CZECH TECHNICAL UNIVERSITY IN PRAGUE

FACULTY OF MECHANICAL ENGINEERING

MATHEMATICS I

Exercises

Linear algebra and analytic geometry

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Experimental version of the collection of exercises for the academic year 2026/2027

Based on: *E. Brožíková, M. Kittlerová: Lineární algebra a analytická geometrie. (2004).*

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CHAPTER I.

Linear algebra

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I.1 Vectors, vector spaces

***Example* 1**: Given the vectors $\vec{a} = (-4; 7)$, $\vec{b} = (-2; 0)$, $\vec{c} = (5; 1)$. Write the coordinates of the vectors:

a) $(-\vec{a})$ b) $3\,\vec{c}$ c) $\vec{c} + \vec{a}$

d) $2\,\vec{a} + \vec{b} - 4\vec{c}$ e) $2\,\vec{c} - 3\,\vec{b} + \vec{a}$

Solution:

a) The vector $(-\vec{a})$ is the opposite vector to the vector $\vec{a}$ and by the definition its coordinates will be $(-\vec{a}) = -\vec{a} = -(-4; 7) = (4; -7)$.

b) By the definition of the multiplication of a vector by a real number we compute $3\,\vec{c} = 3\,(5; 1) = (15; 3)$.

The following items concern a linear combination of vectors:

c) $\vec{c} + \vec{a} = (5; 1) + (-4; 7) = (5 - 4; 1 + 7) = (1; 8)$

d) $2\,\vec{a} + \vec{b} - 4\,\vec{c} = (-8; 14) + (-2; 0) - (20; 4) = (-8 - 2 - 20; 14 + 0 - 4) = (-30; 10)$

e) $2\,\vec{c} - 3\,\vec{b} + \vec{a} = (10; 2) - (-6; 0) + (-4; 7) = (12; 9)$ ■

***Example* 2**: For which m, n are the vectors $\vec{a} = (2\,m; 3)$ and $\vec{b} = (2; m + n)$ equal?

Solution: Two vectors are equal if and only if their corresponding coordinates are equal:

$$\begin{cases} 2\,m = 2 & \Rightarrow \quad m = 1 \\ 3 = m + n & \Rightarrow \quad n = 2 \end{cases}. \qquad \text{Then} \quad \vec{a} = \vec{b} = (2; 3). \qquad \blacksquare$$

***Example* 3**: Given the vectors $\vec{a} = (y - 1; x)$, $\vec{b} = (x + 1; 3\,y)$.
Find the numbers x and y so that

a) $3\,\vec{a} - \vec{b} = \vec{0}$ b) $\vec{b} - \vec{a} = (0; 8)$

Solution:

a) $\Big(3\,(y - 1); 3\,x\Big) - (x + 1; 3\,y) = (0; 0) \quad \Rightarrow \; (3\,y - 3 - x - 1; 3\,x - 3\,y) = (0; 0),$

$$\begin{cases} 3\,y - 3 - x - 1 = 0 \\ 3\,x - 3\,y = 0 \end{cases} \Rightarrow \; x = y \; \Rightarrow \; 3\,x - 3 - x - 1 = 0 \; \Rightarrow \; 2\,x = 4,$$

$$x = y = 2 \; \Rightarrow \; \vec{a} = (1; 2),\; \vec{b} = (3; 6) \quad \Rightarrow \quad \vec{b} = 3\,\vec{a};$$

b) $(x+1;3y)-(y-1;x)=(0;8) \quad \Rightarrow$

$$\left\{\begin{array}{r} x+1-y+1=0 \\ 3\,y-x=8 \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{r} x-y=-2 \\ -x+3\,y=8 \end{array}\right. .$$

Adding both equations we obtain $2y=6$. Hence $y=3$ and $x=1$. ■

***Example* 4**: Given the vectors $\vec{u}=(1;-2;3)$, $\vec{v}=(4;1;0)$, $\vec{w}=(-3;5;1)$. Find the vector $\vec{x}$ which satisfies the equation:

a) $\vec{x}=2\,\vec{u}+\vec{v}-3\,\vec{w}$

b) $\vec{x}-2\,\vec{u}=3\,\vec{x}+2\,(\vec{u}-\vec{v}-2\,\vec{w})$

Solution:

a) We write the linear combination on the right hand side in coordinates, and in this way we compute the coordinates of the vector $\vec{x}$:

$$\begin{aligned}\vec{x} &= 2\,(1;-2;3)+(4;1;0)-3\,(-3;5;1)=(2;-4;6)+(4;1;0)+(9;-15;-3)= \\ &= (2+4+9;\ -4+1-15;\ 6+0-3)=(15;-18;3).\end{aligned}$$

b) From the given equation we express the vector $\vec{x}$ as a linear combination of the remaining vectors:

$$-2\,\vec{x}=2\,\vec{u}+2\,\vec{u}-2\,\vec{v}-4\,\vec{w} \Rightarrow \ \vec{x}=-2\,\vec{u}+\vec{v}+2\,\vec{w},$$

then in coordinates we obtain

$$\begin{aligned}\vec{x} &= -2(1;-2;3)+(4;1;0)+2(-3;5;1)=(-2+4-6;4+1+10;-6+0+2)= \\ &= (-4;15;-4).\end{aligned}$$

■

***Example* 5**: Given the vectors $\vec{a}=(2;3;0)$, $\vec{b}=(1;2;-3;-5)$, $\vec{c}=(0;-1;4;2)$, $\vec{d}=(1;3;-4;0)$. Find the vector $\vec{x}$ satisfying

a) $\vec{x}=2(\vec{a}+\vec{b})$ b) $2\,\vec{b}=\vec{c}-3\,\vec{x}+\vec{d}$

Solution:

a) $\vec{x}$ does not exist, because the vector $\vec{a}\in V(\mathbb{E}_3)$ and the vector $\vec{b}\in V(\mathbb{E}_4)$ (we know that $\vec{a}+\vec{b}$ exists only if $\vec{a}\in V(\mathbb{E}_k)$ and also $\vec{b}\in V(\mathbb{E}_k)$).

b) $3\,\vec{x}=\vec{c}+\vec{d}-2\,\vec{b} \quad \Rightarrow \quad \vec{x}=\dfrac{1}{3}\left(\vec{c}+\vec{d}-2\,\vec{b}\right) \quad \Rightarrow$

$$\begin{aligned}\vec{x} &= \frac{1}{3}\Big((0;-1;4;2)+(1;3;-4;0)-(2;4;-6;-10)\Big)=\frac{1}{3}\Big(-1;-2;6;12\Big)= \\ &= \Big(-\frac{1}{3};-\frac{2}{3};2;4\Big)\end{aligned}$$

■

***Example* 6**: Express the vectors $\vec{a}$ and $\vec{b}$ as a linear combination of the vectors $\vec{u}$ and $\vec{v}$:

a) $\vec{a}=(0;-7)$, $\vec{b}=(7;0)$, $\vec{u}=(1;-3)$, $\vec{v}=(2;1)$

b) $\vec{a}=(10;9;4)$, $\vec{b}=(0;3;1)$, $\vec{u}=(-1;0;2)$, $\vec{v}=(4;3;0)$

Solution:

a) We set up the linear combination and then express it in coordinates:

$$\vec{a} = \alpha_1\vec{u} + \alpha_2\vec{v} \quad \Rightarrow \quad (0;-7) = \alpha_1(1;-3) + \alpha_2\,(2;1),$$

$$(0;-7) = (\alpha_1 + 2\,\alpha_2; -3\,\alpha_1 + \alpha_2) \quad \Rightarrow \quad \left\{ \begin{array}{c} \alpha_1 + 2\,\alpha_2 = 0 \\ -3\,\alpha_1 + \alpha_2 = -7 \end{array} \right. .$$

Multiplying the first equation by the number 3 and adding both equations we obtain

$$7\,\alpha_2 = -7 \quad \Rightarrow \quad \alpha_2 = -1, \alpha_1 = 2. \qquad \text{Then} \quad \vec{a} = 2\,\vec{u} - \vec{v}.$$

Similarly $\vec{b} = \beta_1\vec{u} + \beta_2\vec{v} \quad \Rightarrow \quad (7;0) = \beta_1(1;-3) + \beta_2(2;1) \quad \Rightarrow \left\{ \begin{array}{c} \beta_1 + 2\,\beta_2 = 7 \\ -3\,\beta_1 + \beta_2 = 0 \end{array} \right. .$

We proceed as for the vector $\vec{a}$ and we successively obtain

$$7\,\beta_2 = 21 \quad \Rightarrow \quad \beta_2 = 3,\ \beta_1 = 1 \quad \Rightarrow \quad \vec{b} = \vec{u} + 3\,\vec{v}.$$

b) $\vec{a} = \alpha_1\vec{u} + \alpha_2\vec{v} \quad \Rightarrow \quad (10;9;4) = \alpha_1(-1;0;2) + \alpha_2(4;3;0) \quad \Rightarrow$

$\Rightarrow \quad (10;9;4) = (-\alpha_1 + 4\,\alpha_2; 3\,\alpha_2; 2\,\alpha_1) \quad \Rightarrow$

$$\left\{ \begin{array}{rcrcrcl} -\alpha_1 & + & 4\,\alpha_2 & = & 10 & & \\ & & 3\,\alpha_2 & = & 9 & \Rightarrow & \alpha_2 = 3 \\ 2\,\alpha_1 & & & = & 4 & \Rightarrow & \alpha_1 = 2 \end{array} \right. .$$

Substituting we verify that the first equation is satisfied as well, hence $\vec{a} = 2\,\vec{u} + 3\,\vec{v}$. Similarly

$$\begin{array}{rcl} \vec{b} = \beta_1\vec{u} + \beta_2\vec{v} & \Rightarrow & (0;3;1) = \beta_1(-1;0;2) + \beta_2(4;3;0) \\ & \Rightarrow & (0;3;1) = (-\beta_1 + 4\,\beta_2; 3\,\beta_2; 2\,\beta_1) \quad \Rightarrow \end{array}$$

$$\left\{ \begin{array}{rcrcrcl} -\beta_1 & + & 4\,\beta_2 & = & 0 & & \\ & & 3\,\beta_2 & = & 3 & \Rightarrow & \beta_2 = 1 \\ 2\,\beta_1 & & & = & 1 & \Rightarrow & \beta_1 = 1/2 \end{array} \right. .$$

Now, substituting into the first equation, we obtain the false statement $-1/2 + 4 = 0$. From this it follows that there exist no numbers $\beta_1,\ \beta_2$ satisfying the required relation, and the vector $\vec{b}$ cannot be expressed as a linear combination of the vectors $\vec{u}$ and $\vec{v}$.
In other words: the vectors $\vec{b},\ \vec{u}$ and $\vec{v}$ are linearly independent. ■

***Example* 7**: Decide whether the following groups of vectors are linearly dependent or linearly independent.

a) $\vec{a} = (2;3),\ \vec{b} = (1;-4),\ \vec{c} = (-1;5)$

b) $\vec{a} = (2;3;-1;0),\ \vec{b} = (0;4;3;-5),\ \vec{c} = (0;0;0;0)$

c) $\vec{a} = (3;-1;5),\ \vec{b} = (0;2;3;),\ \vec{c} = (0;0;2)$

d) $\vec{a} = (2;-1;3),\ \vec{b} = (1;0;2),\ \vec{c} = (-4;2;-6)$

e) $\vec{a} = (1;1;-1;-1),\ \vec{b} = (2;0;1;0),\ \vec{c} = (0;1;0;3)$

Solution:

a) In a two–dimensional space every group of three or more vectors is **dependent**.

b) Every group of vectors containing the zero vector is **dependent**.

c) Although the vectors are obviously linearly independent, we shall verify it by the definition. We set up the linear combination $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$ and determine the constants $\alpha,\ \beta,\ \gamma$.

$$\begin{aligned}\alpha(3;-1;5)+\beta(0;2;3)+\gamma(0;0;2)=(0;0;0) \quad &\Rightarrow\\ (3\,\alpha;-\alpha+2\,\beta;5\,\alpha+3\,\beta+2\,\gamma)=(0;0;0) \quad &\Rightarrow\end{aligned}$$

$$\left\{\begin{array}{rcl} 3\,\alpha &=& 0\\ -\alpha+2\,\beta &=& 0\\ 5\,\alpha+3\,\beta+2\,\gamma &=& 0\end{array}\right. .$$

Step by step we compute that only $\alpha=0$, then $\beta=0$ and $\gamma=0$ satisfy the relation $\alpha\,\vec{a}+\beta\,\vec{b}+\gamma\,\vec{c}=\vec{0}$, which by the definition means that the vectors $\vec{a},\vec{b},\vec{c}$ are **independent**.

d) Looking carefully at the vectors $\vec{a},\ \vec{b},\ \vec{c}$, we easily find that $\vec{c}=-2\,\vec{a}$ and therefore the whole triple is **linearly dependent**.

e) From the vector equation $\alpha\,\vec{a}+\beta\,\vec{b}+\gamma\,\vec{c}=\vec{0}$ written in coordinates

$$\alpha\,(1;1;-1;-1)+\beta\,(2;0;1;0)+\gamma\,(0;1;0;3)=(0;0;0;0)$$

we obtain a system for the constants $\alpha,\ \beta,\ \gamma$:

$$\left\{\begin{array}{rcrcrcl} \alpha &+& 2\,\beta &&&=& 0\\ \alpha &&&+&\gamma &=& 0\\ -\alpha &+& \beta &&&=& 0\\ -\alpha &&&+&\gamma &=& 0\end{array}\right. ,$$

from which we find that the only solution is $\alpha=\beta=\gamma=0$. This means that the vectors are **linearly independent**. ■

REMARK: In the next chapter **Matrices and determinants** we shall return to the investigation of the linear dependence or independence of a group of vectors.

***Example* 8**: Find out whether the following vectors form a basis of the given vector space:

a) $\vec{a}=(3;5;0),\ \vec{b}=(0;0;4),\ \vec{c}=(0;3;7),\quad V(\mathbb{E}_3)$

b) $\vec{a}=(2;4),\ \vec{b}=(-1;0),\ \vec{c}=(0;3),\quad V(\mathbb{E}_2)$

c) $\vec{a}=(-1;2;4),\ \vec{b}=(3;1;0),\ \vec{c}=(1;5;8),\quad V(\mathbb{E}_3)$

d) $\vec{a}=(1;3;7),\ \vec{b}=(-1;7;0),\quad V(\mathbb{E}_3)$

e) $\vec{a}=(1;3;-2),\ \vec{b}=(4;2;4),\ \vec{c}=(0;0;0),\quad V(\mathbb{E}_3)$

f) $\vec{a}=(2;0;-1),\ \vec{b}=(-1;3;1),\ \vec{c}=(0;1;0),\ \vec{d}=(4;-3;1),\quad V(\mathbb{E}_3)$

g) $\vec{a}=(1;2;3;1),\ \vec{b}=(0;1;2;-3),\ \vec{c}=(0;0;1;2),\ \vec{d}=(0;0;0;1),\ V(\mathbb{E}_4)$

Solution:

a) We easily find that the vectors $\vec{a},\vec{b},\vec{c}$ are linearly independent, and therefore they **form** a basis of the space $V(\mathbb{E}_3)$. Let us recall the definition: *In the space $V(\mathbb{E}_n)$ every linearly independent group of n vectors forms a basis of this space.*

b) Three vectors in a two–dimensional space are always dependent, which means that the given vectors $\vec{a},\ \vec{b},\ \vec{c}$ **do not form** a basis of $V(\mathbb{E}_2)$.

c) Here we find that the vectors satisfy the dependence $2\,\vec{a}+\vec{b}=\vec{c}$ and therefore the given vectors **do not form** a basis of $V(\mathbb{E}_3)$.

d) The answer is obvious: the vectors **do not form** a basis, since a basis of $V(\mathbb{E}_3)$ is formed by three linearly independent vectors and here only two vectors are given.

e) Again they **do not form** one, because every group of vectors containing the zero vector is **always** dependent.

f) They **do not form** a basis, because the number of vectors is greater than 3, which is the dimension of the space, hence the vectors are dependent.

g) From the vector equation $\alpha\,\vec{a}+\beta\,\vec{b}+\gamma\,\vec{c}+\delta\,\vec{d}=\vec{0}$ we obtain the system

$$\left\{\begin{array}{r} \alpha=0 \\ 2\,\alpha+\beta=0 \\ 3\,\alpha+2\,\beta+\gamma=0 \\ \alpha-3\,\beta+2\,\gamma+\delta=0 \end{array}\right. ,$$

from which we easily compute: $\alpha=\beta=\gamma=\delta=0$. Hence the given group of four vectors is linearly independent and **forms** a basis of the space $V(\mathbb{E}_4)$. ■

***Example* 9**: Verify whether the set V together with the given operations is a vector space:

a) $V=\{a_1+a_2x+a_3x^2;\ a_1,a_2,a_3,x\in\mathbb{R}\}$, where the operations the addition of polynomials and the multiplication by a real number are defined in the usual way

b) $V=\{(a;1-a;b);\ a,b\in\mathbb{R}\}$, where the addition and the multiplication by a number are the usual operations between the vectors

c) V is the set of all increasing functions defined on the interval $\langle a;b\rangle$.

Solution: In order that a non–empty set V forms a vector space, the following conditions must hold for every $\vec{u},\vec{v}\in V$ and $k,k_1,k_2\in\mathbb{R}$:

1) $\vec{u},\vec{v}\in V\Longrightarrow\vec{u}+\vec{v}\in V$
2) $\vec{u}\in V\Longrightarrow k\,\vec{u}\in V$
3) $\vec{0}=(0;\cdots;0)\in V$
4) $\vec{u}\in V\Longrightarrow(-\vec{u})\in V$ and $\vec{u}+(-\vec{u})=\vec{0}$
5) $k_1\,(k_2\,\vec{u})=(k_1\,k_2)\,\vec{u}$,
$(k_1+k_2)\,\vec{u}=k_1\,\vec{u}+k_2\,\vec{u}$,
$k\,(\vec{u}+\vec{v})=k\,\vec{u}+k\,\vec{v}$

a) If we assign to the polynomial $a_1+a_2x+a_3x^2$ the vector $\vec{a}=(a_1;a_2;a_3)$, then very easily we verify that all the conditions hold. For example,

$$\vec{a}=(a_1;a_2;a_3),\ \vec{b}=(b_1;b_2;b_3):$$

$$\vec{a}+\vec{b}=(a_1+b_1;a_2+b_2;a_3+b_3)\quad\longleftrightarrow$$

$$a_1+a_2x+a_3x^2+b_1+b_2x+b_3x^2=a_1+b_1+(a_2+b_2)\,x+(a_3+b_3)\,x^2;$$

$$k\,\vec{a}=(k\,a_1;k\,a_2;k\,a_3)\quad\longleftrightarrow\quad k(a_1+a_2x+a_3x^2)=k\,a_1+k\,a_2x+k\,a_3x^2;$$

$$\vec{0}=(0;0;0)\quad\longleftrightarrow\quad 0+0\,x+0\,x^2\qquad\text{etc.}$$

Here we have considered the set of polynomials of the 2nd degree, but we can easily verify that the set of all polynomials of the n-th degree **forms** a vector space as well.

b) Let us choose two elements $\vec{u} = (a_1; 1 - a_1; b_1)$ and $\vec{v} = (a_2; 1 - a_2; b_2)$ of the set V and let us try to form $\vec{u} + \vec{v} = (a_1 + a_2; 2 - a_1 - a_2; b_1 + b_2)$. It is obvious that $\vec{u} + \vec{v} \notin V$, because $2 - a_1 - a_2 \neq 1 - (a_1 + a_2)$. There is no need to verify the further conditions.
The set V **does not form** a vector space.

c) Let us have two increasing functions f and g on $\langle a; b\rangle$. Then $f + g$ is an increasing function as well, however $(-f)$ is a decreasing function, that is $(-f) \notin V$ and the given set V **does not form** a vector space. ■

10. Given the vectors $\vec{a} = (1; 2; 0)$, $\vec{b} = (0; -1; 3)$, $\vec{c} = (4; 5; -3)$. Compute the vector $\vec{x}$:

a) $\vec{x} = 2\,\vec{a} + 3\,\vec{b} - \vec{c}$ b) $2\,(\vec{x} + \vec{a}) = 3\,(\vec{c} + \vec{b}) - \vec{x}$

$\left[\text{a) } (-2; -4; 12);\ \text{b) } \frac{1}{3}\big(10; 8; 0\big)\right]$

11. Find the numbers $x,\ y,\ z \in \mathbb{R}$ so that for the vectors $\vec{a} = (x; y; 1)$, $\vec{b} = (1; x; z)$, $\vec{c} = (4; 3; -3)$ the equality $\vec{a} + 2\,\vec{b} = \vec{c}$.

$[x = 2,\ y = -1,\ z = -2]$

12. Find the vector $\vec{x}$ satisfying the equation $2\,(\vec{x} - \vec{u}) = \vec{x} - 4\,(\vec{x} + 2\,\vec{u} + 3\,\vec{v})$, where $\vec{u} = (-1; 2; 5; 1)$ and $\vec{v} = (4; -2; -3; 0)$.

$\left[\frac{1}{5}\big(-42; 12; 6; -6\big)\right]$

13. Find the vectors $\vec{a}$ and $\vec{b}$ as linear combinations of the vectors $\vec{u}$ and $\vec{v}$ and conversely $\vec{u}$ and $\vec{v}$ as a linear combination of $\vec{a}$ and $\vec{b}$, if $\vec{a} = (0; -11)$, $\vec{b} = (11; 0)$, $\vec{u} = (3; -1)$, $\vec{v} = (2; 3)$.

$$\left[\begin{array}{c} \vec{a} = 2\vec{u} - 3\vec{v},\ \ \vec{b} = 3\vec{u} + \vec{v} \\ \vec{u} = \frac{1}{11}\big(3\vec{b} + \vec{a}\big),\ \ \vec{v} = \frac{1}{11}\big(2\vec{b} - 3\vec{a}\big) \end{array}\right]$$

14. Find out whether the vectors form a basis of the vector space $V(\mathbb{E}_3)$.

a) $(2; 2; 4),\ (0; 3; 7),\ (3; 5; 0)$ b) $(3; 8; 2),\ (4; -7; 3)$

c) $(1; 5; -3),\ (0; 0; 0),\ (4; 0; 1)$ d) $(2; 3; -1),\ (1; 0; 0),\ (2; 3; 0),\ (0; 1; -2)$

e) $(2; -1; 4),\ (0; 1; 3),\ (0; 0; 5)$

[a) yes; b) no; c) no; d) no; e) yes]

I.2 Matrices and determinants

***Example* 15**: Given the matrices $\boldsymbol{A} = \begin{pmatrix} 3 & 4 & 0 \\ -1 & 2 & 5 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 2 & -7 \end{pmatrix}$.
Compute the matrix:

a) $3\,\boldsymbol{A}$ b) $\boldsymbol{A} + \boldsymbol{B}$ c) $2\,\boldsymbol{A} - 3\,\boldsymbol{B}$ d) $\boldsymbol{A} \cdot \boldsymbol{B}$

Solution:

a) $3\,\boldsymbol{A} = 3 \begin{pmatrix} 3 & 4 & 0 \\ -1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 12 & 0 \\ -3 & 6 & 15 \end{pmatrix}$

b) The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are of the same type $(2,3)$, therefore they can be added:

$$\begin{aligned}\boldsymbol{A}+\boldsymbol{B} &= \begin{pmatrix} 3 & 4 & 0 \\ -1 & 2 & 5 \end{pmatrix} + \begin{pmatrix} -1 & 2 & 3 \\ 0 & 2 & -7 \end{pmatrix} = \begin{pmatrix} 3-1 & 4+2 & 0+3 \\ -1+0 & 2+2 & 5-7 \end{pmatrix} = \\ &= \begin{pmatrix} 6 & 8 & 0 \\ -2 & 4 & 10 \end{pmatrix} - \begin{pmatrix} -3 & 6 & 9 \\ 0 & 6 & -21 \end{pmatrix} = \begin{pmatrix} 6+3 & 8-6 & 0-9 \\ -2-0 & 4-6 & 10+21 \end{pmatrix} = \\ &= \begin{pmatrix} 9 & 2 & -9 \\ -2 & -2 & 31 \end{pmatrix};\end{aligned}$$

c) $\boldsymbol{A}\cdot\boldsymbol{B}$ does not exist, because the product $\boldsymbol{A}\cdot\boldsymbol{B}$ is defined only if the matrix $\boldsymbol{A}$ is of the type $(m,\underline{n})$ and the matrix $\boldsymbol{B}$ is of the type $(\underline{n},p)$, then the matrix $\boldsymbol{A}\cdot\boldsymbol{B}$ is of the type (m,p). ■

***Example* 16**: Let $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -1 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ -2 & 5 \end{pmatrix}$. Compute $\boldsymbol{A}\cdot\boldsymbol{B}$, $\boldsymbol{B}\cdot\boldsymbol{A}$, $\boldsymbol{A}^2$, $\boldsymbol{AB}-\boldsymbol{BA}$, if such matrices exist.

Solution: The products $\boldsymbol{A}\cdot\boldsymbol{B}$ and $\boldsymbol{B}\cdot\boldsymbol{A}$ exist. Let us denote $\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{C}$, $\boldsymbol{B}\cdot\boldsymbol{A}=\boldsymbol{D}$.

The matrix $\boldsymbol{A}$ is of the type $(2,3)$ and the matrix $\boldsymbol{B}$ is of the type $(3,2)$, hence the matrix $\boldsymbol{C}$ will be of the type $(2,2)$ and the matrix $\boldsymbol{D}$ will be of the type $(3,3)$.

The entries of the matrix $\boldsymbol{C}$ are computed according to the rule for the product:

$$\begin{aligned} c_{11} &= a_{11}\,b_{11}+a_{12}\,b_{21}+a_{13}\,b_{31} = 1\cdot 2+(-2)\cdot 0+3\cdot(-2) = -4 \\ c_{12} &= a_{11}\,b_{12}+a_{12}\,b_{22}+a_{13}\,b_{32} = 1\cdot(-1)+(-2)\cdot 3+3\cdot 5 = 8 \\ c_{21} &= a_{21}\,b_{11}+a_{22}\,b_{21}+a_{23}\,b_{31} = 4\cdot 2+(-1)\cdot 0+2\cdot(-2) = 4 \\ c_{22} &= a_{21}\,b_{12}+a_{22}\,b_{22}+a_{23}\,b_{32} = 4\cdot(-1)+(-1)\cdot 3+2\cdot 5 = 3. \end{aligned}$$

This product can also be computed very clearly by means of a table:

	$\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ -2 & 5 \end{pmatrix} = \boldsymbol{B}$	
$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -1 & 2 \end{pmatrix}$	$1\cdot 2-2\cdot 0+3\cdot(-2)$	$1\cdot(-1)-2\cdot 3+3\cdot 5$
	$4\cdot 2-1\cdot 0+2\cdot(-2)$	$4\cdot(-1)-1\cdot 3+2\cdot 5$

$= \begin{pmatrix} -4 & 8 \\ 4 & 3 \end{pmatrix} = \boldsymbol{C}$

Similarly we compute $\boldsymbol{B}\cdot\boldsymbol{A}=\boldsymbol{D}$ again by means of a table.

