graph.

5. Find the following integrals and intervals of their existence.

a) $\displaystyle\int x^2\ \sqrt[3]{8 - x^3}\,\mathrm{d}x$ b) $\displaystyle\int \sin^2 x\ \cos^3 x\,\mathrm{d}x$

6. Given function $f(x) = x^3\ \ln x$. a) Compute the integral $\int f(x)\,\mathrm{d}x$.

b) Find the surface area of the region that is for $x \in \langle\, 2, 4 \,\rangle$ bounded by the x axis and by the curve $y = f(x)$. *Simplify the result*.

c) Decide by computation if the following improper integral converges $\int_0^1 f(x)\,\mathrm{d}x$.