	$\begin{pmatrix} 1 & -2 & 3 \\ 4 & -1 & 2 \end{pmatrix} = \boldsymbol{A}$		
$\boldsymbol{B} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ -2 & 5 \end{pmatrix}$	2-4	-4+1	6-2
	0+12	0-3	0+6
	-2+20	4-5	-6+10

$= \begin{pmatrix} -2 & -3 & 4 \\ 12 & -3 & 6 \\ 18 & -1 & 4 \end{pmatrix} = \boldsymbol{D}$

The product $\boldsymbol{A}^2=\boldsymbol{A}\cdot\boldsymbol{A}$ does not exist and similarly the difference $\boldsymbol{AB}-\boldsymbol{BA}$ does not exist either, because the products $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are not matrices of the same type. ■

***Example* 17**: Let $\boldsymbol{A} = \begin{pmatrix} 3 & 5 \\ 6 & -1 \end{pmatrix}$. Compute $\boldsymbol{A}^2$ and $\boldsymbol{A}\cdot\boldsymbol{A}^T$.

Solution:

$$\boldsymbol{A}^2 = \boldsymbol{A}\cdot\boldsymbol{A} = \begin{pmatrix} 3 & 5 \\ 6 & -1 \end{pmatrix}\cdot\begin{pmatrix} 3 & 5 \\ 6 & -1 \end{pmatrix} = \begin{pmatrix} 3\cdot 3+5\cdot 6 & 3\cdot 5+5\cdot(-1) \\ 6\cdot 3-1\cdot 6 & 6\cdot 5+(-1)\cdot(-1) \end{pmatrix} = \\ = \begin{pmatrix} 39 & 10 \\ 12 & 31 \end{pmatrix};$$

$$\boldsymbol{A}\cdot\boldsymbol{A}^T = \begin{pmatrix} 3 & 5 \\ 6 & -1 \end{pmatrix}\cdot\begin{pmatrix} 3 & 6 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 9+25 & 18-5 \\ 18-5 & 36+1 \end{pmatrix} = \begin{pmatrix} 34 & 13 \\ 13 & 37 \end{pmatrix}.$$ ■

***Example* 18**: Compute $\boldsymbol{A}\cdot\boldsymbol{B}$, $\boldsymbol{B}\cdot\boldsymbol{C}$, $\boldsymbol{C}\cdot\boldsymbol{B}$, where

$$\boldsymbol{A} = \begin{pmatrix} 3 & 4 & -5 & 1 \\ 2 & 6 & -3 & -4 \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} 2 \\ 1 \\ -5 \\ 5 \end{pmatrix},\ \boldsymbol{C} = \begin{pmatrix} 4 & 5 & 6 & -3 \end{pmatrix}.$$

Solution:

$$\boldsymbol{A}\cdot\boldsymbol{B} = \begin{pmatrix} 3 & 4 & -5 & 1 \\ 2 & 6 & -3 & -4 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 1 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 6+4+25+5 \\ 4+6+15-20 \end{pmatrix} = \begin{pmatrix} 40 \\ 5 \end{pmatrix};$$

$$\boldsymbol{B}\cdot\boldsymbol{C} = \begin{pmatrix} 2 \\ 1 \\ -5 \\ 5 \end{pmatrix}\cdot\begin{pmatrix} 4 & 5 & 6 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 & -6 \\ 4 & 5 & 6 & -3 \\ -20 & -25 & -30 & 15 \\ 20 & 25 & 30 & -15 \end{pmatrix}$$

$$\boldsymbol{C}\cdot\boldsymbol{B} = \begin{pmatrix} 4 & 5 & 6 & -3 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 1 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 8+5-30-15 \end{pmatrix} = (-32).$$ ■

***Example* 19**: Compute the matrix $\boldsymbol{X}$ for which there holds $\boldsymbol{A}\cdot\boldsymbol{B} = 2\,\boldsymbol{X} + \boldsymbol{A}^T$, where

$$\boldsymbol{A} = \begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} 3 & 1 \\ 2 & -5 \end{pmatrix}.$$

Solution: $2\,\boldsymbol{X} = \boldsymbol{A}\cdot\boldsymbol{B} - \boldsymbol{A}^T \Rightarrow \boldsymbol{X} = \frac{1}{2}\left(\boldsymbol{A}\cdot\boldsymbol{B} - \boldsymbol{A}^T\right)$,

$$\boldsymbol{X} = \frac{1}{2}\left(\begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix}\cdot\begin{pmatrix} 3 & 1 \\ 2 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -3 & 4 \end{pmatrix}\right) = \frac{1}{2}\left(\begin{pmatrix} 0 & 17 \\ 8 & -20 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -3, & 4 \end{pmatrix}\right) = \\ = \frac{1}{2}\begin{pmatrix} -2 & 17 \\ 11 & -24 \end{pmatrix} = \begin{pmatrix} -1 & \frac{17}{2} \\ \frac{11}{2} & -12 \end{pmatrix}.$$ ■

***Example* 20**: Find $x, y \in \mathbb{R}$ such that the equation holds

$$\begin{pmatrix} 7 & x+y \\ x-y & 12 \end{pmatrix} = \left(\begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix}\cdot\begin{pmatrix} 1 & -1 \\ 4 & 5 \end{pmatrix}\right)^T.$$

Solution: First of all we carry out the product on the right hand side, then we determine the transposed matrix and finally we compare the resulting matrix with the matrix on the left hand side.

$$\left(\begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix}\cdot\begin{pmatrix} 1 & -1 \\ 4 & 5 \end{pmatrix}\right)^T = \begin{pmatrix} 3+4 & -3+5 \\ -2+8 & 2+10 \end{pmatrix}^T = \begin{pmatrix} 7 & 2 \\ 6 & 12 \end{pmatrix}^T = \begin{pmatrix} 7 & 6 \\ 2 & 12 \end{pmatrix},$$
$$\begin{pmatrix} 7 & x+y \\ x-y & 12 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 2 & 12 \end{pmatrix}.$$

From this we obtain the system $\begin{cases} x+y=6 \\ x-y=2 \end{cases} \quad \Rightarrow \quad x=4,\ y=2.$ ■

***Example* 21**: Find the rank of the matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 1 & 1 & -1 \\ 3 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 4 & -3 & 0 & 0 & -2 \end{pmatrix}.$$

Solution: The matrix is of the type $(4,5)$, and that means that its rank will be $\mathrm{r}(\boldsymbol{A}) \le 4$. In order to determine the rank $\mathrm{r}(\boldsymbol{A})$, we transform the matrix to the triangular form by operations which preserve its rank. The first operation will be a change of the order of the columns: as the first we write the fourth, then the fifth, the third, the second and the first column.

$$\boldsymbol{A} \sim \begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 3 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & 0 & -3 & 4 \end{pmatrix} \overset{1)}{\sim} \begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 2 & 1 & 2 & -1 \\ 0 & -2 & 0 & -3 & 4 \end{pmatrix} \overset{2)}{\sim} \begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & -2 & 0 & -3 & 4 \end{pmatrix} \overset{3)}{\sim}$$
$$\overset{3)}{\sim} \begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 & 3 \end{pmatrix} \overset{4)}{\sim} \begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathrm{r}(\boldsymbol{A}) = \min\{3;5\} = 3.$$

Description of the operations:

1) To the third row we add the first row multiplied by (-1), the other rows are copied.
2) We interchange the second and the third row.
3) To the fourth row we add the second row.
4) To the fourth row we add the third one multiplied by (-1).

In the triangular form three nonzero rows are left, which means that the rank is $\mathrm{r}(\boldsymbol{A}) = 3$. The last operation 4) was rather unnecessary. It was enough to notice that the last two rows are dependent and one of them could be immediately omitted and we could write $\mathrm{r}(\boldsymbol{A}) = 3$. ■

***Example* 22**: Find the rank of the matrix depending on the choice of the parameter $k \in \mathbb{R}$

$$\boldsymbol{A} = \begin{pmatrix} 0 & 3 & 1-k \\ 3 & 1+k & 0 \\ k & 2 & 0 \end{pmatrix}.$$

Solution: We write the third column as the first one and shift the others one place to the right

$$\boldsymbol{A} \sim \begin{pmatrix} 1-k & 0 & 3 \\ 0 & 3 & 1+k \\ 0 & k & 2 \end{pmatrix} \overset{1)}{\sim} \begin{pmatrix} 1-k & 0 & 3 \\ 0 & 3 & 1+k \\ 0 & 0 & 6-k-k^2 \end{pmatrix}.$$

1) To three times the third row we add the second row multiplied by $-k$.

For $k = 0$ the matrix is regular. In this way we obtain a triangular matrix whose rank depends on the entries on the main diagonal.

a) $\mathrm{r}(\boldsymbol{A}) = 3 \iff \left\{ \begin{array}{r} 1-k \neq 0 \\ 6-k-k^2 \neq 0 \end{array} \right\} \iff \left\{ \begin{array}{l} k \neq 1 \\ k \neq \{2; -3\} \end{array} \right.$,

$$\mathrm{r}(\boldsymbol{A}) = 3 \iff k \in \mathbb{R} \backslash \{-3; 1; 2\}.$$

b) If we substitute into the triangular matrix some $k \in \{-3; 1; 2\}$, there always remain two nonzero independent rows, or columns. Hence

$$\mathrm{r}(\boldsymbol{A}) = 2 \iff k \in \{-3; 1; 2\}. \qquad \blacksquare$$

***Example* 23**: Find the rank of the matrix $\boldsymbol{A}$ depending on the choice of the parameter $a \in \mathbb{R}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 3 & 4 & 0 \\ -2 & 1 & -1 & -10 & 1 \\ 1 & -1 & a-3 & 6 & -1 \\ 3 & -2 & a-2 & 16 & 0 \\ 2 & 0 & 7-a & a^2-a+8 & 0 \end{pmatrix}.$$

Solution: We write the columns in the following order: 5th col., 2nd col., 1st col., 3rd col., 4th column:

$$\boldsymbol{A} \sim \begin{pmatrix} 0 & 0 & 1 & 3 & 4 \\ 1 & 1 & -2 & -1 & -10 \\ -1 & -1 & 1 & a-3 & 6 \\ 0 & -2 & 3 & a-2 & 16 \\ 0 & 0 & 2 & 7-a & a^2-a+8 \end{pmatrix} \overset{1)}{\sim} \begin{pmatrix} 1 & 1 & -2 & -1 & -10 \\ 0 & -2 & 3 & a-2 & 16 \\ 0 & 0 & 1 & 3 & 4 \\ -1 & -1 & 1 & a-3 & 6 \\ 0 & 0 & 2 & 7-a & a^2-a+8 \end{pmatrix} \overset{2)}{\sim}$$

$$\overset{2)}{\sim} \begin{pmatrix} 1 & 1 & -2 & -1 & -10 \\ 0 & -2 & 3 & a-2 & 16 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & -1 & a-4 & -4 \\ 0 & 0 & 2 & 7-a & a^2-a+8 \end{pmatrix} \overset{3)}{\sim} \begin{pmatrix} 1 & 1 & -2 & -1 & -10 \\ 0 & -2 & 3 & a-2 & 16 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & 1-a & a^2-a \end{pmatrix} \overset{4)}{\sim}$$

$$\overset{4)}{\sim} \begin{pmatrix} 1 & 1 & -2 & -1 & -10 \\ 0 & -2 & 3 & a-2 & 16 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & 0 & a(a-1) \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} \mathrm{r}(\boldsymbol{A}) = \min\{3; 5\} = 3 \text{ for } a = 1, \\ \mathrm{r}(\boldsymbol{A}) = \min\{4; 5\} = 4 \text{ for } a = 0, \\ \mathrm{r}(\boldsymbol{A}) = \min\{5; 5\} = 5 \text{ for } a \in \mathbb{R} \backslash \{0; 1\}. \end{array}$$

Description of the operations:

1) the new order of the rows: 2nd r., 4th r., 1st r., 3rd r., 5th row,
2) to the 4th row we add the 1st row,
3) to the 4th row we add the 3rd row and to the 5th row we add the 3rd row multiplied by (-2),
4) to the 5th row we add the 4th row. $\blacksquare$

***Example* 24**: Let us have the vectors $\vec{a} = (1;3;0;2)$, $\vec{b} = (-1;0;1;1)$, $\vec{c} = (0;1;4;-1)$, $\vec{d} = (0;4;5;2)$. Determine whether the vectors are linearly independent or dependent. What is the dimension of the vector space V which is spanned by the given vectors?

Solution: From the given vectors we form the matrix $\boldsymbol{A}$ and determine its rank $\mathrm{r}(\boldsymbol{A})$. If $\mathrm{r}(\boldsymbol{A}) = 4$ (the number of the vectors), then the vectors are independent; if $\mathrm{r}(\boldsymbol{A}) < 4$, then the vectors are dependent and in both cases $\dim V = \mathrm{r}(\boldsymbol{A})$.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 0 & 2 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 4 & 5 & 2 \end{pmatrix} \overset{1)}{\sim} \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 1 & 4 & -1 \\ 0 & 4 & 5 & 2 \end{pmatrix} \overset{2)}{\sim} \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 3 & 1 & 3 \\ 0 & 4 & 5 & 2 \end{pmatrix} \overset{3)}{\sim}$$

$$\sim \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & -11 & 6 \\ 0 & 0 & -11 & 6 \end{pmatrix} \Rightarrow \quad \mathrm{r}(\boldsymbol{A}) = \min\{3;4\} = 3 \Rightarrow \text{ the vectors are linearly dependent and } \dim V = 3.$$

Description of the operations:

1) to the 2nd row we add the 1st row,

2) we interchange the 2nd and 3rd row,

3) to the 3rd row we add the 2nd row multiplied by (-3) and to the 4th row we add the 2nd row multiplied by (-4). ■

***Example* 25**: Determine for which values of the parameter $k \in \mathbb{R}$ the vector $\vec{c}$ will be a linear combination of the vectors $\vec{a}$ and $\vec{b}$: $\vec{a} = (-1;-1;2)$, $\vec{b} = (2;1;-1)$, $\vec{c} = (k;0;2\,k+1)$.

Solution: The vector $\vec{c}$ is a linear combination of the vectors $\vec{a}$ and $\vec{b}$ if and only if the triple of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is linearly dependent. That happens if and only if the matrix formed from the coordinates of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ has the rank $\mathrm{r}(\boldsymbol{A}) < 3$:

$$\boldsymbol{A} = \begin{pmatrix} -1 & -1 & 2 \\ 2 & 1 & -1 \\ k & 0 & 2\,k+1 \end{pmatrix} \overset{1)}{\sim} \begin{pmatrix} -1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & k & 2\,k+1 \end{pmatrix} \overset{2)}{\sim} \begin{pmatrix} -1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & k & 2\,k+1 \end{pmatrix} \overset{3)}{\sim}$$

$$\sim \begin{pmatrix} -1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & k+1 \end{pmatrix} \Rightarrow\ k+1=0 \ \Rightarrow\ k=-1 \ \Rightarrow \quad \vec{c} = (-1;0;-1) = -\vec{a} - \vec{b}.$$

Description of the operations:

1) we interchange the 1st and 2nd column,

2) to the 2nd row we add the 1st row,

3) to the 3rd row we add the 2nd row multiplied by $(-k)$. ■

***Example* 26**: Compute the determinants:

a) $\begin{vmatrix} 3 & 5 \\ -4 & 2 \end{vmatrix}$; b) $\begin{vmatrix} 2+i & 3-2\,i \\ 3+2\,i & 2-i \end{vmatrix}$; c) $\begin{vmatrix} \sin\alpha & \cos\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$

Solution:

a) $\begin{vmatrix} 3 & 5 \\ -4 & 2 \end{vmatrix} = 3 \cdot 2 - (-4 \cdot 5) = 26$

b) $\begin{vmatrix} 2+i & 3-2\,i \\ 3+2\,i & 2-i \end{vmatrix} = (2+i)(2-i) - (3+2\,i)(3-2\,i) = 4+1-(9+4) = -8$

c) $\begin{vmatrix} \sin\alpha & \cos\alpha \\ \sin\beta & \cos\beta \end{vmatrix} = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$ ■

***Example* 27**: Compute the determinants by Sarrus' rule:

$$\text{a)} \begin{vmatrix} 1 & 2 & 1 \\ 2 & -3 & 4 \\ -3 & 5 & -7 \end{vmatrix}; \qquad \text{b)} \begin{vmatrix} 1 & a & a\,c \\ 1 & b & b\,c \\ 1 & c & c^2 \end{vmatrix}; \qquad \text{c)} \begin{vmatrix} 4 & 49 & 80 \\ -1 & 7 & 0 \\ 2 & 21 & 40 \end{vmatrix}$$

Solution:

a) To the determinant we append the first two rows, then we add the products in the direction of the main diagonal and subtract the products in the direction of the antidiagonal:

$$\begin{array}{c} \searrow + \qquad\quad - \swarrow \\ \begin{vmatrix} 1 & 2 & 1 \\ 2 & -3 & 4 \\ -3 & 5 & -7 \end{vmatrix} \\ \begin{matrix} 1 & 2 & 1 \\ 2 & -3 & 4 \end{matrix} \end{array} = 1\cdot(-3)\cdot(-7) + 2\cdot 5\cdot 1 + (-3)\cdot 2\cdot 4 - \Big(1\cdot(-3)\cdot(-3) + 4\cdot 5\cdot 1 + (-7)\cdot 2\cdot 2\Big) = 21 + 10 - 24 - 9 - 20 + 28 = 6.$$

b) $$\begin{array}{c} \searrow + \qquad\quad - \swarrow \\ \begin{vmatrix} 1 & a & a\,c \\ 1 & b & b\,c \\ 1 & c & c^2 \end{vmatrix} \\ \begin{matrix} 1 & a & a\,c \\ 1 & b & b\,c \end{matrix} \end{array} = b\,c^2 + a\,c^2 + a\,b\,c - a\,b\,c - b\,c^2 - a\,c^2 = 0.$$

We could arrive at the same result after transforming the corresponding matrix. It is enough to notice that the third column is a multiple of the second column, that is, the columns are linearly dependent and the determinant has the value 0.

c) We surely notice that from the second column we can factor out 7 in front of the determinant and from the third column 40. In this way we simplify the expression and then we compute:

$$\begin{vmatrix} 4 & 49 & 80 \\ -1 & 7 & 0 \\ 2 & 21 & 40 \end{vmatrix} = 7\cdot 40 \cdot \begin{array}{c} \begin{vmatrix} 4 & 7 & 2 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{vmatrix} \\ \begin{matrix} 4 & 7 & 2 \\ -1 & 1 & 0 \end{matrix} \end{array} = 7\cdot 40\cdot(4-6-4+7) = 280.$$ ■

***Example* 28**: Compute the determinant by the expansion along some row or column:

$$\det \boldsymbol{A} = \begin{vmatrix} 7 & 6 & 9 & -7 \\ 3 & 1 & 3 & 3 \\ -1 & -1 & 2 & -2 \\ -1 & -2 & -1 & 3 \end{vmatrix}.$$

Solution: We choose the expansion along the first row, since its entries are relatively high numbers

$$\det \boldsymbol{A} = a_{11}\,A_{11} + a_{12}\,A_{12} + a_{13}\,A_{13} + a_{14}\,A_{14},$$

where $A_{ij} = (-1)^{i+j}\,A^*_{ij}$ and A^*_{ij} is the determinant of the matrix of the type (3, 3) which arises from the original one $\boldsymbol{A}$ by omitting the i - th row and the j - th column.

$$\det \boldsymbol{A} = 7\cdot(-1)^2\begin{vmatrix} 1 & 3 & 3 \\ -1 & 2 & -2 \\ -2 & -1 & 3 \end{vmatrix} + 6\cdot(-1)^3\begin{vmatrix} 3 & 3 & 3 \\ -1 & 2 & -2 \\ -1 & -1 & 3 \end{vmatrix} +$$

$$+ 9\cdot(-1)^4\begin{vmatrix} 3 & 1 & 3 \\ -1 & -1 & -2 \\ -1 & -2 & 3 \end{vmatrix} - 7\cdot(-1)^5\begin{vmatrix} 3 & 1 & 3 \\ -1 & -1 & 2 \\ -1 & -2 & -1 \end{vmatrix} =$$

$$= 7(6+3+12+12-2+9) - 6(18+6+3+6-6+9) + 9(-9+2+6-3-12+3) +$$
$$+ 7(3-2+6-3+12-1) = 7\cdot 40 - 6\cdot 36 + 9\cdot(-13) + 7\cdot 15 = 280 - 216 - 117 + 105 = 52.$$

We hope that the reader has verified that such a direct expansion is very lengthy.
In the following examples we shall proceed more cleverly. ■

***Example* 29**: Compute the determinant by a suitable transformation and a suitable expansion along a row or along a column:

$$\det \boldsymbol{A} = \begin{vmatrix} 4 & 4 & 8 & 2 \\ 1 & 3 & 4 & 1 \\ 1 & 2 & 4 & 3 \\ -3 & -3 & -4 & -1 \end{vmatrix}.$$

Solution: From the first row we factor out 2 in front of the determinant and from the third column 4, then we transform the third column so that in the first place there remains 1 and further there are only zeros:

$$\det \boldsymbol{A} = 2\cdot 4\begin{vmatrix} 2 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 3 \\ -3 & -3 & -1 & -1 \end{vmatrix} = 8\begin{vmatrix} 2 & 2 & \boxed{1} & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ -1 & -1 & 0 & 0 \end{vmatrix} =$$

$$= 8\cdot\boxed{2}\cdot(-1)^{2+3}\cdot\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = -16\cdot(1+1) = -32.$$ ■

***Example* 30**: Compute the determinant by transforming the matrix to the triangular form:

$$\det \boldsymbol{A} = \begin{vmatrix} 1 & 3 & 4 & -3 \\ -2 & -4 & -7 & 8 \\ 2 & 10 & 4 & -9 \\ 2 & 12 & 5 & 7 \end{vmatrix}.$$

Solution:

$$\det \boldsymbol{A} = \begin{vmatrix} 1 & 3 & 4 & -3 \\ -2 & -4 & -7 & 8 \\ 2 & 10 & 4 & -9 \\ 2 & 12 & 5 & 7 \end{vmatrix} \overset{1)}{=} \begin{vmatrix} 1 & 3 & 4 & -3 \\ 0 & 2 & 1 & 2 \\ 0 & 4 & -4 & -3 \\ 0 & 6 & -3 & 13 \end{vmatrix} \overset{2)}{=} \begin{vmatrix} 1 & 3 & 4 & -3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & -6 & 7 \end{vmatrix} \overset{3)}{=}$$

$$= \begin{vmatrix} 1 & 3 & 4 & -3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 0 & 14 \end{vmatrix} \overset{4)}{=} 1 \cdot \begin{vmatrix} 2 & 1 & 2 \\ 0 & -6 & -7 \\ 0 & 0 & 14 \end{vmatrix} \overset{5)}{=} 1 \cdot 2 \begin{vmatrix} -6 & -7 \\ 0 & 14 \end{vmatrix} = 1 \cdot 2 \cdot (-6) \cdot 14 = -168.$$

Description of the operations:

1) to the 2nd row we add the 1st row multiplied by 2, to the 3rd row we add the 1st row multiplied by (-2) and to the 4th row we add the 1st row multiplied by (-2);

2) to the 3rd row we add the 2nd row multiplied by (-2) and to the 4th row we add the 2nd row multiplied by (-3);

3) from the 4th row we subtract the 3rd row;

4) we carry out the expansion along the 1st column;

5) we carry out the expansion along the 1st column.

If we look carefully at the triangular form of the matrix $\boldsymbol{A}$, we find that the value of the determinant equals the product of the numbers on the main diagonal, which is a well known statement.

Hence the operations 4) and 5) were unnecessary and

$$\det \boldsymbol{A} = \begin{vmatrix} \boxed{1} & 3 & 4 & -3 \\ 0 & \boxed{2} & 1 & 2 \\ 0 & 0 & \boxed{-6} & -7 \\ 0 & 0 & 0 & \boxed{14} \end{vmatrix} = 1 \cdot 2 \cdot (-6) \cdot 14 = -168. \qquad \blacksquare$$

***Example* 31**: For the given matrices compute the inverse matrices, if they exist.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}; \ \boldsymbol{B} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}; \ \boldsymbol{C} = \begin{pmatrix} 3 & -2 \\ -6 & 4 \end{pmatrix}; \ \boldsymbol{D} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \end{pmatrix}.$$

Solution: The matrix $\boldsymbol{A}^{-1}$ does not exist, because $\boldsymbol{A}$ is not square,

$$\begin{array}{rcll} \det \boldsymbol{B} & = & 6 - 2 = 4 \neq 0 & \Longrightarrow \ \boldsymbol{B}^{-1} \text{ exists,} \\ \det \boldsymbol{C} & = & 12 - 12 = 0 & \Longrightarrow \ \boldsymbol{C}^{-1} \text{ does not exist,} \\ \det \boldsymbol{D} & = & -10 + 3 = -7 \neq 0 & \Longrightarrow \ \boldsymbol{D}^{-1} \text{ exists.} \end{array}$$

If we denote the matrix $\boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then $\boldsymbol{B}^{-1} = \dfrac{1}{\det \boldsymbol{B}} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^T =$

$$= \frac{1}{\det \boldsymbol{B}} \begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\boldsymbol{D}^{-1} = \frac{1}{-7} \begin{pmatrix} -5 & -3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} & \frac{3}{7} \\ -\frac{1}{7} & -\frac{2}{7} \end{pmatrix}.$$

■

***Example* 32**: Compute the inverse matrix $\boldsymbol{A}^{-1}$, if $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 0 & 3 & -1 \end{pmatrix}$.

Solution:

a) We use the expression

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T, \ \det \boldsymbol{A} = -2 + 6 - 2 \neq 0 \Rightarrow \text{the matrix } \boldsymbol{A}^{-1} \text{ exists.}$$

Now we compute the individual co–factors:

$$A_{11} = \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} = -2, \quad A_{12} = -\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = 2, \quad A_{13} = \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 6,$$
$$A_{21} = -\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = 2, \quad A_{22} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1, \quad A_{23} = -\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = -3,$$
$$A_{31} = \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = -2, \quad A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2, \quad A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = 4.$$

We substitute into $\boldsymbol{A}^{-1}$ and compute

$$\boldsymbol{A}^{-1} = \frac{1}{2} \begin{pmatrix} -2 & 2 & 6 \\ 2 & -1 & -3 \\ -2 & 2 & 4 \end{pmatrix}^T = \frac{1}{2} \begin{pmatrix} -2 & 2 & -2 \\ 2 & -1 & 2 \\ 6 & -3 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -\frac{1}{2} & 1 \\ 3 & -\frac{3}{2} & 2 \end{pmatrix}.$$

b) We compute $\boldsymbol{A}^{-1}$ once more by means of the Gaussian algorithm, according to which we transform the matrix $(\boldsymbol{A}|\,\boldsymbol{E})$ so that the matrix $(\boldsymbol{E}|\boldsymbol{A}^{-1})$:

$$(\boldsymbol{A}|\,\boldsymbol{E}) = \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \overset{1)}{\sim} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 4 & -2 & -2 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \overset{2)}{\sim}$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \overset{3)}{\sim} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{4} & 1 \end{array}\right) \overset{4)}{\sim}$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -2 & \frac{3}{2} & -2 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{4} & 1 \end{array}\right) \overset{5)}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 3 & -\frac{3}{2} & 2 \end{array}\right) = (\boldsymbol{E}|\boldsymbol{A}^{-1})$$

$$\Rightarrow \qquad \boldsymbol{A}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -\frac{1}{2} & 1 \\ 3 & -\frac{3}{2} & 2 \end{pmatrix}.$$

Description of the procedure: All the operations are row operations,

1) we multiply the 1st row by (-2) and add it to the 2nd row
2) we divide the 2nd row by 4
3) we multiply the 2nd row by (-3) and add it to the 3rd row
4) we add the 3rd row to the 2nd row and to the 1st row we add the 3rd row multiplied by (-2)
5) we multiply the 3rd row by (2) and to the 1st row we add the 2nd row

REMARK: If the matrix $\boldsymbol{A}$ is of the type $(4,4)$ or higher, then to find $\boldsymbol{A}^{-1}$ the Gaussian algorithm is more advantageous.

■

***Example* 33**: Find the matrix $\boldsymbol{X}$ for which there holds $\boldsymbol{A}\cdot\boldsymbol{X}\cdot\boldsymbol{B}=2\,\boldsymbol{A}+\boldsymbol{E}$, where

$$\boldsymbol{A}=\begin{pmatrix}1&3\\2&4\end{pmatrix};\quad \boldsymbol{B}=\begin{pmatrix}2&-5\\1&-3\end{pmatrix}.$$

Solution: We multiply the given equation from the left by the matrix $\boldsymbol{A}^{-1}$

$$\boldsymbol{A}^{-1}\cdot\boldsymbol{A}\cdot\boldsymbol{X}\cdot\boldsymbol{B}=\boldsymbol{A}^{-1}\Big(2\,\boldsymbol{A}+\boldsymbol{E}\Big)$$

then we use the equality $\boldsymbol{A}^{-1}\boldsymbol{A}=\boldsymbol{E}$ and step by step we obtain

$$\begin{aligned}\boldsymbol{E}\cdot\boldsymbol{X}\cdot\boldsymbol{B}&=2\,\boldsymbol{A}^{-1}\boldsymbol{A}+\boldsymbol{A}^{-1}\cdot\boldsymbol{E}\\ \boldsymbol{X}\cdot\boldsymbol{B}&=2\,\boldsymbol{E}+\boldsymbol{A}^{-1}.\end{aligned}$$

Now we multiply both sides of the equation from the right by the matrix $\boldsymbol{B}^{-1}$:

$$\begin{aligned}\boldsymbol{X}\,\boldsymbol{B}\,\boldsymbol{B}^{-1}&=\Big(2\,\boldsymbol{E}+\boldsymbol{A}^{-1}\Big)\,\boldsymbol{B}^{-1}\quad\text{and after the transformation we obtain}\\ \boldsymbol{X}&=\Big(2\,\boldsymbol{E}+\boldsymbol{A}^{-1}\Big)\,\boldsymbol{B}^{-1}.\end{aligned}$$

We compute the matrices $\boldsymbol{A}^{-1}$, $\boldsymbol{B}^{-1}$, substitute and compute the matrix $\boldsymbol{X}$.

$$\begin{aligned}\boldsymbol{A}^{-1}&=\frac{1}{-2}\begin{pmatrix}4&-3\\-2&1\end{pmatrix}=\begin{pmatrix}-2&\frac{3}{2}\\1&-\frac{1}{2}\end{pmatrix},\ \boldsymbol{B}^{-1}=\frac{1}{-1}\begin{pmatrix}-3&5\\-1&2\end{pmatrix}=\begin{pmatrix}3&-5\\1&-2\end{pmatrix},\\ \boldsymbol{X}&=\left(2\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}-2&\frac{3}{2}\\1&-\frac{1}{2}\end{pmatrix}\right)\cdot\begin{pmatrix}3&-5\\1&-2\end{pmatrix}=\\ &=\begin{pmatrix}0&\frac{3}{2}\\1&\frac{3}{2}\end{pmatrix}\cdot\begin{pmatrix}3&-5\\1&-2\end{pmatrix}=\begin{pmatrix}\frac{3}{2}&-3\\\frac{5}{2}&-8\end{pmatrix}.\end{aligned}$$

■

***Example* 34**: Find the matrix $\boldsymbol{X}$ for which there holds $\boldsymbol{A}\cdot\boldsymbol{X}=\boldsymbol{B}$, where

$$\boldsymbol{A}=\begin{pmatrix}3&2\\5&4\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}-2&10&-3\\-6&20&-7\end{pmatrix}.$$

Solution: From the given equation we find that the matrix $\boldsymbol{X}$ must be of the type $(2,3)$, so that the relation $\boldsymbol{A}\cdot\boldsymbol{X}=\boldsymbol{B}$ holds. In order to compute the matrix sought, we multiply the given equation from the left by the matrix $\boldsymbol{A}^{-1}$ and we obtain $\boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{B}$.

$$\boldsymbol{A}^{-1}=\frac{1}{12-10}\begin{pmatrix}4&-2\\-5&3\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4&-2\\-5&3\end{pmatrix},\ \text{then}$$

$$\boldsymbol{X}=\frac{1}{2}\begin{pmatrix}4&-2\\-5&3\end{pmatrix}\cdot\begin{pmatrix}-2&10&-3\\-6&20&-7\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4&0&2\\-8&10&-6\end{pmatrix}=\begin{pmatrix}2&0&1\\-4&5&-3\end{pmatrix}.$$

■

***Example* 35**: Solve the matrix equation $\boldsymbol{B}\cdot\boldsymbol{X}+\boldsymbol{X}=\boldsymbol{A}\cdot\boldsymbol{B}^T$, where

$$\boldsymbol{A}=\begin{pmatrix}3&1\\5&4\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}2&1\\8&2\end{pmatrix}.$$

Solution: We transform the equation by factoring the matrix $\boldsymbol{X}$ out of the bracket from the right

$$\Big(\boldsymbol{B}+\boldsymbol{E}\Big)\,\boldsymbol{X}=\boldsymbol{A}\,\boldsymbol{B}^T,$$

then, multiplying from the left by the matrix $\Big(\boldsymbol{B}+\boldsymbol{E}\Big)^{-1}$ we obtain the matrix

$$\boldsymbol{X}=\Big(\boldsymbol{B}+\boldsymbol{E}\Big)^{-1}\boldsymbol{A}\,\boldsymbol{B}^T.$$

Now we compute the necessary matrices and then substitute:

$$\Big(\boldsymbol{B}+\boldsymbol{E}\Big)=\begin{pmatrix}2&1\\8&2\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}3&1\\8&3\end{pmatrix}\;\Rightarrow\;\Big(\boldsymbol{B}+\boldsymbol{E}\Big)^{-1}=\frac{1}{1}\begin{pmatrix}3&-1\\-8&3\end{pmatrix}$$

$$\boldsymbol{A}\cdot\boldsymbol{B}^T=\begin{pmatrix}3&1\\5&4\end{pmatrix}\cdot\begin{pmatrix}2&8\\1&2\end{pmatrix}=\begin{pmatrix}7&26\\14&48\end{pmatrix}$$

$$\boldsymbol{X}=\Big(\boldsymbol{B}+\boldsymbol{E}\Big)^{-1}\boldsymbol{A}\,\boldsymbol{B}^T=\begin{pmatrix}3&-1\\-8&3\end{pmatrix}\cdot\begin{pmatrix}7&26\\14&48\end{pmatrix}=\begin{pmatrix}7&30\\-14&-64\end{pmatrix}.$$

■

***Example* 36**: Find the matrix $\boldsymbol{X}$ from the equation

$$\begin{pmatrix}2&1\\4&2\end{pmatrix}\cdot\boldsymbol{X}\cdot\begin{pmatrix}1&2\\-1&-3\end{pmatrix}+\begin{pmatrix}3&8\\6&16\end{pmatrix}=0.$$

Solution: We write the given equation in matrix form:

$$\boldsymbol{A}\cdot\boldsymbol{X}\cdot\boldsymbol{B}+\boldsymbol{C}=0\quad\Rightarrow\quad\boldsymbol{A}\cdot\boldsymbol{X}\cdot\boldsymbol{B}=-\boldsymbol{C}.$$

According to the preceding example we could express the matrix $\boldsymbol{X}$ by means of the matrix product $\boldsymbol{X}=-\boldsymbol{A}^{-1}\boldsymbol{C}\,\boldsymbol{B}^{-1}$, however it is obvious that $\boldsymbol{A}^{-1}$ does not exist, because the rows are dependent, and hence $\det\boldsymbol{A}=0$. Therefore we use only the matrix $\boldsymbol{B}^{-1}$, which exists.

We compute $\boldsymbol{B}^{-1}$ and the product $-\boldsymbol{C}\cdot\boldsymbol{B}^{-1}$, so that $\boldsymbol{A}\cdot\boldsymbol{X}=-\boldsymbol{C}\cdot\boldsymbol{B}^{-1}$.

$$\boldsymbol{B}=\begin{pmatrix}1&2\\-1&-3\end{pmatrix}\quad\Rightarrow\quad\boldsymbol{B}^{-1}=\frac{1}{-1}\begin{pmatrix}-3&-2\\1&1\end{pmatrix}=\begin{pmatrix}3&2\\-1&-1\end{pmatrix}$$

$$-\boldsymbol{C}\cdot\boldsymbol{B}^{-1}=-\begin{pmatrix}3&8\\6&16\end{pmatrix}\cdot\begin{pmatrix}3&2\\-1&-1\end{pmatrix}=-\begin{pmatrix}1&-2\\2&-4\end{pmatrix}=\begin{pmatrix}-1&2\\-2&4\end{pmatrix}.$$

In this way we have obtained the equation $\begin{pmatrix}2&1\\4&2\end{pmatrix}\cdot\boldsymbol{X}=\begin{pmatrix}-1&2\\-2&4\end{pmatrix}$.

We write the matrix $\boldsymbol{X}$ in entries: $\boldsymbol{X}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and let us write out the preceding product, from which we obtain a system for the unknowns $a,\ b,\ c,\ d$:

$$\begin{pmatrix}2&1\\4&2\end{pmatrix}\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}-1&2\\-2&4\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix}2\,a+c&2\,b+d\\4\,a+2\,c&4\,b+2\,d\end{pmatrix}=\begin{pmatrix}-1&2\\-2&4\end{pmatrix}\quad\Rightarrow$$

$$\begin{cases}2\,a&+&c&=&-1\\4\,a&+&2\,c&=&-2\\2\,b&+&d&=&2\\4\,b&+&2\,d&=&4\end{cases}.$$

This system has infinitely many solutions:
$c=-1-2\,a,\ d=2-2\,b,\ \ a,b\in\mathbb{R}.$

Then the solution of the given problem is infinitely many matrices

$$\boldsymbol{X} = \begin{pmatrix} a & b \\ -1-2\,a & 2-2\,b \end{pmatrix}, \quad \text{where } a,\ b\ \in \mathbb{R}. \qquad \blacksquare$$

37. Compute the products:

a) $\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

b) $\left(\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 4 & 2 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -5 & 8 \end{pmatrix}$

e) $\begin{pmatrix} -1 & 0 & 3 \\ 2 & 1 & 2 \\ -1 & 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 3 \\ -5 & 3 & -1 \\ 2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$;

f) $\begin{pmatrix} 1 & 2 & 0 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

$$\left[\text{a) } \begin{pmatrix} 3 \\ 23 \end{pmatrix};\ \text{b) } \begin{pmatrix} 7 & 5 \\ 23 & 5 \end{pmatrix};\ \text{c) } \begin{pmatrix} 7 & 15 \\ 7 & 20 \end{pmatrix};\ \text{d) does not exist; e) } \begin{pmatrix} 10 \\ 21 \\ 23 \end{pmatrix};\ \text{f) } \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right]$$

38. Given the matrices $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Find the products $\boldsymbol{A} \cdot \boldsymbol{B}$ and $\Big(\boldsymbol{B} \cdot \boldsymbol{A}\Big)^T$.

$$\left[\boldsymbol{AB} = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 2 & -1 \\ 4 & -1 & 7 \end{pmatrix}, \quad \Big(\boldsymbol{BA}\Big)^T = \begin{pmatrix} 8 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & -1 & 2 \end{pmatrix} \right]$$

39. Compute the matrix $\boldsymbol{X}$ which satisfies the equation $\boldsymbol{A}\Big(\boldsymbol{B}^T - \boldsymbol{E}\Big) = 2\boldsymbol{X} - \boldsymbol{B}$, if $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$.

$$\left[\boldsymbol{X} = \frac{1}{2} \begin{pmatrix} 7 & -1 \\ -7 & 7 \end{pmatrix} \right]$$

40. Find the rank of the matrices:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -3 \\ -1 & 2 & 3 & 5 \\ 1 & 5 & 11 & 10 \end{pmatrix}, \ \boldsymbol{B} = \begin{pmatrix} 1 & 2 & 3 & -1 & 5 \\ 0 & 2 & -1 & 1 & 3 \\ 2 & 1 & 0 & 1 & -4 \end{pmatrix}, \ \boldsymbol{C} = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 1 & 0 \\ 1 & -3 & 1 & 3 \end{pmatrix}.$$

$$[\mathrm{r}(\boldsymbol{A}) = 3,\ \mathrm{r}(\boldsymbol{B}) = 3,\ \mathrm{r}(\boldsymbol{C}) = 2]$$

41. Find the rank of the matrix depending on the choice of the parameter $k \in \mathbb{R}$

$$\begin{pmatrix} 2k-2 & 3 & 0 & 0 \\ 0 & k-1 & 3 & 0 \\ 0 & 2 & k & 0 \\ 0 & 0 & 0 & k-3 \end{pmatrix}.$$

$$\begin{bmatrix} \mathrm{r}(\boldsymbol{A}) = 4 \text{ for } k \in \mathbb{R}\setminus\{-2;1;3\} \\ \mathrm{r}(\boldsymbol{A}) = 3 \text{ for } k \in \{-2;1\} \\ \mathrm{r}(\boldsymbol{A}) = 2 \text{ for } k = 3 \end{bmatrix}$$

42. Decide whether the given vectors form a basis of the space $V(\mathbb{E}_4)$ (use the rank of a matrix):

a) $(2;-1;1;3),\ (1;0;2;-1),\ (0;1;-1;1),\ (3;0;2;3)$

b) $(1;2;-1;-2),\ (0;1;1;1),\ (2;1;0;1),\ (1;-2;-2;1)$

c) $(1;2;-3;0),\ (1;1;0;1),\ (0;0;2;1),\ (0;1;0;1),\ (3;-1;1;0)$

[a) no; b) yes; c) no]

43. Determine for which values of the parameter $\lambda \in \mathbb{R}$ the given vectors are linearly dependent:

a) $(2;-1;3),\ (4;1;5),\ (-1;2;\lambda)$

b) $(1;2;4),\ (2;\lambda;-4),\ (-1;-2;2\,\lambda)$

c) $(2;-1;-1;3),\ (3;1;2;-1),\ (1;-3;-4;7),\ (1;3;\lambda;4)$

[a) $\lambda = -2$; b) $\lambda \in \{-2;4\}$; c) $\lambda \in \mathbb{R}$]

44. Find out whether the vectors are linearly dependent or independent and find the dimension of the vector space which is generated by them:

a) $(2;1;4),\ (3;-2;1),\ (3;5;11)$

b) $(2;-1;1;3),\ (1;1;3;1),\ (-1;3;2;1),\ (-1;0;-3;2)$

[a) lin. indep., $\dim = 3$; b) lin. dep., $\dim = 3$]

45. Compute the determinants:

a) $\begin{vmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{vmatrix}$; b) $\begin{vmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ 4 & 5 & 6 \end{vmatrix}$; c) $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$

[a) 1; b) 0; c) 9]

46. Compute the determinants:

a) $\begin{vmatrix} 1 & 0 & 2 & 2 \\ 2 & 1 & 3 & -4 \\ -2 & 2 & 1 & 4 \\ -2 & 2 & 0 & 4 \end{vmatrix}$ b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -3 \\ -2 & 2 & 3 & 5 \\ -1 & 3 & 8 & 6 \end{vmatrix}$ c) $\begin{vmatrix} 12 & 16 & 24 & 33 \\ 20 & 25 & 35 & 45 \\ 20 & 27 & 36 & 55 \\ 28 & 38 & 51 & 78 \end{vmatrix}$

[a) 24; b) 0; c) -20]

47. Given the matrices $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ -4 & -2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$. Find $\boldsymbol{A}^{-1}, \boldsymbol{B}^{-1}, (\boldsymbol{AB})^{-1}$.

$$\left[\boldsymbol{A}^{-1} = -\frac{1}{2}\begin{pmatrix} -2 & -1 \\ 4 & 3 \end{pmatrix},\ \boldsymbol{B}^{-1} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix},\ (\boldsymbol{AB})^{-1} = -\frac{1}{2}\begin{pmatrix} -14 & -10 \\ 18 & 13 \end{pmatrix}\right]$$

48. Find the inverse matrices of the matrices:

a) $\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 5 & 4 & 5 \end{pmatrix}$; b) $\begin{pmatrix} 1 & 1 & -1 \\ -4 & -5 & 6 \\ -3 & -3 & 4 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$

$\left[\text{a) does not exist; b) } \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 2 \\ 3 & 0 & 1 \end{pmatrix}\text{; c) } \begin{pmatrix} 10 & -1 & -4 & -9 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -3 & 0 & 1 & 3 \\ -1 & 0 & 0 & 1 \end{pmatrix}\right]$

49. Compute the matrix $\boldsymbol{X}$ from the equation

a) $\begin{pmatrix} 2 & 1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}\boldsymbol{X} = \begin{pmatrix} 15 & 25 \\ -10 & 20 \end{pmatrix}$

b) $\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}\boldsymbol{X}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 9 & -20 \\ 4 & -10 \end{pmatrix}^T$

c) $2\boldsymbol{A} + \boldsymbol{A}\boldsymbol{X} = \boldsymbol{B}\boldsymbol{X} - \boldsymbol{X}$, where $\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ 6 & 4 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix}$.
(use $2\boldsymbol{A} = (\boldsymbol{B} - \boldsymbol{A} - \boldsymbol{E})\boldsymbol{X}$)

$\left[\text{a) } \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\text{; b) } \begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}\text{; c) } \begin{pmatrix} -3 & -2 \\ 4 & -2 \end{pmatrix}\right]$

I.3 Systems of linear equations

***Example* 50**: Solve the given system of linear equations

$$\begin{cases} x + 2y + z = 1 \\ 2x + z = -4 \\ 5x - 3y + z = -13 \end{cases}$$

by a) Cramer's rule;
b) the inverse matrix.

Solution: In order to be able to use both Cramer's rule and the inverse matrix, the matrix of the system $\boldsymbol{A}$ must be regular (in other words: square with a nonzero determinant). Let us recall that a regular matrix $\boldsymbol{A}$ guarantees the existence of a unique solution. We write the given system in the form $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 5 & -3 & 1 \end{pmatrix}, \quad \boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 \\ -4 \\ -13 \end{pmatrix}.$$

We compute $\det \boldsymbol{A} = 0 + 10 - 6 - 0 + 3 - 4 = 3 \neq 0$. Now we know that $\boldsymbol{A}$ is regular and we pass on to the solution of the system.

a) We compute the determinants $\det \boldsymbol{A}_x$, $\det \boldsymbol{A}_y$, $\det \boldsymbol{A}_z$ corresponding to the individual unknowns:

$$\det \boldsymbol{A}_x = \begin{vmatrix} 1 & 2 & 1 \\ -4 & 0 & 1 \\ -13 & -3 & 1 \end{vmatrix} = -26 + 12 + 3 + 8 = -3,$$

where $\det \boldsymbol{A}_x$ is the determinant of the matrix which arises from the matrix $\boldsymbol{A}$ in such a way that the column corresponding to the unknown x is replaced by the column $\boldsymbol{B}$. Similarly

$$\det \boldsymbol{A}_y = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -4 & 1 \\ 5 & -13 & 1 \end{vmatrix} = -4 - 26 + 5 + 20 + 13 - 2 = 6$$

$$\det \boldsymbol{A}_z = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & -4 \\ 5 & -3 & -13 \end{vmatrix} = -40 - 6 - 12 + 52 = -6.$$

Then $x = \dfrac{\det \boldsymbol{A}_x}{\det \boldsymbol{A}} = \dfrac{-3}{3} = -1, \quad y = \dfrac{\det \boldsymbol{A}_y}{\det \boldsymbol{A}} = \dfrac{6}{3} = 2, \quad z = \dfrac{\det \boldsymbol{A}_z}{\det \boldsymbol{A}} = \dfrac{-6}{3} = -2.$

b) We return to the matrix equation $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$, from which we compute $\boldsymbol{X} = \boldsymbol{A}^{-1} \cdot \boldsymbol{B}$. We determine

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T =$$

$$= \frac{1}{3} \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ -3 & 1 \end{vmatrix}, & -\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix}, & \begin{vmatrix} 2 & 0 \\ 5 & -3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix}, & \begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix}, & -\begin{vmatrix} 1 & 2 \\ 5 & -3 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}, & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}, & \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \end{pmatrix}^T = \frac{1}{3} \begin{pmatrix} 3 & 3 & -6 \\ -5 & -4 & 13 \\ 2 & 1 & -4 \end{pmatrix}^T,$$

$$\boldsymbol{A}^{-1} = \frac{1}{3} \begin{pmatrix} 3 & -5 & 2 \\ 3 & -4 & 1 \\ -6 & 13 & -4 \end{pmatrix};$$

$$\boldsymbol{X} = \boldsymbol{A}^{-1} \cdot \boldsymbol{B} = \frac{1}{3} \begin{pmatrix} 3 & -5 & 2 \\ 3 & -4 & 1 \\ -6 & 13 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ -13 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 + 20 - 26 \\ 3 + 16 - 13 \\ -6 - 52 + 52 \end{pmatrix} =$$

$$= \frac{1}{3} \begin{pmatrix} -3 \\ 6 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}; \qquad x = -1,\ y = 2,\ z = -2. \qquad \blacksquare$$

REMARK: The reader has certainly verified that both procedures used are, even in this simple example, very laborious and usable only in the case of the existence of exactly one solution.

***Example* 51**: Find out whether Cramer's rule, or the inverse matrix, can be used when solving the following systems of linear algebraic equations $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$.

a) $\begin{cases} 2x + 3y = 7 \\ 3x - y = 5 \end{cases}$; b) $\begin{cases} 6x - 2y = 10 \\ -3x + y = 2 \end{cases}$;

c) $\begin{cases} x + 3y - 5z = -10 \\ -y + 2z = 5 \\ x + 2y - 4z = -7 \end{cases}$; d) $\begin{cases} 2x - y + 3z = 5 \\ -x + 2y - z = 3 \\ 4x - 5y + 5z = 0 \end{cases}$.

Solution: All the systems have square matrices of the systems.

If the determinant of the matrix of the given system $\begin{cases} = 0 \\ \neq 0 \end{cases}$ then $\begin{cases} \textbf{it is not possible} \\ \textbf{it is possible} \end{cases}$

to use both Cramer's rule and the inverse matrix.

a) $\det \boldsymbol{A} = \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -11 \neq 0 \quad \Rightarrow \quad$ it can be used;

b) $\det \boldsymbol{A} = \begin{vmatrix} 6 & -2 \\ -3 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad$ it cannot be used;

c) $\det \boldsymbol{A} = \begin{vmatrix} 1 & 3 & -5 \\ 0 & -1 & 2 \\ 1 & 2 & -4 \end{vmatrix} = 4 + 6 - 5 - 4 = 1 \neq 0 \quad \Rightarrow \quad$ it can be used;

d) $\det \boldsymbol{A} = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -1 \\ 4 & -5 & 5 \end{vmatrix} = 20 + 4 + 15 - 24 - 10 - 5 = 0 \quad \Rightarrow \quad$ it cannot be used.

REMARK: In the case that $\det \boldsymbol{A} = 0$ we know that the system cannot have exactly one solution, but we have not decided whether it has infinitely many solutions or whether no solution exists.

■

***Example* 52**: Solve the system $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$ by the Gaussian elimination method:

$$\begin{cases} x - y + 2z + 3u = -6 \\ \quad\; y + 2z - u = -1 \\ 2x + y + z + 3u = 3 \\ 3x - y + z - u = -2 \end{cases}.$$

Solution: According to the Gaussian elimination method we need to transform the augmented matrix of the system by equivalent operations to the triangular form:

$$(\boldsymbol{A}|\,\boldsymbol{B}) = \begin{matrix} x \\ 1 \\ 0 \\ 2 \\ 3 \end{matrix} \begin{matrix} y \\ -1 \\ 1 \\ 2 \\ -1 \end{matrix} \begin{matrix} z \\ 2 \\ 2 \\ 1 \\ 1 \end{matrix} \begin{matrix} u \\ 3 \\ -1 \\ 3 \\ -1 \end{matrix} \left|\begin{matrix} -6 \\ -1 \\ 3 \\ -2 \end{matrix}\right. \sim \left(\begin{array}{cccc|c} 1 & -1 & 2 & 3 & -6 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 3 & -3 & -3 & 15 \\ 0 & 2 & -5 & -10 & 16 \end{array}\right) \sim \text{(divide the } 3^{\text{rd}} \text{ row by 3)} \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 2 & 3 & -6 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 1 & -1 & -1 & 5 \\ 0 & 2 & -5 & -10 & 16 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 2 & 3 & -6 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & -3 & 0 & 6 \\ 0 & 0 & -9 & -8 & 18 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 2 & 3 & -6 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & -8 & 0 \end{array}\right).$$

From the resulting triangular form of the matrix we conclude, by Frobenius' theorem, that the system $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$ has a unique solution, because $\mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 4$ (the number of the unknowns).

We write the system corresponding to this triangular matrix

$$\begin{cases} x - y + 2z + 3u = -6 \\ y + 2z - u = -1 \\ -z = 2 \\ -8u = 0 \end{cases}, \text{ from which we easily read the solution: } u = 0,\ z = -2,\ y = 3,\ x = 1. \quad \blacksquare$$

***Example* 53**: Solve the system $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$

$$\begin{cases} -x + y + z + 2u = 9 \\ 3x - y - z + 3u = -5 \\ 2x + y + u = -3 \\ 4x + y + 6u = 2 \end{cases}.$$

Solution: We solve the system by the Gaussian elimination method, but before we write the augmented matrix, we look carefully at the system and choose the order of the columns so that the transformation is easier.

1. the 1st column will be formed by the coefficients of the unknown z,
2. the 2nd column will be formed by the coefficients of the unknown y,
3. the 3rd column will be formed by the coefficients of the unknown x,
4. the 4th column will be formed by the coefficients of the unknown u.

$$(\boldsymbol{A}|\,\boldsymbol{B}) \sim \begin{array}{c} \begin{matrix} z & y & x & u \end{matrix} \\ \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 9 \\ -1 & -1 & 3 & 3 & -5 \\ 0 & 1 & 2 & 1 & -3 \\ 0 & 1 & 4 & 6 & 2 \end{array}\right) \end{array} \sim \left| \begin{array}{c} \text{we put the second row as the last one} \\ \text{and then transform in the usual way} \end{array} \right| \sim$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 9 \\ 0 & 1 & 2 & 1 & -3 \\ 0 & 1 & 4 & 6 & 2 \\ -1 & -1 & 3 & 3 & -5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 9 \\ 0 & 1 & 2 & 1 & -3 \\ 0 & 1 & 4 & 6 & 2 \\ 0 & 0 & 2 & 5 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 9 \\ 0 & 1 & 2 & 1 & -3 \\ 0 & 0 & 2 & 5 & 5 \\ 0 & 0 & 2 & 5 & 4 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 9 \\ 0 & 1 & 2 & 1 & -3 \\ 0 & 0 & 2 & 5 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad \Rightarrow \quad \mathrm{r}(\boldsymbol{A}) = \min\{3;4\} = 3 \neq \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = \min\{4;5\} = 4.$$

By Frobenius' theorem the system has no solution. The same result follows from the last row rewritten in the form of an equation: $0\,u = 1$, which is a false statement. $\blacksquare$

***Example* 54**: Solve the system $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$

$$\begin{cases} x + 2y - z + u = -2 \\ 2x - y + z - u = 4 \\ -x + y + 2z - 3u = -1 \\ 5x - y - 4z + 6u = 4 \end{cases}.$$

Solution: We write the columns in the order of the unknowns $z,\ u,\ x,\ y$:

$$(\boldsymbol{A}|\,\boldsymbol{B}) \sim \begin{array}{c} \begin{matrix} z & u & x & y \end{matrix} \\ \left(\begin{array}{cccc|c} -1 & 1 & 1 & 2 & -2 \\ 1 & -1 & 2 & -1 & 4 \\ 2 & -3 & -1 & 1 & -1 \\ -4 & 6 & 5 & -1 & 4 \end{array}\right) \end{array} \sim \left(\begin{array}{cccc|c} -1 & 1 & 1 & 2 & -2 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & -1 & 1 & 5 & -5 \\ 0 & 2 & 1 & -9 & 12 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} -1 & 1 & 1 & 2 & -2 \\ 0 & -1 & 1 & 5 & -5 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 2 & 1 & -9 & 12 \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 1 & 1 & 2 & -2 \\ 0 & -1 & 1 & 5 & -5 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 1 & 1 & 2 & -2 \\ 0 & -1 & 1 & 5 & -5 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 3 < 4 \ \Rightarrow$ by Frobenius' theorem the system has infinitely many solutions depending on one parameter. We write the system corresponding to the last matrix:

$$\left\{\begin{array}{rcrcrcrcl} -z & + & u & + & x & + & 2\,y & = & -2 \\ & & -u & + & x & + & 5\,y & = & -5 \\ & & & & 3\,x & + & y & = & 2 \end{array}\right. .$$

We choose $x = t$, where $t \in \mathbb{R}$ and successively determine

$$y = 2 - 3\,t,\ u = x + 5\,y + 5 = t + 5\,(2 - 3\,t) + 5 = 15 - 14\,t,$$
$$z = u + x + 2\,y + 2 = 15 - 14\,t + t + 2\,(2 - 3\,t) + 2 = 21 - 19\,t.$$

We write the resulting solution as a quadruple, which can be written in vector form

$$(x; y; z; u) = (t; 2 - 3\,t; 21 - 19\,t; 15 - 14\,t) = (0; 2; 21; 15) + (1; -3; -19; -14) \cdot t, \quad t \in \mathbb{R}.$$

REMARK: If we chose another unknown as the parameter, then the quadruple of the solution $(x; y; z; u)$ would have a completely different form.

■

***Example* 55**: Solve the system $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$.

$$\left\{\begin{array}{rcrcrcrcrcl} 3\,x_1 & + & 4\,x_2 & + & 5\,x_3 & + & x_4 & - & 2\,x_5 & = & 3 \\ x_1 & - & x_2 & + & 3\,x_3 & + & 2\,x_4 & + & 3\,x_5 & = & 5 \\ 2\,x_1 & + & 3\,x_2 & + & x_3 & - & x_4 & + & x_5 & = & 2 \\ 2\,x_1 & & & + & 7\,x_3 & + & 4\,x_4 & & & = & 6 \end{array}\right. .$$

Solution: Let us choose the following order of the rows: 2nd, 3rd, 4th, 1st and transform the first column

$$(\boldsymbol{A}|\,\boldsymbol{B}) \sim \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \left(\begin{array}{ccccc|c} 1 & -1 & 3 & 2 & 3 & 5 \\ 2 & 3 & 1 & -1 & 1 & 2 \\ 2 & 0 & 7 & 4 & 0 & 6 \\ 3 & 4 & 5 & 1 & -2 & 3 \end{array}\right) \end{array} \sim \left(\begin{array}{ccccc|c} 1 & -1 & 3 & 2 & 3 & 5 \\ 0 & 5 & -5 & -5 & -5 & -8 \\ 0 & 2 & 1 & 0 & -6 & -4 \\ 0 & 7 & -4 & -5 & -11 & -12 \end{array}\right).$$

We easily notice that the 4th row is the sum of the 2nd and the 3rd row, and therefore we can omit it.

$$(\boldsymbol{A}|\,\boldsymbol{B}) \sim \begin{array}{c} \begin{matrix} x_1 & x_4 & x_3 & x_2 & x_5 \end{matrix} \\ \left(\begin{array}{rrrrr|r} 1 & -1 & 3 & 2 & 3 & 5 \\ 0 & 5 & -5 & -5 & -5 & -8 \\ 0 & 2 & 1 & 0 & -6 & -4 \end{array}\right) \end{array} \sim \left(\begin{array}{rrrrr|r} 1 & 2 & 3 & -1 & 3 & 5 \\ 0 & -5 & -5 & 5 & -5 & -8 \\ 0 & 0 & 1 & 2 & -6 & -4 \end{array}\right) \Rightarrow$$

$$\Longrightarrow \mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 3 < 5.$$

We choose $x_2 = t,\ x_5 = s; \quad t,\ s \in \mathbb{R}$ and then

$$\begin{aligned} x_3 &= -2\,x_2 + 6\,x_5 - 4 = -2\,t + 6\,s - 4, \\ -5\,x_4 &= 5\,x_3 - 5\,x_2 + 5\,x_5 - 8 \\ &= 5\,(-2\,t + 6\,s - 4) - 5\,t + 5\,s - 8 = -15\,t + 35\,s - 28, \\ x_4 &= 3\,t - 7\,s + \frac{28}{5} \\ x_1 &= -2\,x_4 - 3\,x_3 + x_2 - 3\,x_5 + 5 \\ &= -2\,(3\,t - 7\,s + \frac{28}{5}) - 3\,(-2\,t + 6\,s - 4) + t - 3\,s + 5 = \\ &= t - 7\,s - \frac{56}{5} + 17 = t - 7\,s + \frac{29}{5} \\ (x_1; x_2; x_3; x_4; x_5) &= \Big(\frac{29}{5} + t - 7\,s; t; -4 - 2\,t + 6\,s; \frac{28}{5} + 3\,t - 7\,s; s\Big) = \\ &= \Big(\frac{29}{5}; 0; -4; \frac{28}{5}; 0\Big) + (1; 1; -2; 3; 0) \cdot t + (-7; 0; 6; -7; 1) \cdot s, \\ & \quad t, s \in \mathbb{R}. \end{aligned}$$

■

***Example* 56**: Using Frobenius' theorem, determine for which parameter values each system $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$ has no solution, a unique solution, or infinitely many solutions:

a) $\left\{\begin{array}{rcrcrcr} 2\,x & + & 3\,y & + & z & = & 6 \\ -x & + & 3\,y & - & z & = & -5 \\ a\,x & + & 6\,y & + & 2\,z & = & c \end{array}\right.$; b) $\left\{\begin{array}{rcrcrcr} x & + & y & - & z & = & c \\ a\,x & + & 2\,y & - & 3\,z & = & 5 \\ 2\,x & - & b\,y & + & b\,z & = & 2 \end{array}\right.$;

c) $\left\{\begin{array}{rcrcrcr} x & + & 2\,y & + & 3\,z & = & 5 \\ 3\,x & + & y & + & 2\,z & = & k \\ 2\,x & - & y & - & z & = & 0 \end{array}\right.$;

d) $\left\{\begin{array}{rcrcr} a\,x & - & 2\,y & = & a \\ 3\,x & + & (a-5)\,y & = & 3 \end{array}\right.$ solve the system for $a = 2, a = 3$;

e) $\left\{\begin{array}{rcrcr} x & + & y & = & 1 \\ x & - & a\,y & = & a^2 \\ a\,x & - & y & = & a \end{array}\right.$.

Solution:

a) Let us choose the following order of the columns: $z,\ y,\ x$

$$(\boldsymbol{A}|\,\boldsymbol{B}) \sim \begin{array}{c} \begin{matrix} z & y & x \end{matrix} \\ \left(\begin{array}{rrr|r} 1 & 3 & 2 & 6 \\ -1 & 3 & -1 & -5 \\ 2 & 6 & a & c \end{array}\right) \end{array} \sim \left(\begin{array}{rrr|r} 1 & 3 & 2 & 6 \\ 0 & 6 & 1 & 1 \\ 0 & 0 & a-4 & c-12 \end{array}\right).$$

$$\text{The system}\ \begin{cases} \text{has a unique solution for } a \neq 4, \\ \qquad \mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 3, \\ \text{has infinitely many solutions for } a = 4 \ \text{ and } \ c = 12, \\ \qquad \mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 2, \\ \text{has no solution for } a = 4 \ \text{ and } \ c \neq 12, \\ \qquad \mathrm{r}(\boldsymbol{A}) = 2 \neq \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 3. \end{cases}$$

b)

$$(\boldsymbol{A}|\,\boldsymbol{B}) \sim \left(\begin{array}{ccc|c} \scriptstyle y & \scriptstyle z & \scriptstyle x & \\ 1 & -1 & 1 & c \\ 2 & -3 & a & 5 \\ -b & b & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & c \\ 0 & -1 & a-2 & 5-2\,c \\ 0 & 0 & 2+b & 2+b\,c \end{array}\right).$$

$$\text{The system}\ \begin{cases} \text{has a unique solution for } b \neq -2, \\ \qquad \mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 3, \\ \text{has infinitely many solutions for } b = -2 \ \text{ and } \ c = 1, \\ \qquad \mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 2, \\ \text{has no solution for } b = -2 \text{ and } c \neq 1, \\ \qquad \mathrm{r}(\boldsymbol{A}) = 2 \neq \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 3. \end{cases}$$

The parameter a does not influence the number of solutions, that is, $a \in \mathbb{R}$.

c)

$$(\boldsymbol{A}|\,\boldsymbol{B}) = \left(\begin{array}{ccc|c} \scriptstyle x & \scriptstyle y & \scriptstyle z & \\ 1 & 2 & 3 & 5 \\ 3 & 1 & 2 & k \\ 2 & -1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -5 & -7 & k-15 \\ 0 & -5 & -7 & -10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -5 & -7 & -10 \\ 0 & 0 & 0 & k-5 \end{array}\right)$$

For $k = 5 \quad \Rightarrow$ infinitely many solutions, $\mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 2$, for $k \neq 5 \quad \Rightarrow$ no solution exists, $\mathrm{r}(\boldsymbol{A}) = 2 \neq \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 3$, for no k can the system have a unique solution.

d) Here it will be better to start with the determinant of the matrix of the system

$$\det \boldsymbol{A} = \begin{vmatrix} a & -2 \\ 3 & a-5 \end{vmatrix} = a\,(a-5) + 6 = a^2 - 5\,a + 6 = (a-2)\,(a-3),$$

$\det \boldsymbol{A} \neq 0 \quad \Rightarrow \quad a \in \mathbb{R}\setminus\{2;3\} \quad \Rightarrow$ a unique solution exists.

We successively substitute $a = 2$ and $a = 3$ into the given system:

$$a = 2: \ \begin{cases} 2\,x - 2\,y = 2 \\ 3\,x - 3\,y = 3 \end{cases} \quad \Rightarrow \quad \begin{array}{l} \text{infinitely many solutions: } y = x - 1, \\ (x;y) = (t;t-1),\ t \in \mathbb{R}; \end{array}$$

$$a = 3: \ \begin{cases} 3\,x - 2\,y = 3 \\ 3\,x - 2\,y = 3 \end{cases} \quad \Rightarrow \quad \begin{array}{l} \text{infinitely many solutions: } y = \frac{3\,x-3}{2}, \\ (x;y) = \left(t;\frac{3\,t-3}{2}\right),\ t \in \mathbb{R}. \end{array}$$

Conclusion: the system has a solution for all $a \in \mathbb{R}$.

e)

$$(\boldsymbol{A}|\,\boldsymbol{B}) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -a & a^2 \\ a & -1 & a \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -a-1 & a^2-1 \\ 0 & -1-a & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -a-1 & a^2-1 \\ 0 & 0 & 1-a^2 \end{array}\right)$$

For $a \neq \pm 1$ no solution exists, $\mathrm{r}(\boldsymbol{A}) = 2 \neq \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 3$, for $a = 1$ a unique solution exists, $\mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 2$, for $a = -1$ infinitely many solutions exist, $\mathrm{r}(\boldsymbol{A}) = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = 1$. ■

***Example* 57**: Solve the homogeneous systems $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{0}$ by the Gaussian elimination method

a)
$$\left\{\begin{array}{rcrcrcl} x & + & y & - & 4z & = & 0 \\ x & + & 2y & + & 3z & = & 0 \\ x & + & 6y & + & 10z & = & 0 \\ 4x & + & 7y & + & 5z & = & 0 \end{array}\right. ;$$

b)
$$\left\{\begin{array}{rcrcrcrcl} x & - & 3y & + & 5z & - & 5u & = & 0 \\ 2x & + & 4y & - & 5z & + & 3u & = & 0 \\ 4x & - & 2y & + & 5z & - & 7u & = & 0 \\ 6x & + & 2y & & & + & 3u & = & 0 \end{array}\right. ;$$

c)
$$\left\{\begin{array}{rcrcrcrcrcl} x & + & y & - & z & + & u & & & = & 0 \\ 2x & - & 3y & + & 2z & + & 3u & - & v & = & 0 \\ -x & + & y & + & z & + & u & & & = & 0 \\ 2x & - & y & + & 2z & + & 5u & - & v & = & 0 \end{array}\right. .$$

Solution: We know that a homogeneous system always has the trivial solution. It remains to determine whether a nontrivial solution exists as well. In all the cases we write the matrix of the system; the zero column of the right hand sides is not written. If $\mathrm{r}(\boldsymbol{A}) = n$ (the number of the unknowns), then there exists **only the trivial** solution; if $\mathrm{r}(\boldsymbol{A}) < n$ there exists **a nontrivial** solution as well.

The set of all the solutions forms a vector space V of the dimension $\Big(n - \mathrm{r}(\boldsymbol{A})\Big)$.

a)
$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & -4 \\ 1 & 2 & 3 \\ 1 & 6 & 10 \\ 4 & 7 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & 7 \\ 0 & 5 & 14 \\ 0 & 3 & 21 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & 7 \\ 0 & 0 & -21 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathrm{r}(\boldsymbol{A}) = \min\{4;3\} = 3$$

$\Rightarrow$ the only solution is the **trivial** one $x = y = z = 0$.

b)
$$\boldsymbol{A} \sim \begin{array}{c} \begin{matrix} z & y & x & u \end{matrix} \\ \begin{pmatrix} 5 & -3 & 1 & -5 \\ -5 & 4 & 2 & 3 \\ 5 & -2 & 4 & -7 \\ 0 & 2 & 6 & 3 \end{pmatrix} \end{array} \sim \begin{pmatrix} 5 & -3 & 1 & -5 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & 3 \end{pmatrix} \sim \begin{pmatrix} 5 & -3 & 1 & -5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 7 \end{pmatrix},$$

$\mathrm{r}(\boldsymbol{A}) = \min\{3;4\} = 3 \quad \Rightarrow \quad$ the system has a **nontrivial** solution, $\dim V = 1$,

$$\left\{\begin{array}{rcrcrcl} 5z & - & 3y & + & x & - & 5u & = & 0 \\ & & y & + & 3x & - & 2u & = & 0 \\ & & & & & & 7u & = & 0 \end{array}\right. \Rightarrow \; u = 0,\; x = t,\; y = -3t,\; z = -2t,$$

$(x;y;z;u) = (1;-3;-2;0)\,t, \quad t \in \mathbb{R}$.

c)

$$\boldsymbol{A} = \begin{array}{c} \begin{matrix} x & y & z & u & v \end{matrix} \\ \begin{pmatrix} 1 & 1 & -1 & 1 & 0 \\ 2 & -3 & 2 & 3 & -1 \\ -1 & 1 & 1 & 1 & 0 \\ 2 & -1 & 2 & 5 & -1 \end{pmatrix} \end{array} \sim \begin{pmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & -5 & 4 & 1 & -1 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & -3 & 4 & 3 & -1 \end{pmatrix} \sim$$

$$\begin{array}{c} \begin{matrix} x & v & z & u & y \end{matrix} \\ \begin{pmatrix} 1 & 0 & -1 & 1 & 1 \\ 0 & -1 & 4 & 1 & -5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & -1 & 4 & 3 & -3 \end{pmatrix} \end{array} \sim \begin{pmatrix} 1 & 0 & -1 & 1 & 1 \\ 0 & -1 & 4 & 1 & -5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}; \quad \mathrm{r}(\boldsymbol{A}) = \min\{3;5\} = 3 \quad \Rightarrow$$

the system has a **nontrivial** solution and $\dim V = 2$.

$$\begin{cases} x & & & -z & + & u & + & y & = & 0 \\ & -v & + & 4\,z & + & u & - & 5\,y & = & 0 \\ & & & & & u & + & y & = & 0 \end{cases} \qquad \begin{array}{l} y = t,\ z = s \\ u = -t, \\ v = 4\,z + u - 5\,y = 4\,s - 6\,t, \\ x = z - u - y = s + t - t = s \end{array}$$

$$(x;y;z;u;v) = (s;t;s;-t;4\,s-6\,t) = (0;1;0;-1;-6)\,t + (1;0;1;0;4)\,s, \quad t,s \in \mathbb{R}. \quad \blacksquare$$

***Example* 58**: Let us have the vectors $\vec{a} = (2;1;-3;1)$, $\vec{b} = (1;2;3;4)$, $\vec{c} = (3;-1;0;-2)$, $\vec{d} = (10;7;3;12)$. Find all the constants $\alpha, \beta, \gamma, \delta$ satisfying the equation $\alpha\,\vec{a} + \beta\,\vec{b} + \gamma\,\vec{c} + \delta\,\vec{d} = \vec{0}$.

Solution: We compute the constants and carry out the discussion:

a) if only the trivial solution exists $\alpha = \beta = \gamma = \delta = 0$, then the vectors will be linearly independent.

b) if at least one nontrivial solution exists (at least one of the constants $\alpha, \beta, \gamma, \delta$ is different from zero), then the vectors will be linearly dependent.

To find $\alpha, \beta, \gamma, \delta$ means to solve a homogeneous system. We write the vectors in columns

$$\alpha\,\vec{a} + \beta\,\vec{b} + \gamma\,\vec{c} + \delta\,\vec{d} = \alpha \begin{pmatrix} 2 \\ 1 \\ -3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \gamma \begin{pmatrix} 3 \\ -1 \\ 0 \\ -2 \end{pmatrix} + \delta \begin{pmatrix} 10 \\ 7 \\ 3 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and after comparing the coordinates we obtain the homogeneous system $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{0}$:

$$\begin{cases} 2\,\alpha & + & \beta & + & 3\,\gamma & + & 10\,\delta & = & 0 \\ \alpha & + & 2\,\beta & - & \gamma & + & 7\,\delta & = & 0 \\ -3\,\alpha & + & 3\,\beta & & & + & 3\,\delta & = & 0 \\ \alpha & + & 4\,\beta & - & 2\,\gamma & + & 12\,\delta & = & 0 \end{cases},$$

which we solve by the Gaussian elimination method: we divide the 3rd row by three and write it as the first one

$$\boldsymbol{A} \sim \begin{pmatrix} -1 & 1 & 0 & 1 \\ 2 & 1 & 3 & 10 \\ 1 & 2 & -1 & 7 \\ 1 & 4 & -2 & 12 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 3 & 3 & 12 \\ 0 & 3 & -1 & 8 \\ 0 & 5 & -2 & 13 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 3 & -1 & 8 \\ 0 & 5 & -2 & 13 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & -7 & -7 \end{pmatrix} \quad \Rightarrow \quad \mathrm{r}(\boldsymbol{A}) = \min\{3;4\} = 3 \quad \Rightarrow \text{ a } \textbf{nontrivial} \text{ solution exists,}$$

hence the vectors are **dependent** and

$$\delta = -t,\ \gamma = t,\ \beta = 3t,\ \alpha = 2t, \quad (\alpha;\beta;\gamma;\delta) = (2;3;1;-1)t.$$

The concrete dependence of the vectors will be $2\,\vec{a} + 3\,\vec{b} + \vec{c} - \vec{d} = \vec{0}$ or $\vec{d} = 2\,\vec{a} + 3\,\vec{b} + \vec{c}$. ■

***Example* 59**: How many solutions do the following homogeneous systems $\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{0}$ have depending on the values of the parameters which occur in them? Solve the systems for the given values of the parameters:

a) $\left\{ \begin{array}{rcrcrcl} x & + & y & - & 2\,z & = & 0 \\ x & + & a\,y & + & z & = & 0 \\ x & - & y & - & z & = & 0 \end{array} \right.$, $\quad a = 2;$

b) $\left\{ \begin{array}{rcrcrcl} a\,x & - & 2\,y & + & z & = & 0 \\ x & - & 2\,a\,y & + & z & = & 0 \\ x & - & 2\,y & + & a\,z & = & 0 \end{array} \right.$, $\quad a = 1;$

c) $\left\{ \begin{array}{rcrcrcl} x & - & 3\,y & + & 2\,z & = & 0 \\ 2\,x & + & y & + & z & = & 0 \\ 3\,x & + & a\,y & + & b\,z & = & 0 \end{array} \right.$, $\quad a = 2,\ b = 6.$

Solution:

a) $$\boldsymbol{A} \sim \begin{matrix} x \\ \\ \\ \end{matrix} \begin{matrix} z \\ \\ \\ \end{matrix} \begin{matrix} y \\ \\ \\ \end{matrix} \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & a \\ 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & a \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -2 \\ 0 & 3 & a-1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & a+5 \end{pmatrix} \left\{ \begin{array}{l} \text{for } a = -5 \text{ infinitely many solutions exist} \\ \text{for } a \neq -5 \text{ only the trivial solution exists} \end{array} \right. .$$

For the given $a = 2 \neq -5$ there holds $x = y = z = 0$.

b) $$\boldsymbol{A} \sim \begin{matrix} z \\ \\ \\ \end{matrix} \begin{matrix} y \\ \\ \\ \end{matrix} \begin{matrix} x \\ \\ \\ \end{matrix} \begin{pmatrix} 1 & -2 & a \\ 1 & -2\,a & 1 \\ a & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & a \\ 0 & -2\,a+2 & 1-a \\ 0 & -2+2\,a & 1-a^2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & a \\ 0 & 2\,(1-a) & 1-a \\ 0 & 0 & 2-a-a^2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -2 & a \\ 0 & 2\,(1-a) & 1-a \\ 0 & 0 & -(a-1)(a+2) \end{pmatrix} \quad \Rightarrow \left\{ \begin{array}{l} \text{for } a \in \mathbb{R}\backslash\{-2;1\} \text{ there is } \mathrm{r}(\boldsymbol{A}) = 3 \\ \text{and the system has only the trivial solution,} \\ \text{for } a \in \{-2;1\} \text{ there is } \mathrm{r}(\boldsymbol{A}) < 3, \\ \text{infinitely many solutions exist.} \end{array} \right.$$

For the given $a = 1$ there is $\mathrm{r}(\boldsymbol{A}) = 1$, the dimension of the space of the solutions is 2. The only nonzero row gives us the equation $z - 2\,y + x = 0$, from which we obtain $z = 2\,y - x$ and then all the solutions will be $(x; y; z) = (t; s; 2\,s - t) = (1; 0; -1)\,t + (0; 1; 2)\,s, \qquad s, t \in \mathbb{R}$.

c) $\boldsymbol{A} = \begin{matrix} x \\ \\ \\ \end{matrix} \begin{matrix} y \\ \\ \\ \end{matrix} \begin{matrix} z \\ \\ \\ \end{matrix} \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & 1 \\ 3 & a & b \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & 7 & -3 \\ 0 & a+9 & b-6 \end{pmatrix} \Rightarrow \mathrm{r}(\boldsymbol{A}) = \begin{cases} 2 \\ 3 \end{cases}$, if the last two rows are $\begin{cases} \text{dependent} \\ \text{independent} \end{cases}$ or if the determinant $\begin{vmatrix} 7 & -3 \\ a+9 & b-6 \end{vmatrix} =$

$$= 7\,(b-6) + 3\,(a+9) = 7\,b + 3\,a - 15 \begin{cases} = 0, & a = 5 - \frac{7}{3}\,b \\ \neq 0, & a \neq 5 - \frac{7}{3}\,b \end{cases}.$$

For $a = 2,\ b = 6$ we have $2 \neq 5 - 7 \cdot 2 \quad \Rightarrow \quad$ the solution will be the trivial one $x = y = z = 0$.

■

• Solve the systems:

60. by Cramer's rule

$$\begin{cases} 2\,x - y + 3\,z = 13 \\ x + 3\,y - z = -8 \\ x + 2\,y + z = 0 \end{cases};$$

$[x = 1,\ y = -2,\ z = 3]$

61. By means of the inverse matrix

$$\begin{cases} x + y + z = 1 \\ 2\,y + z = -1 \\ 3\,x + 2\,y + 3\,z = 3 \end{cases};$$

$[x = 2,\ y = 0,\ z = -1]$

62. By the Gaussian elimination method

$$\begin{cases} x - y + z + 2\,v = 9 \\ -x + 3\,y - z + 3\,v = -5 \\ x + 2\,y + v = -3 \\ y + 3\,z - v = -2 \end{cases};$$

$[x = 1,\ y = -3,\ z = 1,\ v = 2]$

63.

$$\begin{cases} x + 2\,y - z + u = -3 \\ 2\,x - y + 2\,u = 3 \\ 3\,x + 2\,z - u = 7 \\ 3\,y + z + u = -1 \end{cases};$$

$[x = 1,\ y = -1,\ z = 2,\ u = 0]$

64.

$$\left\{\begin{array}{rcrcrcrcl} x & - & 4\,y & + & 2\,z & & & = & 1 \\ 3\,x & - & 7\,y & + & 11\,z & - & 15\,u & = & 3 \\ & & y & + & z & - & 3\,u & = & 0 \\ 4\,x & - & 11\,y & + & 13\,z & - & 15\,u & = & 4 \end{array}\right. ;$$

$$[(x;y;z;u) = (1;0;0;0) + (12;3;0;1)\,t + (-6;-1;1;0)\,s]$$

65.

$$\left\{\begin{array}{rcrcrcrcl} x_1 & + & 2\,x_2 & - & 3\,x_3 & + & x_4 & - & x_5 = 5 \\ -2\,x_1 & + & x_2 & + & 4\,x_3 & - & x_4 & + & 2\,x_5 = 7 \\ & & x_2 & + & 3\,x_3 & + & x_4 & - & x_5 = 2 \\ -x_1 & + & 2\,x_2 & - & 2\,x_3 & - & x_4 & + & 2\,x_5 = 0 \end{array}\right. .$$

[no solution exists]

• Using Frobenius' theorem carry out the discussion of the solutions of the systems depending on the values of the parameters which occur in them:

66.

$$\left\{\begin{array}{rcrcrcl} x & - & 2\,y & + & 4\,z & = & 7 \\ 2\,x & + & 3\,y & - & 3\,z & = & -4 \\ 3\,x & + & a\,y & + & z & = & c+1 \end{array}\right. ;$$

$$\left[\begin{array}{l} \text{for } a \neq 1 \text{ there is a unique solution} \\ \text{for } a = 1, c = 2 \text{ there are infinitely many solutions} \\ \text{for } a = 1, c \neq 2 \text{ no solution exists} \end{array}\right]$$

67.

$$\left\{\begin{array}{rcrcl} a\,x & - & y & = & 2 \\ x & + & a\,y & = & a \end{array}\right. ;$$

[a unique solution for all $a \in \mathbb{R}$]

68.

$$\left\{\begin{array}{rcrcrcl} x & + & y & + & a\,z & = & 1 \\ x & + & a\,y & + & z & = & 1 \\ a\,x & + & y & + & z & = & 1 \end{array}\right. .$$

$$\left[\begin{array}{l} \text{a unique solution for } a \in \mathbb{R}\setminus\{-2;1\} \\ \text{infinitely many solutions for } a = 1 \\ \text{no solution for } a = -2 \end{array}\right]$$

• Solve the homogeneous systems:

69.

$$\left\{\begin{array}{rcrcrcrcl} -x & + & y & + & z & + & u & = & 0 \\ x & - & y & + & z & + & u & = & 0 \\ x & + & y & - & z & + & u & = & 0 \\ x & + & y & + & z & + & 3\,u & = & 0 \end{array}\right. ;$$

$$[\dim V = 1, \quad (x;y;z;u) = (1;1;1;-1)\,t; \quad t \in \mathbb{R}]$$

70.

$$\left\{\begin{array}{rcrcrcrcl} x & + & 2\,y & + & 4\,z & - & 3\,u & = & 0 \\ 3\,x & + & 5\,y & + & 6\,z & - & 4\,u & = & 0 \\ 4\,x & + & 5\,y & - & 3\,z & + & 3\,u & = & 0 \\ 3\,x & + & 8\,y & + & 24\,z & - & 19\,u & = & 0 \end{array}\right. ;$$

$$[\dim V = 2, \quad (x;y;z;u) = (8;-6;1;0)\,t + (-7;5;0;1)\,s; \quad t,s \in \mathbb{R}]$$

71.

$$\begin{cases} 2x + 3y - z = 0 \\ x - y + 2z = 0 \\ x + 9y - 8z = 0 \\ x + y + z = 0 \end{cases}.$$

$$[(x;y;z) = (0;0;0)]$$

72. Let us have the vectors $\vec{a},\ \vec{b},\ \vec{c},\ \vec{d}$. Find all the numbers $\alpha, \beta, \gamma, \delta$ so that $\alpha\,\vec{a} + \beta\,\vec{b} + \gamma\,\vec{c} + \delta\,\vec{d} = \vec{0}$.

a) $\vec{a} = (1;2;-3),\ \vec{b} = (0;-2;-1),\ \vec{c} = (2;4;0),\ \vec{d} = (2;-2;3)$;

b) $\vec{a} = (1;0;2;3),\ \vec{b} = (4;2;-6;1),\ \vec{c} = (6;1;1;2),\ \vec{d} = (-1;1;-5;2)$;

c) $\vec{a} = (3;0;0;0),\ \vec{b} = (4;-1;0;0),\ \vec{c} = (2;3;-1;0),\ \vec{d} = (1;2;-2;5)$.

$$\left[\begin{array}{l} \text{a) } (\alpha;\beta;\gamma;\delta) = (2;-3;-2;1)\,t,\ t \in \mathbb{R}; \\ \text{b) } (\alpha;\beta;\gamma;\delta) = (1;1;-1;-1)\,t,\ t \in \mathbb{R}; \\ \text{c) } (\alpha;\beta;\gamma;\delta) = (0;0;0;0) \end{array}\right]$$

73. For which a does the system have a nonzero solution?

$$\begin{cases} ax + 4y + 7z = 0 \\ 3x - 4y + 5z = 0 \\ x + ay + 4z = 0 \end{cases}$$

$$[a \in \{0;1\}]$$

74. How many solutions do the following systems have depending on the values of the parameters which occur in them? Solve the systems for the given values of the parameters:

a) $\begin{cases} ax + 2y = 0 \\ x + by = 0 \end{cases}$, $a = 1,\ b = 3$;

b) $\begin{cases} ax - y + 7z = 0 \\ 3x + 4y + 20z = 0 \\ x - 4ay + 16z = 0 \end{cases}$, $a = 2$;

c) $\begin{cases} x - 3y + 2z = 0 \\ ax + by + z = 0 \\ 2y + z = 0 \end{cases}$, $a = 1,\ b = -5$.

$$\left[\begin{array}{l} \text{a) } \infty \text{ solutions for } a \cdot b = 2, \quad \text{for } a = 1,\ b = 3 \text{ only the zero solution;} \\ \text{b) } \infty \text{ solutions for } a \in \left\{0; \frac{1}{4}\right\}, \quad \text{for } a = 2 \text{ only the zero solution;} \\ \text{c) } \infty \text{ solutions for } b = 2 - 7a, \quad \text{for } a = 1,\ b = -5,\ (x;y;z) = (7;1;-2)\,t,\ t \in \mathbb{R} \end{array}\right]$$

I.4* Linear mappings of $\mathbb{E}_n$ into $\mathbb{E}_m$ or of $V(\mathbb{E}_n)$ into $V(\mathbb{E}_m)$

***Example* 75**: Let L be the linear mapping of $V(\mathbb{E}_n)$ into $V(\mathbb{E}_m)$ given by the matrix $\boldsymbol{A}$. Find the images of the vectors $\vec{u}$ and $\vec{v}$:

a) $\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix};$

b) $\boldsymbol{A} = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix};$

c) $\boldsymbol{A} = \begin{pmatrix} 1, 2, -3, 4 \end{pmatrix}, \quad \vec{u} = (-2; 3; 0; 2)^T, \quad \vec{v} = (0; 0; 0; 1)^T.$

Solution:

$$L: V(\mathbb{E}_n) \xrightarrow[\text{of the type } (m,n)]{\boldsymbol{A}} V(\mathbb{E}_m)$$

a) The matrix $\boldsymbol{A}$ is of the type (3, 2), and therefore it represents a mapping L from the space $V(\mathbb{E}_2)$ into the space $V(\mathbb{E}_3)$.

$$L(\vec{u}) = \boldsymbol{A} \cdot \vec{u} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix},$$

$$L(\vec{v}) = \boldsymbol{A} \cdot \vec{v} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ -4 \end{pmatrix};$$

b) $L: V(\mathbb{E}_3) \xrightarrow[\text{of the type } (2,3)]{\boldsymbol{A}} V(\mathbb{E}_2),$

$$L(\vec{u}) = \boldsymbol{A} \cdot \vec{u} = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 7 \end{pmatrix},$$

$$L(\vec{v}) = \boldsymbol{A} \cdot \vec{v} = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix};$$

c) $L: V(\mathbb{E}_4) \xrightarrow[\text{of the type } (1,4)]{\boldsymbol{A}} V(\mathbb{E}_1),$

$$L(\vec{u}) = \boldsymbol{A} \cdot \vec{u} = \begin{pmatrix} 1, 2, -3, 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ 0 \\ 2 \end{pmatrix} = (-2 + 6 + 8) = (12),$$

$$L(\vec{v}) = \boldsymbol{A} \cdot \vec{v} = (1; 2; -3; 4) \cdot (0; 0; 0; 1)^T = (4). \qquad \blacksquare$$

***Example* 76**: Let L be the mapping of $\mathbb{E}_3$ into $\mathbb{E}_3$ given by the system:

$$\text{a)}\quad \left\{\begin{array}{rcrcrcl} x' &=& x &+& 2\,y &-& z \\ y' &=& x &+& 2\,y &-& z \\ z' &=& x &+& 2\,y &-& z \end{array}\right. ;$$

$$\text{b)}\quad \left\{\begin{array}{rcrcrcl} x' &=& x &+& y &+& z \\ y' &=& & & y &-& z \\ z' &=& x &+& 2\,y & & \end{array}\right. ;$$

$$\text{c)}\quad \left\{\begin{array}{rcrcrcl} x' &=& x &-& y &+& 3\,z \\ y' &=& & & y &+& 2\,z \\ z' &=& & & & & -z \end{array}\right. .$$

Find the matrix $\boldsymbol{A}$ corresponding to the mapping L and determine the dimension of the image $L(\mathbb{E}_3)$. Find out whether the mapping L^{-1} exists and if so, find the matrix of the inverse mapping.

Solution:

a) The matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{pmatrix}$ is formed from the coefficients of the transformation formulas.

We know that $\mathrm{r}(\boldsymbol{A}) = \dim L(\mathbb{E}_n)$. In our case $\mathrm{r}(\boldsymbol{A}) = 1 = \dim L(\mathbb{E}_3)$. Indeed, from the expression it is seen that $x' = y' = z'$, hence $L(\mathbb{E}_3) = \{[t;t;t], t \in \mathbb{R}\}$. The matrix $\boldsymbol{A}$ is not regular $(\det \boldsymbol{A} = 0)$, $\boldsymbol{A}^{-1}$ does not exist and the inverse mapping L^{-1} does not exist either.

b)

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \mathrm{r}(\boldsymbol{A}) = 2,\ \dim L(\mathbb{E}_3) = 2.$$

Again the matrix $\boldsymbol{A}$ is not regular and $\boldsymbol{A}^{-1}$ together with L^{-1} do not exist.

c)

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \mathrm{r}(\boldsymbol{A}) = 3 = \dim L(\mathbb{E}_3).$$

The matrix $\boldsymbol{A}$ is regular and $\boldsymbol{A}^{-1}$ and L^{-1} exist. Between regular mappings and their matrices there holds the relation:

$$\begin{array}{rcl@{\qquad}rcl} L &\longleftrightarrow& \boldsymbol{A}, & & L^{-1} &\longleftrightarrow& \boldsymbol{A}^{-1}, \\ L^2 &\longleftrightarrow& \boldsymbol{A}^2, & L \circ L^{-1} = \text{identity} &\longleftrightarrow& \boldsymbol{E} = \text{the identity matrix.} \end{array}$$

$$\vdots$$

Now we compute $\boldsymbol{A}^{-1}$ by the Gaussian elimination method:

$$(\boldsymbol{A}|\,\boldsymbol{E}) = \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) = \left(\boldsymbol{E}|\boldsymbol{A}^{-1}\right) \quad\Longrightarrow\quad \boldsymbol{A}^{-1} = \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

We choose the vector $\vec{u} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ and we compute $L(\vec{u}),\ L^{-1}\Big(L\,(\vec{u})\Big)$:

$$L\,(\vec{u}) = \boldsymbol{A} \cdot \vec{u} = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix},$$

$$L^{-1}\Big(L\,(\vec{u})\Big) = \boldsymbol{A}^{-1}\Big(\boldsymbol{A}\,\vec{u}\Big) = \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \vec{u}. \qquad \blacksquare$$

***Example* 77**: The linear mapping L is given by the matrix $\boldsymbol{A}$. Compute $L(\vec{a}),\quad L(2\,\vec{a}),\ L(2\,\vec{a}+3\,\vec{b}),\quad L\big(L(\vec{a})\big)$, where $\boldsymbol{A} = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix},\quad \vec{a} = (2;-1;3)^T,$ $\vec{b} = (-2;1;-1)^T$.

Solution:

$$L(\vec{a}) = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \end{pmatrix};$$

When computing $L(2\,\vec{a})$ and $L(2\,\vec{a}+3\,\vec{b})$ we use the property that the mapping L is linear:

$$L(2\,\vec{a}) = 2\,L(\vec{a}) = 2 \cdot \begin{pmatrix} 9 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix}$$

$$L(2\,\vec{a}+3\,\vec{b}) = 2\,L(\vec{a}) + 3\,L(\vec{b}) = \begin{pmatrix} 18 \\ -8 \end{pmatrix} + 3 \cdot \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 18 \\ -8 \end{pmatrix} + 3 \cdot \begin{pmatrix} -7 \\ 4 \end{pmatrix} = \begin{pmatrix} 18-21 \\ -8+12 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

The last image $L\big(L(\vec{a})\big)$ does not exist, because the corresponding matrix product $\boldsymbol{A} \cdot (\boldsymbol{A}\,\vec{a})$ does not exist either. $\blacksquare$

***Example* 78**: Let us have the linear mappings L_1, L_2 given by the matrices $\boldsymbol{A}_1, \boldsymbol{A}_2$. Find the matrices of the mappings

$L_1 \circ L_2,\ L_2 \circ L_1,\ L_1^{-1},\ L_2^{-1},\ (L_1 \circ L_2)^{-1}$ and $(L_2 \circ L_1)^{-1}$ (if they exist), where $\boldsymbol{A}_1 = \begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 2 \end{pmatrix},\quad \boldsymbol{A}_2 = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}.$

Solution:

$$\begin{array}{rcl} L_1 & \longleftrightarrow & \boldsymbol{A}_1, \\ L_1 \circ L_2 & \longleftrightarrow & \boldsymbol{A}_1\,\boldsymbol{A}_2, \\ L_1^{-1} & \longleftrightarrow & \boldsymbol{A}_1^{-1}. \end{array} \qquad \begin{array}{rcl} L_2 & \longleftrightarrow & \boldsymbol{A}_2, \\ L_2 \circ L_1 & \longleftrightarrow & \boldsymbol{A}_2\,\boldsymbol{A}_1, \end{array}$$

Because the product $\boldsymbol{A}_1\,\boldsymbol{A}_2$ does not exist $(2;3) \times (2;2)$, the composite mapping does not exist either $L_1 \circ L_2$.

$$\boldsymbol{A}_2\,\boldsymbol{A}_1 = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 10 & 4 \\ 6 & 5 & 5 \end{pmatrix} \longleftrightarrow L_2 \circ L_1.$$

Of the inverse mappings only one exists, namely L_2^{-1} with the matrix

$$\boldsymbol{A}_2^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -5 \\ -1 & 2 \end{pmatrix}, \qquad \text{the other inverse mappings do not exist.} \quad \blacksquare$$

***Example* 79**: Let us have a mapping of the points of $\mathbb{E}_2 \longmapsto \mathbb{E}_2$. Write the matrices corresponding to

a) the symmetry with respect to the axis x,

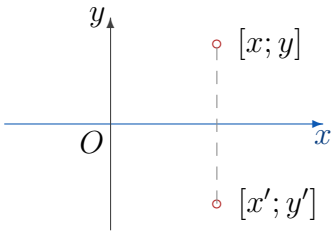
b) the symmetry with respect to the axis y,

c) the rotation of the coordinate system about the origin through the angle α,

d) the reflection in the straight line $y = k\,x$.

Solution:

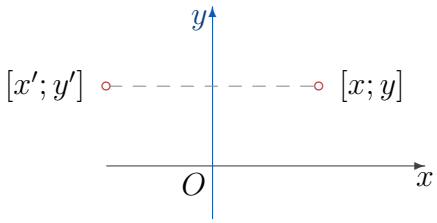
a)



$$\begin{cases} x' = & x \\ y' = & -y \end{cases}.$$

The symmetry with respect to the axis x has the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

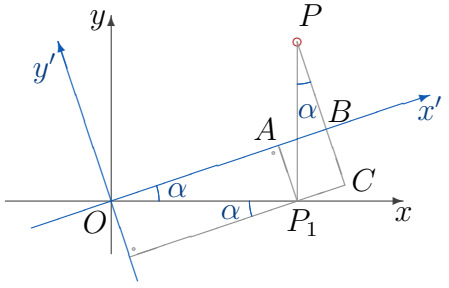
b)



$$\begin{cases} x' = & -x \\ y' = & y \end{cases}.$$

The symmetry with respect to the axis y has the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

c) We derive the transformation formulas for the coordinates of a fixed point P under the rotation of the rectangular coordinate system about its origin through the angle α. Let us denote the original coordinates of the point $P = [x; y]$ and after the rotation $P = [x'; y']$, where



$$\begin{array}{llllllll} x & = & OP_1, & x' & = & OB, & AB & = & P_1C, \\ y & = & P_1P, & y' & = & BP, & CB & = & P_1A. \end{array}$$

Now we derive x' and y' by means of x, y and α:

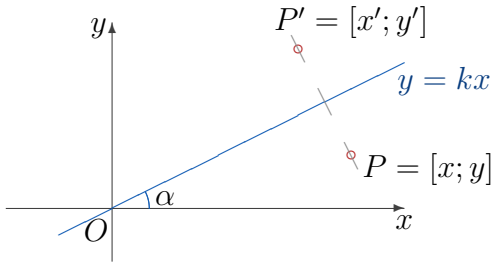
$$x' = OB = OA + AB = OA + P_1C = OP_1 \cdot \cos\alpha + P_1P \cdot \sin\alpha = x \cdot \cos\alpha + y \cdot \sin\alpha,$$
$$y' = BP = CP - CB = P_1P \cdot \cos\alpha - OP_1 \cdot \sin\alpha = y \cdot \cos\alpha - x \cdot \sin\alpha.$$

The rotation of the axis x through the angle α into the position of the axis x' is the same as the rotation of the point P about the origin through the angle $(-\alpha)$ with the axis x fixed. This mapping satisfies the transformation formulas:

$$\begin{cases} x' = \ x \cdot \cos\alpha + y \cdot \sin\alpha \\ y' = -x \cdot \sin\alpha + y \cdot \cos\alpha \end{cases} \quad \text{and has the matrix } \boldsymbol{A}(-\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

The rotation of the point P through the angle α (of the axis x through the angle $-\alpha$) has the matrix $\boldsymbol{A}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, which arose from the matrix $\boldsymbol{A}(-\alpha)$, where for α we substituted $(-\alpha)$ and used the properties $\cos(-\alpha) = \cos\alpha$ and $\sin(-\alpha) = -\sin\alpha$.

d)



$\tan\alpha = k.$

At first sight it seems that the reflection in the straight line $y = kx$ is a rather complicated mapping. However, we use the mappings from a) and from c). Let us denote the mapping sought $L = L_3 \circ L_2 \circ L_1$, $L(P) = L_3\Big(L_2\big(L_1(P)\big)\Big)$, where the individual mappings are:

L_1 is the rotation of the point P through the angle $(-\alpha)$, that is, the straight line $y = k\,x$ gets into the position of the axis x,

L_2 is the symmetry of the points with respect to the axis x,

L_3 is the rotation of the points through the angle α, that is, the straight line $y = k\,x$ gets back to its original position.

$$\begin{aligned} L = L_3 \circ L_2 \circ L_1 \quad &\longleftrightarrow \quad \boldsymbol{A} = \boldsymbol{A}_3 \cdot \Big(\boldsymbol{A}_2 \cdot \boldsymbol{A}_1\Big) = \\ &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \right) = \\ &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} = \\ &= \begin{pmatrix} \cos^2\alpha - \sin^2\alpha & \sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha + \sin\alpha\cos\alpha & \sin^2\alpha - \cos^2\alpha \end{pmatrix} = \\ &= \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}. \end{aligned}$$

Now we compute $\sin 2\,\alpha$ and $\cos 2\,\alpha$ by means of $\tan\alpha = k$:

$$\sin 2\,\alpha = \frac{2\,\sin\alpha\cos\alpha}{\sin^2\alpha+\cos^2\alpha} = \frac{\frac{2\,\sin\alpha\cos\alpha}{\cos^2\alpha}}{\frac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha}} = \frac{2\,\tan\alpha}{\tan^2\alpha+1},$$

$\sin 2\,\alpha = \dfrac{2\,k}{k^2+1}$, similarly $\cos 2\,\alpha = \dfrac{1-k^2}{k^2+1}$.

The resulting form of the matrix $\boldsymbol{A}$ corresponding to the symmetry with respect to the straight line $y = k\,x$ is

$$\boldsymbol{A} = \begin{pmatrix} \frac{1-k^2}{k^2+1} & \frac{2\,k}{k^2+1} \\ \frac{2\,k}{k^2+1} & \frac{k^2-1}{k^2+1} \end{pmatrix}, \text{ where } \det\boldsymbol{A} = -1. \qquad \blacksquare$$

80. Given the mappings L_1, L_2 with the corresponding matrices $\boldsymbol{A}_1, \boldsymbol{A}_2$. Find the matrices of the mappings

$L_1 \circ L_1,\ L_1 \circ L_2,\ L_2 \circ L_1,\ L_1^{-1},\ (L_2 \circ L_1)^{-1}$, if they exist, where

a) $\boldsymbol{A}_1 = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} -1 & 2 & 0 \\ 4 & -1 & 3 \end{pmatrix};$

b) $\boldsymbol{A}_1 = \begin{pmatrix} 4 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 1 & 1 \end{pmatrix}.$

$$\left[\begin{array}{c} \text{a)}\boldsymbol{A}^2 = \begin{pmatrix} 1 & -4 \\ 12 & 1 \end{pmatrix},\ \boldsymbol{A}_1\boldsymbol{A}_2 = \begin{pmatrix} -6 & 5 & -3 \\ 5 & 4 & 6 \end{pmatrix},\ \boldsymbol{A}_1^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} \\ \text{the others do not exist} \\ \text{b)}\boldsymbol{A}_1\boldsymbol{A}_2 = \begin{pmatrix} 0 & -2 & 2 \\ -3 & 2 & 1 \\ -3 & 1 & 2 \end{pmatrix},\ \boldsymbol{A}_2\boldsymbol{A}_1 = \begin{pmatrix} 5 & 1 \\ -8 & -1 \end{pmatrix},\ \left(\boldsymbol{A}_2\boldsymbol{A}_1\right)^{-1} = \frac{1}{3}\begin{pmatrix} -1 & -1 \\ 8 & 5 \end{pmatrix} \\ \text{the others do not exist} \end{array}\right]$$

81. The mapping $L : V(\mathbb{E}_3) \longmapsto V(\mathbb{E}_3)$ is given by the matrix $\boldsymbol{A}$. Find the vector $\vec{u}$ whose image under the mapping L is the vector $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{pmatrix}$. [Hint: $\vec{v} = \boldsymbol{A}\vec{u}$.]

$[\vec{u} = \left(-\frac{1}{2}; -9; 4\right)^T]$

82. Let the mapping $L : V(\mathbb{E}_3) \longmapsto V(\mathbb{E}_2)$ be given by the matrix $\boldsymbol{A}$. Find all the vectors which are mapped onto the zero vector, if $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -3 \\ -2 & -3 & 5 \end{pmatrix}$.

[Hint: $\boldsymbol{A}\vec{u} = \vec{0}$.]

$[\vec{u} = \left(1; 1; 1\right)^T \cdot t,\ t \in \mathbb{R}]$

83. The linear mapping $L : \mathbb{E}_n \longmapsto \mathbb{E}_m$ is given by the matrix $\boldsymbol{A}$. Find the numbers n, m and the dimension of the space $L(\mathbb{E}_n) \subset \mathbb{E}_m$, if

a) $\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 1 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 3 & -6 \end{pmatrix};$ b) $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -3 \\ 4 & 1 & -1 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}.$

$$\left[\begin{array}{l}\text{a) } n=4,\ m=3,\ \dim=2\\ \text{b) } n=3,\ m=4,\ \dim=3\end{array}\right]$$

84. Let the mapping L_1 be the rotation of the points through the angle α and L_2 the rotation of the points through the angle β. Compute the matrices of the mappings $L_1 \circ L_2,\ L_1^{-1}$. (Use example **79.)**

$$\left[\begin{pmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{pmatrix}, \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\right]$$

85. Let the mapping L be the reflection in the straight line $y = 2\,x$. Compute the coordinates of the images of the points $P = [3; -1],\ Q = [2; 4]$. (Use example **79.)**

$$\left[\begin{array}{l} L \longleftrightarrow \boldsymbol{A} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{pmatrix}\\ L(P) = \left[-\frac{13}{5}; \frac{9}{5}\right]\\ L(Q) = [2; 4] = Q, \text{the point } Q \text{ lies on the straight line } y = 2\,x\end{array}\right]$$

I.5 Eigenvalues and eigenvectors of a matrix

Throughout this section, eigenvectors are sought in $\mathbb{C}^n$. Every scalar parameter t or t_j multiplying a displayed eigenvector ranges over $\mathbb{C}\setminus\{0\}$, unless stated otherwise. The parameter is chosen independently for each eigenvector family. Thus each displayed multiple describes a family of nonzero eigenvectors.

***Example* 86**: Find the eigenvalues and the corresponding eigenvectors of the matrices:

$$\text{a)}\ \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}; \qquad \text{b)}\ \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}; \qquad \text{c)}\ \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Solution:

a) To find the eigenvalues and the eigenvectors of a square matrix $\boldsymbol{A}$ means to find a vector $\vec{u} \neq \vec{0}$, for which $\boldsymbol{A}\vec{u} = \lambda\vec{u}$ holds. In other words: the preimage and the image are collinear vectors for the linear mapping given by the matrix $\boldsymbol{A}$. Such a vector $\vec{u}$ is called an **eigenvector** and the number λ an **eigenvalue** of the matrix $\boldsymbol{A}$. Let us denote $\vec{u} = (u_1; u_2)^T$ and write the relation in coordinates $\boldsymbol{A}\vec{u} = \lambda\,\vec{u}$:

$$\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \lambda\,u_1 \\ \lambda\,u_2 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} -3\,u_1 & + & 2\,u_2 = & \lambda\,u_1 \\ -2\,u_1 & + & u_2 = & \lambda\,u_2 \end{cases} \quad \Longrightarrow$$

$$(\mathbf{1}) \quad \begin{cases} (-3-\lambda)\,u_1 & + & 2\,u_2 = & 0 \\ -2\,u_1 & + & (1-\lambda)\,u_2 = & 0 \end{cases}.$$

We look for a nontrivial solution of this homogeneous system. For that it is necessary that the determinant of the matrix of the system equals zero:

$$(\mathbf{2}) \quad \begin{vmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (-3-\lambda)(1-\lambda)+4=0 \quad \Rightarrow \quad \lambda^2+2\,\lambda+1=0.$$

The last equation is called the **characteristic equation**.

$$(\lambda+1)^2 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = -1.$$

Now we substitute $\lambda = -1$ into (1) and compute the vector $\vec{u}$:

$$\begin{cases} -2\,u_1 & + & 2\,u_2 = & 0 \\ -2\,u_1 & + & 2\,u_2 = & 0 \end{cases}.$$

Here we have a guarantee (from $\det(\boldsymbol{A}-\lambda\boldsymbol{E}) = 0$) that the equations will be dependent, therefore it is enough to substitute into one of them only. In our case

$$u_1 = u_2 \quad \Rightarrow \quad \vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot t, \quad t \in \mathbb{C}\setminus\{0\}.$$

We can also verify that $\boldsymbol{A}\,\vec{u} = \lambda\,\vec{u}$ holds:

$$\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot t = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot t.$$

REMARK: In the following examples we shall proceed more quickly. From the given matrix $\boldsymbol{A}$ we immediately set up the determinant of the matrix $(\boldsymbol{A} - \lambda\,\boldsymbol{E})$, that is, on the main diagonal of the matrix $\boldsymbol{A}$ we subtract λ. And we put this determinant equal to 0. In this way we obtain the characteristic equation.

b)

$$\begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{vmatrix} 2-\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (2-\lambda)(1-\lambda) - 6 = 0 \quad \Rightarrow$$

$$\lambda^2 - 3\,\lambda - 4 = 0 \quad \Rightarrow \quad (\lambda - 4)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = 4.$$

For $\underline{\lambda_1 = -1}$ we look for the solution of the equation $3\,u_1 + 2\,u_2 = 0$. This equation can be written as the scalar product $(3;2) \cdot (u_1;u_2)^T = 0$. The vector $\vec{u} = (u_1;u_2)^T$ is perpendicular to the vector $(3;2)$ and therefore $\vec{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot t_1, \quad t_1 \in \mathbb{C}\backslash\{0\}$.

For $\underline{\lambda_2 = 4}$ we look for the solution of the equation $-2\,v_1 + 2\,v_2 = 0$. The vector $\vec{v} = (v_1;v_2)^T$ is perpendicular to the vector $(-2;2)$ and therefore $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot t_2, \quad t_2 \in \mathbb{C}\backslash\{0\}$.

For the vectors determined in this way there holds : $\boldsymbol{A}\,\vec{u} = -\vec{u}, \qquad \boldsymbol{A}\,\vec{v} = 4\,\vec{v}.$

REMARK: We always include all nonzero complex multiples of each eigenvector. If an eigenspace has dimension greater than one, all nonzero linear combinations of a basis of that eigenspace must be included.

c) $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)^2 + 4 = 0 \quad \Rightarrow$

$$\lambda^2 - 2\,\lambda + 5 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2\,i.$$

[We know that the given matrix must have the entries $a_{ij} \in \mathbb{R}$, but the eigenvalues and the eigenvectors are looked for in the field of complex numbers.]

$\underline{\lambda_1 = 1 + 2\,i}$:

$$\Big(1 - (1 + 2\,i)\Big)\,u_1 + 2\,u_2 = 0 \;\Rightarrow\; -2\,i\,u_1 + 2\,u_2 = 0 \quad \Rightarrow \quad \vec{u} = \begin{pmatrix} 1 \\ i \end{pmatrix} \cdot t_1,\; t_1 \in \mathbb{C}\backslash\{0\},$$

$\underline{\lambda_2 = 1 - 2\,i}$:

$$\Big(1 - (1 - 2\,i)\Big)\,v_1 + 2\,v_2 = 0 \;\Rightarrow\; 2\,i\,v_1 + 2\,v_2 = 0 \quad \Rightarrow \quad \vec{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot t_2,\; t_2 \in \mathbb{C}\backslash\{0\}.$$

The displayed direction vectors for the two conjugate eigenvalues are complex conjugates. Each may be multiplied independently by any nonzero complex scalar. ■

***Example* 87**: Find the eigenvalues and the eigenvectors of the given matrices:

a) $\begin{pmatrix} 5 & 6 & -3 \\ 6 & 9 & 0 \\ -3 & 0 & 9 \end{pmatrix}$; b) $\begin{pmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$.

Solution:

a) $\begin{vmatrix} 5-\lambda & 6 & -3 \\ 6 & 9-\lambda & 0 \\ -3 & 0 & 9-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (5-\lambda)(9-\lambda)^2 - 9\,(9-\lambda) - 36\,(9-\lambda) = 0,$

$$(9-\lambda)\big((5-\lambda)(9-\lambda) - 45\big) = 0 \quad \Rightarrow \quad (9-\lambda)(\lambda^2 - 14\,\lambda) = 0 \quad \Rightarrow$$
$$\Rightarrow \quad \lambda_1 = 9, \quad \lambda_2 = 0, \quad \lambda_3 = 14.$$

To compute the eigenvectors it is enough to substitute into such two rows of the determinant that the resulting equations are independent.

$\underline{\lambda_1 = 9}$: $\left\{ \begin{array}{rcrcrcl} -4\,u_1 & + & 6\,u_2 & - & 3\,u_3 & = & 0 \\ 6\,u_1 & & & & & = & 0 \end{array} \right. \quad \Rightarrow \quad u_1 = 0,\ u_3 = 2\,u_2 \quad \Rightarrow$

$$\vec{u} = (0;1;2)^T \cdot t_1, \qquad \boldsymbol{A}\,\vec{u} = 9\,\vec{u};$$

$\underline{\lambda_2 = 0}$: $\left\{ \begin{array}{rcrcrcl} 6\,v_1 & + & 9\,v_2 & & & = & 0 \quad \Rightarrow \quad v_2 = -\dfrac{2}{3}\,v_1 \\ -3\,v_1 & & & + & 9\,v_3 & = & 0 \quad \Rightarrow \quad v_3 = \dfrac{1}{3}\,v_1 \end{array} \right.,$

$$\vec{v} = (3;-2;1)^T \cdot t_2, \qquad \boldsymbol{A}\,\vec{v} = \vec{0};$$

$\underline{\lambda_3 = 14}$: $\left\{ \begin{array}{rcrcrcl} 6\,w_1 & - & 5\,w_2 & & & = & 0 \quad \Rightarrow \quad w_2 = \dfrac{6}{5}\,w_1 \\ -3\,w_1 & & & - & 5\,w_3 & = & 0 \quad \Rightarrow \quad w_3 = -\dfrac{3}{5}\,w_1 \end{array} \right.,$

$$\vec{w} = (5;6;-3)^T \cdot t_3, \qquad \boldsymbol{A}\,\vec{w} = 14\,\vec{w}; \quad t_i \in \mathbb{C}\backslash\{0\}\,.$$

b) $\begin{vmatrix} 2-\lambda & 1 & -2 \\ -1 & -\lambda & 0 \\ 1 & 1 & -1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (2-\lambda)\lambda(1+\lambda) + 2 - 2\,\lambda - (1+\lambda) = 0,$

$$(2-\lambda)(\lambda+\lambda^2) + 1 - 3\,\lambda = 0 \quad \Rightarrow \quad 2\lambda + 2\lambda^2 - \lambda^2 - \lambda^3 + 1 - 3\lambda = 0,$$
$$-\lambda^3 + \lambda^2 - \lambda + 1 = 0 \quad \Rightarrow \quad \lambda^3 - \lambda^2 + \lambda - 1 = 0 \quad \Rightarrow \quad \lambda^2(\lambda-1) + (\lambda-1) = 0,$$

$$(\lambda-1)(\lambda^2+1) = 0 \quad \Rightarrow \quad \lambda_1 = 1, \quad \lambda_{2,3} = \pm i.$$

$\underline{\lambda_1 = 1}$: $\left\{ \begin{array}{rcrcrcl} u_1 & + & u_2 & - & 2\,u_3 & = & 0 \\ -u_1 & - & u_2 & & & = & 0 \end{array} \right. \Rightarrow\ u_2 = -u_1 \ \Rightarrow\ u_3 = \dfrac{1}{2}(u_1 + u_2) = 0\ ,$

$$\vec{u} = (1;-1;0)^T \cdot t_1, \qquad \boldsymbol{A}\,\vec{u} = \vec{u};$$

$\underline{\lambda_2 = i}$: $\left\{ \begin{array}{rcrcrcl} (2-i)\,v_1 & + & v_2 & - & 2\,v_3 & = & 0 \\ -v_1 & - & i\,v_2 & & & = & 0 \quad \Rightarrow \quad v_2 = -\dfrac{1}{i}\,v_1 \quad \Rightarrow v_2 = i\,v_1 \\ & & & & & & v_3 = \dfrac{1}{2}\Big((2-i)\,v_1 + v_2\Big) = v_1 \end{array} \right.,$

$$\vec{v} = (1;i;1)^T \cdot t_2.$$

$\underline{\lambda_3 = -i} \quad \Rightarrow \quad \vec{w} = (1;-i;1)^T \cdot t_3.$ The two displayed direction vectors are complex conjugates; the nonzero parameters t_2 and t_3 are independent. ■

***Example* 88**: Let $\boldsymbol{A}$ be a square matrix of the type $(3,3)$ with real entries. Decide whether its eigenvalues and the corresponding eigenvector families could be:

a) $1,\ -1,\ 1+i$; b) $2,\ 2+i,\ 2+i$;

c) $1,\ 1+i,\ 1-i$ and $\begin{pmatrix}1\\2\\0\end{pmatrix}\cdot t,\ \begin{pmatrix}2\\2+i\\1\end{pmatrix}\cdot t,\ \begin{pmatrix}1\\2\\1-i\end{pmatrix}\cdot t$;

d) $1,\ 1+i,\ 1-i$ and $\begin{pmatrix}3\\1\\-1\end{pmatrix}\cdot t,\ \begin{pmatrix}2\\-1\\2+i\end{pmatrix}\cdot t,\ \begin{pmatrix}2\\-1\\2-i\end{pmatrix}\cdot t$;

e) $i,\ 1+i,\ 1-i$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot t,\ \begin{pmatrix}2\\1\\i\end{pmatrix}\cdot t,\ \begin{pmatrix}2\\1\\-i\end{pmatrix}\cdot t$.

Solution:

a) **NO,** because an equation of the 3rd degree with real coefficients has either all its roots real or one real and two complex conjugate ones.
b) **NO;**
c) the numbers $1,\ 1+i,\ 1-i$ could be eigenvalues, but the vectors cannot be eigenvectors. The overall answer is **NO**.
d) **YES** the numbers and the coordinates of the vectors satisfy the requirements.
e) **NO,** see a). ■

***Example* 89**: Let us have a square matrix $\boldsymbol{A}$. Find its eigenvalues and eigenvectors. What are the eigenvalues and the eigenvectors of the matrix $\boldsymbol{A}^2$ and $\boldsymbol{A}^{-1}$, if $\boldsymbol{A}=\begin{pmatrix}2&0\\1&3\end{pmatrix}$?

Solution: First we compute the eigenvalues and the eigenvectors of the matrix $\boldsymbol{A}$.

$$\begin{vmatrix}2-\lambda&0\\1&3-\lambda\end{vmatrix}=0 \quad\Rightarrow\quad (2-\lambda)(3-\lambda)=0 \quad\Rightarrow\quad \lambda_1=2,\ \lambda_2=3,$$

$$\underline{\lambda_1=2}: \quad u_1+u_2=0 \quad\Longleftrightarrow\quad \vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}\cdot t,$$
$$\underline{\lambda_2=3}: \quad -v_1=0 \quad\Longleftrightarrow\quad \vec{v}=\begin{pmatrix}0\\1\end{pmatrix}\cdot t.$$

Now we compute $\boldsymbol{A}^2=\begin{pmatrix}2&0\\1&3\end{pmatrix}\cdot\begin{pmatrix}2&0\\1&3\end{pmatrix}=\begin{pmatrix}4&0\\5&9\end{pmatrix}$.

From the characteristic equation $\begin{vmatrix}4-\lambda&0\\5&9-\lambda\end{vmatrix}=0 \quad\Rightarrow\quad \lambda_1=4,\ \lambda_2=9,$

$$\underline{\lambda_1=4}: \quad 5\,u_1+5\,u_2=0 \quad\Longleftrightarrow\quad \vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}\cdot t,$$
$$\underline{\lambda_2=9}: \quad -5\,v_1=0 \quad\Longleftrightarrow\quad \vec{v}=\begin{pmatrix}0\\1\end{pmatrix}\cdot t.$$

For the matrix $\boldsymbol{A}^{-1}$ we determine its eigenvalues

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}.$$

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 \\ -\frac{1}{6} & \frac{1}{3}-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{3},$$

$$\underline{\lambda_1 = \frac{1}{2}}: \quad -\frac{1}{6}u_1 - \frac{1}{6}u_2 = 0 \quad \Rightarrow \quad u_1 + u_2 = 0 \quad \Longleftrightarrow \quad \vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot t,$$

$$\underline{\lambda_2 = \frac{1}{3}}: \quad -\frac{1}{6}v_1 = 0 \quad \Longleftrightarrow \quad \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot t.$$

Let us sum it all up:

The matrix $\boldsymbol{A}$ has the eigenvalues $\begin{cases} \lambda_1 = 2 \\ \lambda_2 = 3 \end{cases}$ and the eigenvectors $\begin{cases} \vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot t \\ \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot t \end{cases}$.

The matrix $\boldsymbol{A}^2$ has the eigenvalues $\begin{cases} \lambda_1 = 4 = 2^2 \\ \lambda_2 = 9 = 3^2 \end{cases}$

and the eigenvectors $\begin{cases} \vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot t \\ \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot t \end{cases}$.

The matrix $\boldsymbol{A}^{-1}$ has the eigenvalues $\begin{cases} \lambda_1 = \frac{1}{2} \\ \lambda_2 = \frac{1}{3} \end{cases}$

and the eigenvectors $\begin{cases} \vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot t \\ \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot t \end{cases}$.

We have computed everything by direct computation on purpose. Of course we could have computed also in this way:

$$\boldsymbol{A}\,\vec{u} = \lambda\,\vec{u} \quad \Rightarrow \quad \boldsymbol{A}^2\,\vec{u} = \boldsymbol{A}(\boldsymbol{A}\,\vec{u}) = \boldsymbol{A}(\lambda\vec{u}) = \lambda \cdot \boldsymbol{A}\,\vec{u} = \lambda^2\,\vec{u},$$

further, similarly there holds $\boldsymbol{A}^3\,\vec{u} = \lambda^3\,\vec{u}, \ldots$

If we multiply the relation $\boldsymbol{A}\,\vec{u} = \lambda\,\vec{u}$ by the matrix $\boldsymbol{A}^{-1}$ from the left, then

$$\vec{u} = \lambda\,\boldsymbol{A}^{-1}\,\vec{u} \quad \Rightarrow \quad \boldsymbol{A}^{-1}\,\vec{u} = \frac{1}{\lambda}\,\vec{u}.$$

For this matrix, the eigenvectors of all the mappings $\boldsymbol{A},\ \boldsymbol{A}^2,\ \boldsymbol{A}^3,\ \boldsymbol{A}^{-1}\ldots$ are the same. ■

***Example* 90**: Let us have a matrix $\boldsymbol{A}$ of the type $(3,3)$ with the eigenvalues $-2, 1+i, 1-i$. Find the eigenvalues of the matrix $\boldsymbol{A}^2$ and $\boldsymbol{A}^{-1}$.

Solution: If the matrix $\boldsymbol{A}$ has the eigenvalue λ, then $\boldsymbol{A}^2$ has λ^2 and $\boldsymbol{A}^{-1}$ has $\dfrac{1}{\lambda}$.

Hence the matrix $\boldsymbol{A}^2$ will have the eigenvalues

$$(-2)^2 = 4, \quad (1+i)^2 = 1 + 2\,i + i^2 = 2\,i, \quad (1-i)^2 = 1 - 2\,i + i^2 = -2\,i$$

and the matrix $\boldsymbol{A}^{-1}$ will have the eigenvalues

$$-\frac{1}{2}, \quad \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2}, \quad \frac{1}{1-i} = \frac{1+i}{2}.$$

91. Find the eigenvalues and the eigenvectors of the matrices

a) $\begin{pmatrix} -1 & 5 \\ 1 & 3 \end{pmatrix}$; b) $\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$; c) $\begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}$;

d) $\begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix}$; e) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; f) $\begin{pmatrix} 3 & -1 & 0 \\ 6 & -3 & 2 \\ 8 & -6 & 5 \end{pmatrix}$.

$$\left[\begin{array}{ll} \text{a) } -2, 4;\ \begin{pmatrix} -5 \\ 1 \end{pmatrix} \cdot t, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot t; & \text{b) } \pm i;\ \begin{pmatrix} -2 \\ 1-i \end{pmatrix} \cdot t, \begin{pmatrix} -2 \\ 1+i \end{pmatrix} \cdot t; \\ \text{c) } 3, 2;\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot t, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot t; & \text{d) } 3 \pm 2\,i;\ \begin{pmatrix} 1 \\ 1+2\,i \end{pmatrix} \cdot t, \begin{pmatrix} 1 \\ 1-2\,i \end{pmatrix} \cdot t; \\ \text{e) } 1, -1, 2;\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot t, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \cdot t, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot t; & \text{f) } 1, 2 \pm i;\ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot t, \begin{pmatrix} 1+i \\ 2 \\ 2-2\,i \end{pmatrix} \cdot t, \begin{pmatrix} 1-i \\ 2 \\ 2+2\,i \end{pmatrix} \cdot t \end{array}\right]$$

92. Show that the image of the vector $\vec{u}$ under the mapping given by the matrix $\boldsymbol{A}$ is collinear with $\vec{u}$. Do there exist further vectors with this property ?

Find them, if $\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot t$, $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

$[(-1;1)^T \cdot t$ or $(1;3)^T \cdot s, \quad t, s \in \mathbb{C}\backslash\{0\}]$

93. The matrix has the eigenvalues $-3, i, -i$ and the eigenvectors

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot t, \begin{pmatrix} -1 \\ 0 \\ 1+i \end{pmatrix} \cdot t, \begin{pmatrix} -1 \\ 0 \\ 1-i \end{pmatrix} \cdot t.$$

Answer whether it is possible that the matrix $\boldsymbol{A}^2$ has the following eigenvalues and eigenvectors:

a) $9, -1, -1$; $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot t, \begin{pmatrix} 1 \\ 0 \\ 2+i \end{pmatrix} \cdot t, \begin{pmatrix} 1 \\ 0 \\ 2-i \end{pmatrix} \cdot t$;

b) $-9, 1, -1$; $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot t, \begin{pmatrix} -1 \\ 0 \\ 1+i \end{pmatrix} \cdot t, \begin{pmatrix} -1 \\ 0 \\ 1-i \end{pmatrix} \cdot t$;

c) $9, -1, -1$; $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot t, \begin{pmatrix} -1 \\ 0 \\ 1+i \end{pmatrix} \cdot t, \begin{pmatrix} -1 \\ 0 \\ 1-i \end{pmatrix} \cdot t$.

[a) both the numbers and the vectors yes; b) the numbers no, the vectors yes; c) both the numbers and the vectors yes]

94. Find the eigenvalues and the eigenvectors of the matrices $\boldsymbol{A}, \boldsymbol{A}^2, \boldsymbol{A}^{-1}$, if you know that $\boldsymbol{A} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$.

$$\begin{aligned} \boldsymbol{A}: &\quad \lambda = \pm 3i, \quad \vec{u} = (1; \pm i)^T \cdot t, \quad t \in \mathbb{C}\backslash\{0\}; \\ \boldsymbol{A}^{-1}: &\quad \lambda = \mp i/3, \quad \vec{u} = (1; \pm i)^T \cdot t, \quad t \in \mathbb{C}\backslash\{0\}; \\ \boldsymbol{A}^2 = -9\boldsymbol{E}: &\quad \lambda = -9, \quad \vec{w} = (a; b)^T, \quad a, b \in \mathbb{C},\ (a, b) \neq (0, 0). \end{aligned}$$

CHAPTER II.

Analytic geometry in $\mathbb{E}_3$

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II.1 The scalar product of two vectors

***Example* 95**: Given the points $A = [3; 4; -2]$, $B = [3; 6; 1]$, $C = [2; 4; 0]$. Find the coordinates, the lengths and the angle of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution: The coordinates of the vector $\overrightarrow{AB} = B - A = (3-3; 6-4; 1+2) = (0; 2; 3)$. If we use the unit vectors $\widehat{\imath} = (1; 0; 0)$, $\widehat{\jmath} = (0; 1; 0)$, $\widehat{k} = (0; 0; 1)$, then the same vector can be expressed as a linear combination of these vectors:

$$\overrightarrow{AB} = (0; 2; 3) = 0 \cdot \widehat{\imath} + 2 \cdot \widehat{\jmath} + 3 \cdot \widehat{k} = 2\,\widehat{\jmath} + 3\,\widehat{k}$$

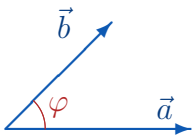
$$\text{Similarly } \overrightarrow{AC} = C - A = (-1; 0; 2) = -\widehat{\imath} + 2\,\widehat{k}$$

The length of the vector $\vec{a} = (a_1; a_2; a_3)$ is computed according to the formula

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

In our example $\|\overrightarrow{AB}\| = \sqrt{4+9} = \sqrt{13}$, $\|\overrightarrow{AC}\| = \sqrt{1+4} = \sqrt{5}$.

The angle of two vectors is determined from the scalar product of these vectors, which we write both by the definition and by means of the coordinates:



$$\varphi \in \langle 0; \pi \rangle,$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\varphi = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \Rightarrow \quad \cos\varphi = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\|\vec{a}\| \cdot \|\vec{b}\|}.$$

We compute the angle of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\cos\varphi = \frac{(0; 2; 3) \cdot (-1; 0; 2)}{\sqrt{13} \cdot \sqrt{5}} = \frac{0 + 0 + 6}{\sqrt{65}} = \frac{6}{\sqrt{65}} \neq 1,$$

which means that $\varphi \in \left(0; \frac{\pi}{2}\right)$. ■

***Example* 96**: Compute the scalar products of the vectors:

a) $(2;1)\cdot(-1;5)$; b) $(3;4;-2)\cdot(3;0;1)$;

c) $(2\widehat{\imath}+3\widehat{\jmath}+4\widehat{k})\cdot(-2\widehat{\jmath})$; d) $(3\widehat{\imath}+2\widehat{\jmath}+\widehat{k})\cdot(\widehat{\imath}+\widehat{\jmath}-5\widehat{k})$.

Solution:

a) $(2;1)\cdot(-1;5) = -2+5 = 3$;
b) $(3;4;-2)\cdot(3;0;1) = 9+0-2 = 7$;
c) $(2\widehat{\imath}+3\widehat{\jmath}+4\widehat{k})\cdot(-2\widehat{\jmath}) = -4\widehat{\imath}\cdot\widehat{\jmath}-6\widehat{\jmath}\cdot\widehat{\jmath}-8\widehat{k}\cdot\widehat{\jmath} = 4\cdot 0-6\cdot 1-8\cdot 0 = -6$; the vectors can also be written without using $\widehat{\imath},\widehat{\jmath},\widehat{k}$, then
$$(2\widehat{\imath}+3\widehat{\jmath}+4\widehat{k})\cdot(-2\widehat{\jmath}) = (2;3;4)\cdot(0;-2;0) = -6,$$
d) $(3\widehat{\imath}+2\widehat{\jmath}+\widehat{k})\cdot(\widehat{\imath}+\widehat{\jmath}-5\widehat{k}) = (3;2;1)\cdot(1;1;-5) = 3+2-5 = 0$. The result is zero, from which it follows that the given vectors are perpendicular to each other. ■

***Example* 97**: The angle of the vectors $\vec{a}$ and $\vec{b}$ is $\varphi = \frac{\pi}{3}$ and $\|\vec{a}\| = 3$, $\|\vec{b}\| = 4$. Compute :

a) $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$; b) $(3\,\vec{a}-2\,\vec{b})\cdot(2\,\vec{a}+\vec{b})$.

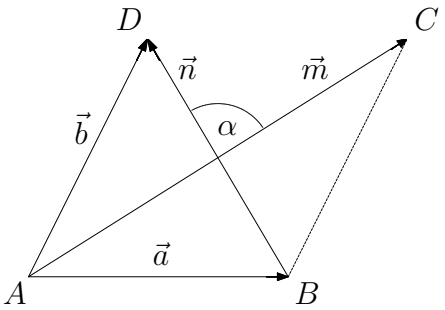
Solution:

a) We expand the product term by term $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b} =$
$= \Big|$ we know that $\vec{a}\cdot\vec{a} = \|\vec{a}\|\cdot\|\vec{a}\|\cdot\cos 0 = \|\vec{a}\|^2$ and also $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}\Big| =$
$= \|\vec{a}\|^2+2\,\vec{a}\cdot\vec{b}+\|\vec{b}\|^2 = 9+2\,\|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\varphi+16 = 9+2\cdot 3\cdot 4\cdot\cos\frac{\pi}{3}+16 = 25+24\cdot\frac{1}{2} = 37.$

b) We expand the product and proceed as in a) $(3\,\vec{a}-2\,\vec{b})\cdot(2\,\vec{a}+\vec{b}) =$
$= 6\,\vec{a}\cdot\vec{a}-4\,\vec{b}\cdot\vec{a}+3\,\vec{a}\cdot\vec{b}-2\,\vec{b}\cdot\vec{b} = 6\,\|\vec{a}\|^2-\vec{a}\cdot\vec{b}-2\,\|\vec{b}\|^2 = 6\cdot 9-3\cdot 4\cdot\cos\frac{\pi}{3}-2\cdot 16 =$
$= 54-6-32 = 16.$ ■

***Example* 98**: Find the angle of the diagonals of the parallelogram $ABCD$, if $\overrightarrow{AB} = 2\widehat{\imath}+\widehat{\jmath}$ and $\overrightarrow{AD} = -\widehat{\jmath}+2\widehat{k}$.

Solution:



The diagonals of a parallelogram can be expressed by means of the vectors of the sides $\vec{m} = \overrightarrow{AB}+\overrightarrow{AD}$, $\vec{n} = \overrightarrow{AB}-\overrightarrow{AD}$.

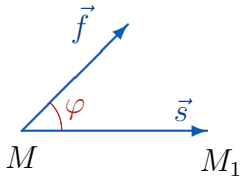
Then $\vec{m} = 2\widehat{\imath}+2\widehat{k}$, $\vec{n} = 2\widehat{\imath}+2\widehat{\jmath}-2\widehat{k}$ a

$$\cos\alpha = \frac{\vec{m}\cdot\vec{n}}{\|\vec{m}\|\cdot\|\vec{n}\|} = \frac{(2;0;2)\cdot(2;2;-2)}{\sqrt{8}\cdot\sqrt{12}} = \frac{4-4}{\sqrt{96}} = 0 \quad\Rightarrow\quad \alpha = \frac{\pi}{2}.$$

The example can also be computed in another way. It is enough to notice that both vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ have the same length $\|\overrightarrow{AB}\| = \|\overrightarrow{AD}\| = \sqrt{5}$, which means that the parallelogram considered is a rhombus, and from plane geometry we know that the diagonals of every rhombus are perpendicular to each other. ■

***Example* 99**: Under the action of the force $\vec{f} = (2; 4; 6)$ [N] the point M was displaced along the oriented segment $\vec{s} = (3; 2; -1)$ [m]. Compute the work W of the force $\vec{f}$.

Solution: From physics we know that the work is computed by means of the scalar product



$$W = \vec{f} \cdot \vec{s} = (2; 4; 6) \cdot (3; 2; -1) = \\ = 6 + 8 - 6 = 8 \text{ [N·m]}.$$

■

***Example* 100**: Find all the vectors of unit length perpendicular to the vector $\vec{a} = (-3; 4)$.

Solution: Let us denote the vector sought $\vec{b} = (b_1; b_2)$, which must satisfy:

$$\vec{b} \perp \vec{a} \iff \vec{b} \cdot \vec{a} = 0, \quad \|\vec{b}\| = 1.$$

In coordinates we obtain the system:

$$\left\{ \begin{array}{l} (b_1; b_2) \cdot (-3; 4) = 0 \\ b_1^2 + b_2^2 = 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} -3\,b_1 + 4\,b_2 = 0 \\ b_1^2 + b_2^2 = 1 \end{array} \right. \Rightarrow \quad b_2 = \frac{3}{4}\,b_1 \quad \Rightarrow$$

$$b_1^2 + \left(\frac{3}{4}\,b_1\right)^2 = 1 \quad \Rightarrow \quad \frac{25}{16}\,b_1^2 = 1 \quad \Rightarrow \quad b_1^2 = \frac{16}{25} \quad \Rightarrow \quad b_1 = \pm\frac{4}{5}.$$

The result is two vectors $\left(\frac{4}{5}; \frac{3}{5}\right)$ and $\left(-\frac{4}{5}; -\frac{3}{5}\right)$. ■

***Example* 101**: For which value of the parameter p are the given vectors perpendicular to each other:

a) $(0; 2; -3),\ (p^2 + p; 2\,p + 1; 2)$; b) $(2\,p; p^2; -1),\ (p; 2; 36)$;

c) $(1; -1; p; 0),\ (p - 1; p^2; 1; 4)$?

Write the coordinates of the individual vectors.

Solution:

a) $(0; 2; -3) \cdot (p^2 + p; 2\,p + 1; 2) = 0 \Rightarrow 2\,(2\,p + 1) - 6 = 0 \Rightarrow 4\,p = 4 \Rightarrow p = 1$;
b) $(2\,p; p^2; -1) \cdot (p; 2; 36) = 0 \Rightarrow 2\,p^2 + 2\,p^2 - 36 = 0 \Rightarrow p^2 = 9 \Rightarrow p_{1,2} = \pm 3$;
c) $(1; -1; p; 0),\ (p - 1; p^2; 1; 4) = 0 \Rightarrow p - 1 - p^2 + p = 0 \Rightarrow (p - 1)^2 = 0 \Rightarrow p = 1$. ■

102. Compute the scalar products:

a) $(2; -5) \cdot (6; 3)$; b) $(2\,\widehat{\imath} + \widehat{\jmath}) \cdot (3\,\widehat{\jmath} + \widehat{k})$;

c) $(1; 3; -5) \cdot (2; 1; 1)$; d) $(1; -2; 0; 3) \cdot (4; 1; -1; 0)$.

[a) -3; b) 3; c) 0; d) 2]

103. Given the ΔABC, where $A = [-1; 2; 3]$, $B = [1; 1; 1]$, $C = [0; 0; 5]$. Prove that this triangle is right–angled and compute the angle β.

$[\alpha = \pi/2,\ \beta = \pi/4]$

104. * Find the angle between the vectors $\vec{a} = 2\,\vec{m} + \vec{n}$ and $\vec{b} = -3\,\vec{m} + 2\,\vec{n}$, if the vectors $\vec{m}, \vec{n}$ form the angle $\pi/3$ and have unit length. (Hint: choose $\vec{m} = (1;0)$.)

$[2\,\pi/3]$

105. Given 3 vectors $\vec{a} = (2;-1;3),\ \vec{b} = (1;-3;2),\ \vec{c} = (3;2;-4)$. Find the vector $\vec{x} = (x_1;x_2;x_3)$ which satisfies the conditions $\vec{x}\cdot\vec{a} = -5,\ \vec{x}\cdot\vec{b} = -11,\ \vec{x}\cdot\vec{c} = 20$.

$[\vec{x} = (2;3;-2)]$

106. Find for which p the vectors are perpendicular to each other:

a) $(2\,p;1-p;4),\ (-1;1;2)$;
b) $(-2;1+p;p^2;0),\ (3-p;p;2;p^2+1)$;
c) $(3;4;-1),\ (4\,p;-3\,p;0)$;
d) $(1+p;1-p;3;-2),\ (-2;2;p;p+3)$;

[a) 3; b) -2, 1; c) $p \in \mathbb{R}$; d) -2]

II.2 The vector product of two vectors in $\mathbb{E}_3$

***Example* 107**: Compute the vector products $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ and verify that $\left(\vec{a}\times\vec{b}\right) \perp \vec{a}$ and $\left(\vec{a}\times\vec{b}\right) \perp \vec{b}$, if $\vec{a} = (5;3;1),\ \vec{b} = (-1;3;2)$.

Solution: To compute the vector product we use its expression in coordinates

$$\vec{a}\times\vec{b} = \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ 5 & 3 & 1 \\ -1 & 3 & 2 \end{vmatrix} = 3\,\widehat{\imath} - 11\,\widehat{\jmath} + 18\,\widehat{k},$$

$$\vec{c} = \vec{a}\times\vec{b} = (3;-11;18), \quad \vec{d} = \vec{b}\times\vec{a} = -\vec{c} = (-3;11;-18),$$

From the scalar products $\vec{a}\cdot\vec{c} = 15 - 33 + 18 = 0$ and $\vec{b}\cdot\vec{c} = -3 - 33 + 36 = 0$ it follows that $\vec{a} \perp \vec{c}$ and $\vec{b} \perp \vec{c}$.

> REMARK: This of course follows directly from the definition of the vector product.

■

***Example* 108**: Find the vector $\vec{x}$ from the relation: $\vec{x} = 2\,\vec{a}\times\widehat{\imath} - 3\,\widehat{k}\times\vec{b}$, where $\vec{a} = 2\widehat{\imath} - \widehat{\jmath} + 3\,\widehat{k}$, $\vec{b} = 4\,\widehat{\jmath} + 5\,\widehat{k}$.

Solution: The products $\vec{a}\times\widehat{\imath}$ and $\widehat{k}\times\vec{b}$ are obtained by direct computation, where we use the relations:

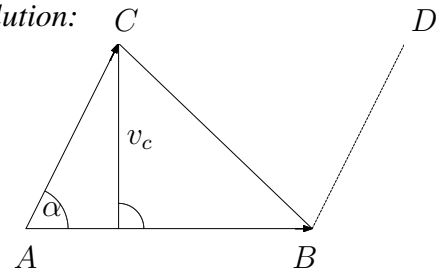
$$\widehat{\imath}\times\widehat{\imath} = \widehat{\jmath}\times\widehat{\jmath} = \widehat{k}\times\widehat{k} = \vec{0} \text{ and } \widehat{\imath}\times\widehat{\jmath} = \widehat{k} = -\widehat{\jmath}\times\widehat{\imath}, \widehat{\jmath}\times\widehat{k} = \widehat{\imath} = -\widehat{k}\times\widehat{\jmath}, \widehat{k}\times\widehat{\imath} = \widehat{\jmath} = -\widehat{\imath}\times\widehat{k},$$

$$\vec{a}\times\widehat{\imath} = (2\widehat{\imath} - \widehat{\jmath} + 3\,\widehat{k})\times\widehat{\imath} = 2\widehat{\imath}\times\widehat{\imath} - \widehat{\jmath}\times\widehat{\imath} + 3\,\widehat{k}\times\widehat{\imath} = 2\,\vec{0} + \widehat{k} + 3\,\widehat{\jmath} = 3\,\widehat{\jmath} + \widehat{k},$$
$$\widehat{k}\times\vec{b} = \widehat{k}\times(4\,\widehat{\jmath} + 5\,\widehat{k}) = 4\,\widehat{k}\times\widehat{\jmath} + 5\,\widehat{k}\times\widehat{k} = -4\widehat{\imath}.$$

Then $\vec{x} = 2\,(3\,\widehat{\jmath} + \widehat{k}) - 3\,(-4\widehat{\imath}) = 12\widehat{\imath} + 6\,\widehat{\jmath} + 2\,\widehat{k}$. ■

***Example* 109**: Compute the area of the ΔABC and the length of the altitude v_c, where $A = [7;3;4],\ B = [1;0;6],\ C = [4;5;-2]$.

Solution:



By the definition we know that

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\| = \|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|\,\sin\alpha = S_{ABCD}.$$

so that $S_{\Delta ABC} = \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$, where $\overrightarrow{AB} = (-6;-3;2),\ \overrightarrow{AC} = (-3;2;-6)$.

$$S_{\Delta ABC} = \frac{1}{2}\left|\begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ -6 & -3 & 2 \\ -3 & 2 & -6 \end{vmatrix}\right| = \frac{1}{2}\left|14\,\widehat{\imath} - 42\,\widehat{\jmath} - 21\,\widehat{k}\right| = \frac{1}{2}\cdot 7\cdot\left|2\,\widehat{\imath} - 6\,\widehat{\jmath} - 3\,\widehat{k}\right| =$$
$$= \frac{7}{2}\sqrt{4+36+9} = \frac{49}{2}.$$

The length of the altitude v_c is computed from the area of the parallelogram (or of the triangle):

$$S_{ABCD} = \left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\| = \left\|\overrightarrow{AB}\right\|\cdot v_c \quad\Rightarrow\quad 49 = \sqrt{36+9+4}\cdot v_c \quad\Rightarrow\quad v_c = \frac{49}{7} = 7. \qquad \blacksquare$$

***Example* 110**: Compute $\left\|\vec{u}\times\vec{v}\right\|$, if $\|\vec{u}\| = 2,\ \|\vec{v}\| = \frac{3}{2},\ \vec{u}\cdot\vec{v} = 3$.

Solution: By the definition $\left\|\vec{u}\times\vec{v}\right\| = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin\varphi$, where the angle φ is computed from the scalar product:

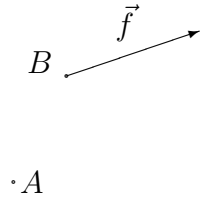
$$\cos\varphi = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{3}{2\cdot\frac{3}{2}} = 1 \quad\Rightarrow\quad \varphi = 0 \quad\Rightarrow\quad \left\|\vec{u}\times\vec{v}\right\| = 0. \qquad \blacksquare$$

***Example* 111**: Find the moment of the force $\vec{f} = 3\,\widehat{\imath} + \widehat{k}$ with respect to the point

a) $A = [1;2;-1]$;

b) $O = [0;0;0]$, if $\vec{f}$ acts at the point $B = [2;-1;3]$.

Solution:



In mechanics the vector product is used to compute the moment $\vec{M}$ of a force $\vec{f}$ acting at a point B with respect to the point A: $\vec{M} = \overrightarrow{AB}\times\vec{f}$.

a) In our case $\overrightarrow{AB} = (1;-3;4),\ \vec{f} = (3;0;1)$ and

$$\vec{M} = \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ 1 & -3 & 4 \\ 3 & 0 & 1 \end{vmatrix} = -3\,\widehat{\imath} + 11\,\widehat{\jmath} + 9\,\widehat{k};$$

b) $\overrightarrow{OB} = (2;-1;3)$, then $\vec{M} = \overrightarrow{OB}\times\vec{f} = \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ 2 & -1 & 3 \\ 3 & 0 & 1 \end{vmatrix} = -\widehat{\imath} + 7\,\widehat{\jmath} + 3\,\widehat{k}.$ $\blacksquare$

112. Compute the vector products:

a) $(3;-1;2)\times(0;4;5)$;

b) $(2\widehat{\imath}+3\widehat{\jmath}-4\widehat{k})\times(2\widehat{\jmath})$;

c) $(\widehat{\imath}+2\widehat{\jmath}-\widehat{k})\times(2\widehat{\imath}-\widehat{\jmath}+\widehat{k})$.

$$\left[\begin{array}{ll} \text{a)} & (-13;-15;12) \\ \text{b)} & 8\widehat{\imath}+4\widehat{k} \\ \text{c)} & (1;-3;-5) \end{array}\right]$$

113. Compute the unit vector $\vec{c}$ which is perpendicular to each of the vectors $\vec{a},\vec{b}$ $\vec{a}=(2;-1;1),\ \vec{b}=(1;2;-1)$.

$$\left[\vec{c}=\pm\frac{1}{\sqrt{35}}(1;-3;-5)\right]$$

114. Compute the area of the ΔABC with the vertices $A=[2;3;4],\ B=[-1;2;-3]$, $C=[5;4;-2]$.

$$\left[\frac{13}{2}\sqrt{10}\right]$$

115. Compute $\left\|(2\,\vec{u}+\vec{v})\times(\vec{u}-3\,\vec{v})\right\|$, if $\|\vec{u}\|=3, \|\vec{v}\|=2$ and the angle of the vectors $\vec{u},\vec{v}$ is $\pi/6$.

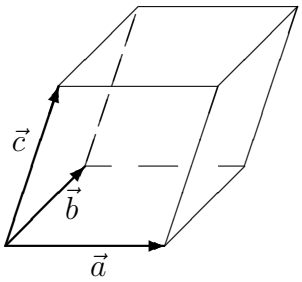
[21]

II.3 The scalar triple product in $\mathbb{E}_3$

***Example* 116**: Compute the volume of the parallelepiped whose edges are formed by the given vectors $\vec{a}=(1;2;0),\ \vec{b}=(-2;1;3),\ \vec{c}=(2;-3;-5)$.

Solution: The scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$ is expressed in coordinates as

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$



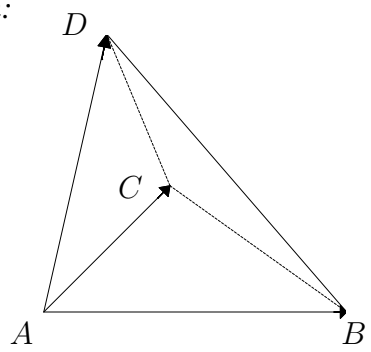
The absolute value of this product equals the volume of the parallelepiped built from the vectors $\vec{a},\vec{b},\vec{c}$. In our case

$$V=\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|=\left|\begin{vmatrix} 1 & 2 & 0 \\ -2 & 1 & 3 \\ 2 & -3 & -5 \end{vmatrix}\right|=$$
$$=|-5+12+9-20|=4.$$

■

***Example* 117**: The vertices of a tetrahedron are $A=[2;-3;5],\ B=[0;2;1],\ C=[-2;-2;3]$, $D=[3;2;4]$. Compute the volume of the tetrahedron and the length of the altitude v_D dropped from the vertex D onto the base ABC.

Solution:



The volume of a tetrahedron equals $\frac{1}{6}$ of the volume of the parallelepiped built from the vectors $\overrightarrow{AB},\ \overrightarrow{AC},\ \overrightarrow{AD}$

$$V_{ABCD} = \frac{1}{6}\left|(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}\right| = \frac{1}{6}\left|\begin{vmatrix} -2 & 5 & -4 \\ -4 & 1 & -2 \\ 1 & 5 & -1 \end{vmatrix}\right| = \frac{1}{6}\left|2-10+80+4-20-20\right| =$$

$$= \frac{1}{6}\cdot 36 = 6.$$

The length of the altitude v_D is computed from the volume of the parallelepiped, which we compute in two ways: by means of the scalar triple product and by means of the well known formula for the volume of a prism (the area of the base multiplied by the length of the altitude of the solid).

$\left|(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}\right| = \|\overrightarrow{AB}\times\overrightarrow{AC}\|\cdot v_D$, where by the preceding

$$\left|(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}\right| = 36 \text{ and } \|\overrightarrow{AB}\times\overrightarrow{AC}\| = \left\|\begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ -2 & 5 & -4 \\ -4 & 1 & -2 \end{vmatrix}\right\| = \|-6\,\widehat{\imath}+12\,\widehat{\jmath}+18\,\widehat{k}\| =$$

$$= 6\,\|-\widehat{\imath}+2\,\widehat{\jmath}+3\,\widehat{k}\| = 6\sqrt{1+4+9} = 6\sqrt{14}.$$

$$v_D = \frac{36}{6\sqrt{14}} = \frac{3\sqrt{14}}{7}.$$ ■

***Example* 118**: Find the parameter α so that the vectors $\vec{a} = (\alpha+1;\alpha;2),\ \vec{b} = (3;1;-4),$ $\vec{c} = (-1;2;2)$ are coplanar (i.e. parallel with one plane).

Solution: Three vectors are coplanar if and only if their scalar triple product equals zero.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} \alpha+1 & \alpha & 2 \\ 3 & 1 & -4 \\ -1 & 2 & 2 \end{vmatrix} = 0 \quad\Rightarrow\quad 2(\alpha+1)+4\alpha+12+2+8(\alpha+1)-6\alpha = 0$$

$$\Rightarrow\quad 8\alpha+24 = 0 \quad\Rightarrow\quad \alpha = -3 \quad\Rightarrow\quad \vec{a} = (-2;-3;2).$$ ■

119. Decide whether the points $A = [1;2;-1], B = [0;1;5], C = [-1;2;1], D = [2;1;3]$ lie in one plane. (Hint: use the scalar triple product of three vectors.)

[yes]

120. Compute the products:

a) $(2\,\widehat{\imath}+\widehat{\jmath})\cdot\Big((3\,\widehat{\jmath}-\widehat{k})\times 2\,\widehat{k}\Big)$;

b) $\big[(3;2;-1)\times(1;0;-1)\big]\cdot(2;1;0)$;

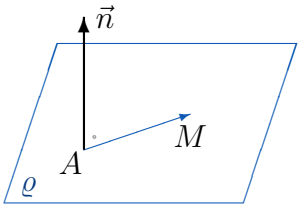
c) find the volume of the parallelepiped built from the vectors in item b).

[a) 12; b) -2; c) 2]

II.4 Planes in $\mathbb{E}_3$

***Example* 121**: Write the equation of the plane ϱ given by the point $A = [3; 4; -5]$ and the normal vector $\vec{n} = (-2; 3; 7)$.

Solution:



Let us denote $M = [x; y; z]$ an arbitrary point lying in the plane ϱ. Then the vectors $\overrightarrow{AM}$ and $\vec{n}$ are perpendicular and their scalar product $\vec{n} \cdot \overrightarrow{AM} = 0$.

We write this scalar product in coordinates under the assumption that $\vec{n} = (a; b; c)$ and the point $A = [x_0; y_0; z_0]$:

$$(a; b; c) \cdot (x - x_0; y - y_0; z - z_0) = 0 \quad \Rightarrow \quad a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$$

we have obtained the so called **general** equation of the plane:
$a\,x + b\,y + c\,z + d = 0$, where $d = -(a\,x_0 + b\,y_0 + c\,z_0)$.

Let us solve our example numerically:

$$(-2; 3; 7) \cdot (x - 3; y - 4; z + 5) = 0 \quad \Longrightarrow \quad -2\,(x - 3) + 3\,(y - 4) + 7\,(z + 5) = 0$$

$$\varrho : \; -2\,x + 3\,y + 7\,z + 29 = 0. \qquad \blacksquare$$

***Example* 122**: Determine the position of the following planes with respect to the coordinate axes and the coordinate planes:

a) $2\,x - 3\,y + z = 0$;
b) $z = 2$;
c) $2\,x + y = 0$;
d) $x - 2\,y + 5 = 0$;
e) $y = -3$;
f) $x + y + z = a,\ a \neq 0$;
g) $x = 0$.

Solution:

a) The plane passes through the origin of the coordinate system $[0; 0; 0]$;
b) the plane is parallel with the coordinate plane (xy) and is therefore perpendicular to the axis z;
c) the plane contains the axis z and of course the origin as well, and is perpendicular to the plane (xy);
d) the plane is parallel with the axis z and is perpendicular to the plane (xy);
e) the plane is parallel with the plane (xz) and is perpendicular to the axis y;
f) the plane cuts off equal segments on the axes x, y, z;
g) the plane is the coordinate plane (yz). $\blacksquare$

***Example* 123**: Write the equation of the plane ϱ which passes through the point $A = [3; 2; -5]$ and

a) contains the axis y;
b) is perpendicular to the axis y;
c) is parallel with the plane $\alpha : \; 2\,x + 3\,y - z = 10$.

Solution:

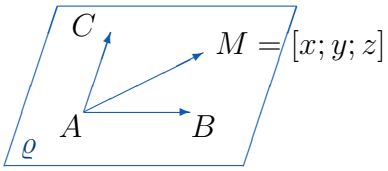
a) A plane containing the axis y has the general equation $a\,x + c\,z = 0$. We use $A \in \varrho$ and compute the coefficients a, c: $3\,a - 5\,c = 0 \implies a = \frac{5}{3}\,c \implies$ we choose the simplest solution $c = 3,\ a = 5 \implies \varrho:\ 5\,x + 3\,z = 0.$

b) $\varrho \perp y \iff y = \text{const.},\ A \in \varrho \implies \varrho:\ y = 2;$

c) $\varrho \parallel \alpha \iff \vec{n}_\varrho = \vec{n}_\alpha:\quad \varrho:\ 2\,x + 3\,y - z + d = 0,$
$A \in \varrho:\ 2 \cdot 3 + 3 \cdot 2 + 5 + d = 0 \implies d = -17 \implies \varrho:\ 2\,x + 3\,y - z - 17 = 0.$ ■

***Example* 124**: Write the equation of the plane ϱ passing through the points $A = [1; 2; -1]$, $B = [3; -3; 2], C = [2; 0; 1]$ and find the distance of the point $P = [-2; 1; 6]$ from the plane ϱ.

Solution:



From the given points and from a general point M of the plane ϱ we form three vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AM}$, which are with the plane parallel, hence they are coplanar and their scalar triple product will equal zero:

$$\overrightarrow{AB} = (2; -5; 3), \quad \overrightarrow{AC} = (1; -2; 2), \quad \overrightarrow{AM} = (x - 1; y - 2; z + 1),$$

$$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AM} = \begin{vmatrix} 2 & -5 & 3 \\ 1 & -2 & 2 \\ x-1 & y-2 & z+1 \end{vmatrix} = 0 \quad \Rightarrow \quad -4\,(x-1) - (y-2) + (z+1) = 0,$$

$\varrho:\ 4\,x + y - z - 7 = 0$, where the coefficients of the variables are the coordinates of the normal vector $\vec{n} = (4; 1; -1)$ of the plane ϱ.

Now we find the distance d of the point P from the plane ϱ according to

$$d = \left|\frac{a\,x_P + b\,y_P + c\,z_P + d}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{-8 + 1 - 6 - 7}{\sqrt{16 + 1 + 1}}\right| = \frac{20}{\sqrt{18}} = \frac{10\sqrt{2}}{3}.$$ ■

***Example* 125**: Through the points $A = [-1; 0; 2],\ B = [3; 2; 0]$ construct the plane

a) α parallel with the axis z;

b) β parallel with the vector $\vec{u} = (3; 1; -2)$.

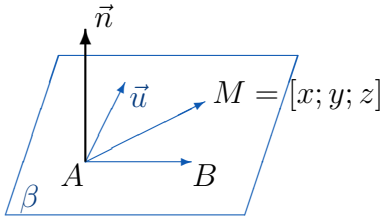
Find the angle φ of the planes α, β.

Solution:

a) $\alpha:\ a\,x + b\,y + d = 0, \qquad A \in \alpha:\ -a + d = 0 \quad \Rightarrow \quad a = d,$
$B \in \alpha:\ 3\,a + 2\,b + d = 0 \quad \Rightarrow \quad b = -2\,d.$

We choose $d = 1$, then $a = 1,\ b = -2$ and we obtain $\alpha:\ x - 2\,y + 1 = 0.$

b)



The equation of the plane β can be found in two ways:

1) from the scalar triple product $\left(\overrightarrow{AB} \times \vec{u}\right) \cdot \overrightarrow{AM} = 0,$

2) by means of the normal vector $\vec{n} = \overrightarrow{AB} \times \vec{u}$ and of the scalar product $\vec{n} \cdot \overrightarrow{AM} = 0.$

We use the second possibility (the first one was used in the preceding example), where $\overrightarrow{AB} = (4;2;-2)$ and then

$$\vec{n} = \overrightarrow{AB} \times \vec{u} = \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ 4 & 2 & -2 \\ 3 & 1 & -2 \end{vmatrix} = (-2;2;-2).$$

As the normal we take the vector $(1;-1;1)$, which is collinear with the vector $(-2;2;-2)$.

The equation of the plane β is obtained from the scalar product $\vec{n} \cdot \overrightarrow{AM} = 0$.

$$(1;-1;1) \cdot (x+1; y; z-2) = 0 \;\Rightarrow\; x+1-y+z-2 = 0 \;\Rightarrow\; \beta:\; x-y+z-1 = 0.$$

The angle φ between two planes is computed as the angle of the corresponding normal vectors.

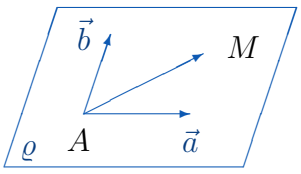
$$\varphi = \angle(\alpha, \beta) = \angle(\vec{n}_\alpha, \vec{n}_\beta) \in \langle 0; \pi/2 \rangle,$$

$$\cos\varphi = \frac{|\vec{n}_\alpha \cdot \vec{n}_\beta|}{\|\vec{n}_\alpha\| \cdot \|\vec{n}_\beta\|}, \quad \vec{n}_\alpha = (1;-2;0), \quad \vec{n}_\beta = (1;-1;1),$$

$$\text{then} \quad \cos\varphi = \frac{|1+2|}{\sqrt{5}\cdot\sqrt{3}} = \frac{3}{\sqrt{15}} = \sqrt{\frac{3}{5}}.$$ ■

***Example* 126**: Through the point $A = [3;0;-1]$ construct the plane ϱ parallel with the vectors $\vec{a} = (1;2;3)$ and $\vec{b} = (-2;1;-5)$.

Solution:



As usual we use a general point $M = [x;y;z]$ and we form the vector $\overrightarrow{AM}$.

The vectors $\overrightarrow{AM}$, $\vec{a}$, $\vec{b}$ must be coplanar.

The condition of coplanarity can be written in two ways:

1) $\overrightarrow{AM} \cdot (\vec{a} \times \vec{b}) = 0$, from which we obtain the general equation of the plane:

$$\begin{vmatrix} x-3 & y & z+1 \\ 1 & 2 & 3 \\ -2 & 1 & -5 \end{vmatrix} = 0 \quad \Rightarrow \quad -13\,(x-3) - y + 5\,(z+1) = 0, \quad \varrho:\; 13\,x + y - 5\,z - 44 = 0.$$

2) $\overrightarrow{AM} = u\,\vec{a} + v\,\vec{b} \quad \Rightarrow \quad \overrightarrow{AM}$ is a linear combination of the vectors $\vec{a}, \vec{b}$, which in coordinates gives us the equation of the plane in parametric form:

$$\begin{pmatrix} x-3 \\ y \\ z+1 \end{pmatrix} = u \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + v \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix} \qquad \left\{ \begin{array}{rcrcrcl} x &=& 3 &+& u &-& 2\,v \\ y &=& & & 2\,u &+& v \\ z &=& -1 &+& 3\,u &-& 5\,v \end{array} \right. ; \quad u,\, v \in \mathbb{R}.$$

Compared with the general equation of a plane the parametric expression is more complicated, because besides the coordinates of the points x, y, z it contains two parameters as well, so that for a choice of x_0, y_0 the corresponding z_0 is computed only after we determine u_0 and v_0 from the first two equations and substitute them into the last equation. ■

***Example* 127**: Find the common points of three planes:

a) $x + 2\,y + 3\,z = 1, \qquad 2\,x - 3\,y + 2\,z = 9, \qquad 5\,x + 8\,y - z = 7;$

b) $x + 3\,y - 5 = 0, \qquad x - y - 2\,z + 4 = 0, \qquad 2\,x + 2\,y - 2\,z = 1.$

Solution: We look for the point (or the points) whose coordinates must satisfy all three equations.

Therefore we solve the corresponding system and from the number of solutions we find out whether the three planes have only one common point or whether they have infinitely many or no common point.

a)

$$\left\{\begin{array}{r} x+2y+3z=1 \\ 2x-3y+2z=9 \\ 5x+8y-z=7 \end{array}\right. \Rightarrow (\boldsymbol{A}|\,\boldsymbol{B}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & -3 & 2 & 9 \\ 5 & 8 & -1 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -7 & -4 & 7 \\ 0 & -2 & -16 & 2 \end{array}\right) \sim$$

$$\sim \left|\begin{array}{c}\text{we divide the 3rd row} \\ \text{by (-2) and write it} \\ \text{as the second one.}\end{array}\right| \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 8 & -1 \\ 0 & -7 & -4 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 8 & -1 \\ 0 & 0 & 52 & 0 \end{array}\right) \Rightarrow$$

$$\mathrm{r}(\boldsymbol{A}) = \min\{3;3\} = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = \min\{3;4\} = 3 \Rightarrow \text{one solution}$$

$$\Rightarrow \left\{\begin{array}{r} x+2y+3z=1 \\ y+8z=-1 \\ 52z=0 \end{array}\right. \Rightarrow z=0, y=-1, x=3.$$

The three given planes have a single common point $P=[3;-1;0]$.

b)

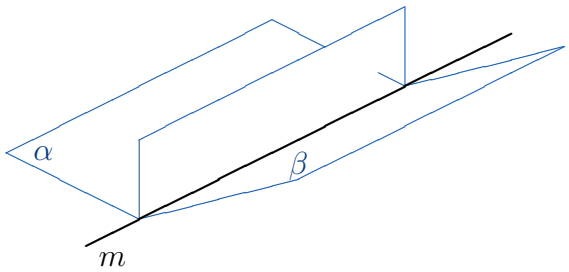
$$\left\{\begin{array}{lr} x+3y & =5 \\ x-y-2z & =-4 \\ 2x+2y-2z & =1 \end{array}\right. \Rightarrow (\boldsymbol{A}|\,\boldsymbol{B}) = \left(\begin{array}{ccc|c} 1 & 3 & 0 & 5 \\ 1 & -1 & -2 & -4 \\ 2 & 2 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 5 \\ 0 & -4 & -2 & -9 \\ 0 & -4 & -2 & -9 \end{array}\right)$$

$$\mathrm{r}(\boldsymbol{A}) = \min\{2;3\} = \mathrm{r}(\boldsymbol{A}|\boldsymbol{B}) = \min\{2;4\} = 2 < 3 \Rightarrow \left\{\begin{array}{c}\text{Infinitely many solutions} \\ \text{depending on one parameter } t:\end{array}\right.$$

$$\left\{\begin{array}{lr} x+3y & =5 \\ 4y+2z & =9 \end{array}\right. \Rightarrow \text{we choose } y=t, \text{ then } x=5-3t, z=\frac{1}{2}(9-4t).$$

The three given planes intersect in the straight line $m:\ X=\left[5;0;\frac{9}{2}\right]+t\,(-3;1;-2)$.

REMARK: The set of all planes which pass through a common straight line m is called **a pencil of planes**.



The straight line m is the so called **axis of the pencil**. A pencil is determined if we know two of its different planes $\alpha:\ a_1x+b_1y+c_1z+d_1=0$ and $\beta:\ a_2x+b_2y+c_2z+d_2=0$.

Any plane of the pencil considered can be written in the form

$$k_1(a_1x+b_1y+c_1z+d_1)+k_2(a_2x+b_2y+c_2z+d_2)=0,$$

where $k_1, k_2 \in \mathbb{R}$ and at least one of the numbers k_1 and k_2 is different from zero. If $k_1 \neq 0$, then the pencil of planes has the equation:

$$a_1x+b_1y+c_1z+d_1+k(a_2x+b_2y+c_2z+d_2)=0, \quad \text{where } k=\frac{k_2}{k_1}.$$

In this example we have found the axis of the pencil. We easily verify that the third plane $2\,x + 2\,y - 2\,z = 1$ belongs to the pencil determined by the equation

$$x + 3\,y - 5 + k(x - y - 2\,z + 4) = 0 \quad \text{and has the coefficient } k = 1:$$
$$x + 3\,y - 5 + 1 \cdot (x - y - 2\,z + 4) = 0 \quad \Rightarrow \quad 2\,x + 2\,y - 2\,z - 1 = 0. \qquad \blacksquare$$

***Example* 128**: Find the equation of the plane ϱ which passes through the point $A = [5; -2; -4]$ and through the line of intersection of the planes $x + y - z - 3 = 0,\ 2\,x - y + 3\,z + 2 = 0$.

Solution: The line of intersection of the given planes is the axis of the pencil:
$x + y - z - 3 + k(2\,x - y + 3\,z + 2) = 0$.

The plane ϱ belongs to this pencil and at the same time passes through the point A. That means that we look for the k for which the plane of the pencil passes through the point A. Substituting the coordinates of the point A we compute k and then also the equation of the plane ϱ:

$$5 - 2 + 4 - 3 + k\,(10 + 2 - 12 + 2) = 0 \quad \Rightarrow \quad 2\,k = -4 \quad \Rightarrow \quad k = -2,$$
$$\varrho:\ x + y - z - 3 - 2(2\,x - y + 3\,z + 2) = 0 \quad \Rightarrow \quad 3\,x - 3\,y + 7\,z + 7 = 0. \qquad \blacksquare$$

***Example* 129**: Through the line of intersection of the planes $4\,x - y + 3\,z - 1 = 0, x + 5\,y - z + 2 = 0$ construct the plane ϱ perpendicular to the plane $\sigma:\ 2\,x - y + 5\,z - 3 = 0$.

Solution: The first two planes determine a pencil and neither of them is perpendicular to the plane σ, therefore the equation of the plane ϱ sought will be

$$4\,x - y + 3\,z - 1 + k\,(x + 5\,y - z + 2) = 0 \quad \Rightarrow$$

$$\Rightarrow\ (4 + k)\,x + (-1 + 5\,k)\,y + (3 - k)\,z - 1 + 2\,k = 0.$$

Because the plane ϱ is perpendicular to the plane σ, their normals are perpendicular as well, hence the scalar product $\vec{n}_\varrho$ and $\vec{n}_\sigma$ equals zero.

The normal vector of the plane of the pencil is $\vec{n}_\varrho = (4 + k; -1 + 5\,k; 3 - k)$ and the normal of the plane σ is $\vec{n}_\sigma = (2; -1; 5)$.

$$\vec{n}_\varrho \perp \vec{n}_\sigma \quad \Longleftrightarrow \quad \vec{n}_\varrho \cdot \vec{n}_\sigma = 0 \quad \Rightarrow \quad (4 + k; -1 + 5\,k; 3 - k) \cdot (2; -1; 5) = 0,$$
$$2\,(4 + k) - (-1 + 5\,k) + 5\,(3 - k) = 0 \quad \Rightarrow \quad -8\,k + 24 = 0 \quad \Rightarrow \quad k = 3.$$

We substitute this parameter into the equation of the pencil and obtain the equation of the plane $\varrho:\ 4\,x - y + 3\,z - 1 + 3\,(x + 5\,y - z + 2) = 0 \quad \Rightarrow \quad 7\,x + 14\,y + 5 = 0.$ $\blacksquare$

130. Given the plane $\varrho:\ 2\,x - y + 2\,z + 6 = 0$. Find:

a) whether the points $A = [3; 10; -1]$ and $B = [-2; 4; 3]$ lie in this plane;

b) the point of intersection of the plane with the axis z;

c) the angle with the plane $\sigma:\ -x + 2\,y - 2\,z + 6 = 0$.

$\Big[$ a) A yes, B no; b) $[0; 0; -3]$; c) $\cos\,\varphi = 8/9\Big]$

131. Write the equation of the plane ϱ passing through the points $A = [1; 3; 2], B = [2; 2; 1]$ and

a) containing the point $[0; 0; 0]$;

b) parallel with the normal vector of the plane $3\,x - y + 10 = 0$;

c) perpendicular to the plane $2\,x - y - z + 4 = 0$.

[a) $x - 3\,y + 4\,z = 0$; b) $x + 3\,y - 2\,z - 6 = 0$; c) $y - z - 1 = 0$]

132. Write the equation of the plane ϱ passing through the point $A = [4; -7; 5]$ and

a) through the axis x;
b) perpendicular to the axis z;
c) parallel with the plane (xz);
d) and cutting off equal segments on the axes x, y, z.

[a) $5\,y + 7\,z = 0$; b) $z = 5$; c) $y = -7$; d) $x + y + z - 2 = 0$]

133. Find the distance of the planes $\varrho:\ 2\,x - y + 2\,z + 5 = 0$ and $\sigma:\ 2\,x - y + 2\,z - 7 = 0$. (Hint: Choose a point A in the plane ϱ and find its distance from the plane σ.)

[4]

134. Find the equation of the plane passing through the line of intersection of the planes $3\,x - 2\,y - 2\,z = -1, x - 3\,y - z + 2 = 0$ and through the point $B = [3; 1; 3]$.

[$5x - 8y - 4z + 5 = 0$]

135. Through the line of intersection of the planes $x + 3\,y - 5 = 0, x - y - 2\,z + 4 = 0$ construct the plane ϱ parallel with the vector $\overrightarrow{AB}$, $A = [2; 1; 3]$, $B = [3; 3; 2]$.

[$6x - 10y - 14z + 33 = 0$]

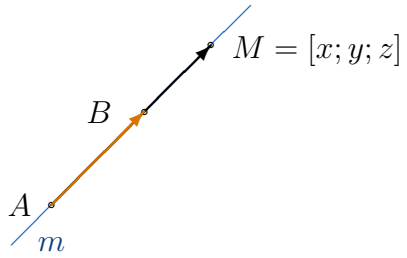
II.5 Straight lines in $\mathbb{E}_3$

***Example* 136**: Write the equation of the straight line p passing through the points $A = [4; 2; -1]$ and $B = [2; 5; 0]$.

a) Decide whether the points $C = [6; -1; -2]$ and $D = [0; 3; 2]$ lie on the straight line p.

b) Find the point of intersection Q of the straight line p with the plane (xz).

Solution:



From the points A, B and a general point $M = [x; y; z]$ we form two vectors $\overrightarrow{AB}$ and $\overrightarrow{AM}$, which are collinear (that is, linearly dependent), which we write by the relation

$$\overrightarrow{AM} = t \cdot \overrightarrow{AB},\ t \in \mathbb{R}.$$

The vector $\overrightarrow{AB}$ is the **directional** vector of the straight line, often denoted $\vec{s} = \overrightarrow{AB}$.

$$\begin{array}{rcl} \overrightarrow{AM} & = & (x-4; y-2; z+1) \\ \overrightarrow{AB} & = & (-2; 3; 1) \end{array} \Rightarrow \left\{ \begin{array}{rcl} x - 4 & = & -2\,t \\ y - 2 & = & 3\,t \\ z + 1 & = & t \end{array} \right. \Rightarrow \quad p: \left\{ \begin{array}{l} x = 4 - 2\,t \\ y = 2 + 3\,t \\ z = -1 + t \end{array} \right. ,$$

we have obtained the so called **parametric** equations of the straight line p.

a) The point C lies on the straight line p if its coordinates satisfy the equations for a certain parameter t_0. Let us try to substitute

$$\left\{ \begin{array}{rcrcll} 6 & = & 4 & - & 2\,t & \Rightarrow\ t_0 = -1 \\ -1 & = & 2 & + & 3\,t & \Rightarrow\ t_0 = -1 \\ -2 & = & -1 & + & t & \Rightarrow\ t_0 = -1 \end{array} \right. \quad \Rightarrow \quad \text{the point } C \text{ lies on the straight line } p;$$

We find the same for the point D :

$$\left\{\begin{array}{rcrcll} 0 &=& 4 &-& 2t & \Rightarrow \ t_0 = 2 \\ 3 &=& 2 &+& 3t & \Rightarrow \ t_0 = \frac{1}{3} \neq 2 \\ 2 &=& -1 &+& t & \end{array}\right. \quad \Rightarrow$$

the point D does not lie on the straight line p;

b) the plane (xz) has the equation $y = 0$. We substitute this value into the equations of the straight line p and compute the remaining coordinates of the point Q:

$$y = 2 + 3t = 0 \quad \Rightarrow \quad t_0 = -\frac{2}{3}, \qquad \begin{cases} x = 4 - 2 \cdot \left(-\frac{2}{3}\right) = \frac{16}{3}, \\ z = -1 - \frac{2}{3} = -\frac{5}{3}. \end{cases}$$

$$Q = \left[\frac{16}{3};\ 0;\ -\frac{5}{3}\right]. \qquad \blacksquare$$

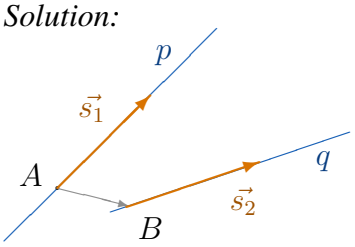
***Example* 137**: Determine the mutual position of the straight lines p and q, if:

a) $p: \begin{cases} x = 3 - t \\ y = 1 + t \\ z = -2 + 3t \end{cases}$, $\quad q: \begin{cases} x = 1 + 2u \\ y = 3 - 2u \\ z = 4 - 6u \end{cases}$, $t,\ u \in \mathbb{R}$;

b) $p: \begin{cases} x = 1 \\ y = 2 + 5t \\ z = 3 + 6t \end{cases}$, $\quad q: \begin{cases} x = 2u \\ y = -1 + u \\ z = 0 \end{cases}$, $t,\ u \in \mathbb{R}$;

c) $p: \begin{cases} x = -1 + 4t \\ y = 4 + t \\ z = -1 + 3t \end{cases}$, $\quad q: \begin{cases} x = 2 + 4u \\ y = 5u \\ z = -5 + 3u \end{cases}$, $t,\ u \in \mathbb{R}$.

Solution:



Let us have two straight lines given by a point and a directional vector:
$p: \ A, \ \vec{s}_1; \quad q: \ B, \ \vec{s}_2$. Then for their mutual position the following cases can occur:

1) $\vec{s}_1 = k\,\vec{s}_2$, $k \neq 0,\ k \in \mathbb{R}$ $\quad \left\{\begin{array}{ll} A \in q \iff & p \text{ coincides with } q, \\ A \notin q \iff & p \parallel q, \text{ they do not coincide,} \end{array}\right.$

2) $\vec{s}_1 \neq k\,\vec{s}_2$, then we pass on to the scalar triple product:

$\overrightarrow{AB} \cdot (\vec{s}_1 \times \vec{s}_2)$ $\quad \left\{\begin{array}{ll} = 0 \iff & p \text{ and } q \text{ are intersecting lines,} \\ \neq 0 \iff & p \text{ and } q \text{ are skew lines.} \end{array}\right.$

Now we investigate the given straight lines:

a) on the straight line p we have the point $A = [3; 1; -2]$ and $\vec{s}_1 = (-1; 1; 3)$;
on the straight line q we have the point $B = [1; 3; 4]$ and $\vec{s}_2 = (2; -2; -6)$.
We see immediately that $\vec{s}_2 = -2\,\vec{s}_1$. Now we find out whether $A \in q$:

$$\left\{\begin{array}{rcrcll} 3 &=& 1 &+& 2u & \Rightarrow \ u_0 = 1 \\ 1 &=& 3 &-& 2u & \Rightarrow \ u_0 = 1 \\ -2 &=& 4 &-& 6u & \Rightarrow \ u_0 = 1 \end{array}\right. \quad \Rightarrow \ A \in q$$

$\Rightarrow$ the straight lines p, q coincide.

b)

$$p:\ A=[1;2;3],\ \vec{s}_1=(0;5;6),$$
$$q:\ B=[0;-1;0],\ \vec{s}_2=(2;1;0),\qquad \vec{s}_1\neq k\,\vec{s}_2,$$

therefore we pass on to the scalar triple product,

$$\overrightarrow{AB}\cdot(\vec{s}_1\times\vec{s}_2)=\begin{vmatrix}-1&-3&-3\\0&5&6\\2&1&0\end{vmatrix}=-36+30+6=0\ \Rightarrow$$ the straight lines $p,\ q$ are intersecting lines.

c)

$$p:\ A=[-1;4;-1],\ \vec{s}_1=(4;1;3),$$
$$q:\ B=[2;0;-5],\ \vec{s}_2=(4;5;3),\qquad \vec{s}_1\neq k\,\vec{s}_2,$$

therefore we pass on to the scalar triple product,

$$\overrightarrow{AB}\cdot(\vec{s}_1\times\vec{s}_2)=\begin{vmatrix}3&-4&-4\\4&1&3\\4&5&3\end{vmatrix}=\begin{vmatrix}3&-4&-4\\4&1&3\\0&4&0\end{vmatrix}=-4\begin{vmatrix}3&-4\\4&3\end{vmatrix}=-4\,(9+16)=$$

$=-100\neq 0\quad\Rightarrow\quad$ the straight lines p and q are skew lines. ■

***Example* 138**: Use the straight lines from example **137** and find the point of intersection Q of the intersecting lines in b)
and the shortest distance of the skew lines in c).

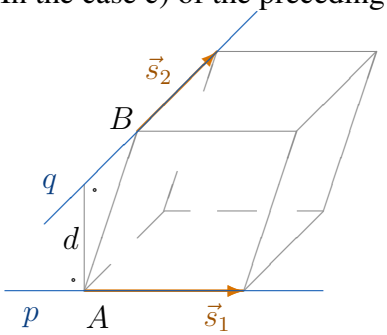
Solution: The point of intersection of two intersecting lines must satisfy the equations of both straight lines p and q:

$$p:\begin{cases}x=&1\\y=&2+5\,t\\z=&3+6\,t\end{cases}\qquad q:\begin{cases}x=&2\,u\\y=&-1+u\\z=&0\end{cases}\quad\Longrightarrow$$

$$\begin{cases}1&=&2\,u&\Rightarrow&u=\frac{1}{2}\\2+5\,t&=&-1+u&&\\3+6\,t&=&0&\Rightarrow&t=-\frac{1}{2},\end{cases}\qquad\Longrightarrow\qquad 2+5\cdot\left(-\tfrac{1}{2}\right)=-1+\tfrac{1}{2}$$

the point of intersection Q has the parameter $t=-\dfrac{1}{2}$ in the equation of the straight line p and the parameter $u=\dfrac{1}{2}$ in the equation of the straight line q and $Q=\left[1;-\dfrac{1}{2};0\right]$.

In the case c) of the preceding example we know that the straight lines are skew lines.



Their shortest distance equals the length of the altitude d of the parallelepiped. We compute this distance from the volume of the parallelepiped:

$$V=\left|\overrightarrow{AB}\cdot(\vec{s}_1\times\vec{s}_2)\right|=\|\vec{s}_1\times\vec{s}_2\|\cdot d\quad\Rightarrow\quad d=\frac{\left|\overrightarrow{AB}\cdot(\vec{s}_1\times\vec{s}_2)\right|}{\|\vec{s}_1\times\vec{s}_2\|}.$$

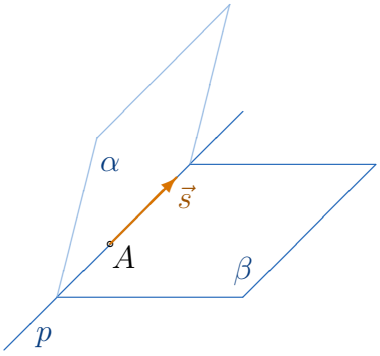
In the preceding example we computed $\left|\overrightarrow{AB}\cdot(\vec{s_1}\times\vec{s_2})\right| = 100$, now we determine

$$\|\vec{s_1}\times\vec{s_2}\| = \left\|\begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ 4 & 1 & 3 \\ 4 & 5 & 3 \end{vmatrix}\right\| = \|(-12;0;16)\| = 4\cdot\|(-3;0;4)\| = 4\cdot\sqrt{9+16} = 20,$$

$$d = \frac{100}{20} = 5. \qquad \blacksquare$$

***Example* 139**: Write the parametric equations of the straight line p which is the line of intersection of the planes $\alpha:\ 2x-3y-3z-9=0$ and $\beta:\ x-2y+z+3=0$.

Solution:



In order to be able to write the parametric equations of the straight line p, we need to determine the point A and the directional vector $\vec{s}$.

1) The point A lies both in the plane α and in the plane β: $A\in\alpha\cap\beta$. We choose for example $y_A=0$, that is, we find the point of intersection of the straight line p with the plane (xz), where x_A and z_A are computed from the system

$$\left\{\begin{array}{rcrcrcrcl} 2x & - & 3y & - & 3z & - & 9 & = & 0 \\ x & - & 2y & + & z & + & 3 & = & 0 \\ & & y & & & & & = & 0 \end{array}\right. \quad\Rightarrow\quad z=-3,\ x=0 \quad\Rightarrow\quad A=[0;0;-3],$$

the point A even lies on the axis z.

2) The directional vector $\vec{s}$ of the straight line p is perpendicular to the normal vector $\vec{n}_\alpha$ of the plane α and to the normal vector $\vec{n}_\beta$ of the plane β. Hence

$$\vec{s} = \vec{n_\alpha}\times\vec{n_\beta} = \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ 2 & -3 & -3 \\ 1 & -2 & 1 \end{vmatrix} = (-9;-5;-1).$$

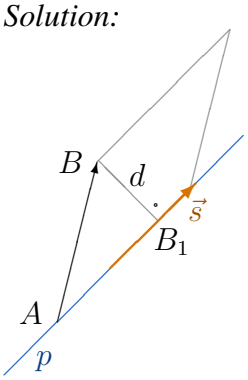
The directional vector of the straight line p is for example the vector $(9;5;1)$ and the parametric equations are

$$p:\ \left\{\begin{array}{rcrcl} x & = & 0 & + & 9\cdot t \\ y & = & 0 & + & 5\cdot t \\ z & = & -3 & + & 1\cdot t \end{array}\right. \quad\Rightarrow\quad \left\{\begin{array}{rcl} x & = & 9t \\ y & = & 5t \\ z & = & -3+t \end{array}\right.,\quad t\in\mathbb{R}. \qquad \blacksquare$$

***Example* 140**: Find the distance of the point $B=[3;4;-1]$ from the straight line p:

$$\left\{\begin{array}{rcrcl} x & = & 2 & + & t \\ y & = & 3 & - & 2t \\ z & = & -4 & + & 3t \end{array}\right.,\ t\in\mathbb{R}.$$

Solution:



The distance d sought is equal to the length of the perpendicular segment BB_1 dropped from the point B onto the straight line p. We express the area of the parallelogram built from the vectors $\vec{s}$ and $\overrightarrow{AB}$ in two ways:

$\left\|\vec{s}\times\overrightarrow{AB}\right\| = \|\vec{s}\|\cdot d$ (the base times the altitude), then $d = \dfrac{\left\|\vec{s}\times\overrightarrow{AB}\right\|}{\|\vec{s}\|}$.

In the given example $A=[2;3;-4]$, $\vec{s}=(1;-2;3)$, $\overrightarrow{AB}=(1;1;3)$,

$$\left\|\vec{s}\times\overrightarrow{AB}\right\| = \left\| \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 1 & -2 & 3 \\ 1 & 1 & 3 \end{vmatrix} \right\| = \|(-9;0;3)\| = 3\,\|(-3;0;1)\| = 3\sqrt{10},$$

$$d = \frac{3\sqrt{10}}{\sqrt{1+4+9}} = 3\sqrt{\frac{10}{14}} = 3\sqrt{\frac{5}{7}}.$$ ■

***Example* 141**:

a) Confirm that the given straight lines p and q are skew.

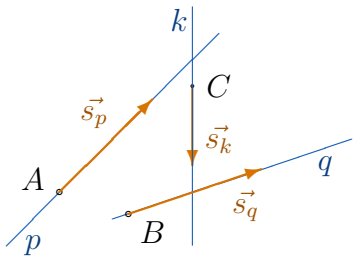
$$p:\ \begin{cases} x &= 7 &+ &t \\ y &= 3 &+ &2t \\ z &= 9 &- &t \end{cases} \quad\text{and}\quad q:\ \begin{cases} x &= 3 &- &7u \\ y &= 1 &+ &2u \\ z &= 2 &+ &3u \end{cases},\ t,\ u\in\mathbb{R}.$$

b) Find the angle φ of these straight lines.

c) Find the equation of their transversal perpendicular to both the straight lines p and q, the so called **common perpendicular of two skew lines**.

Solution:

a) For each straight line we choose a point and a directional vector:



$p:\ A=[7;3;9],\ \vec{s}_p=(1;2;-1),$
$q:\ B=[3;1;2],\ \vec{s}_q=(-7;2;3).$

The straight lines p and q are skew, because the vectors $\vec{s}_p$, $\vec{s}_q$, $\overrightarrow{AB}=(-4;-2;-7)$ are linearly independent,

since $(\vec{s}_p\times\vec{s}_q)\cdot\overrightarrow{AB}) = \begin{vmatrix} 1 & 2 & -1 \\ -7 & 2 & 3 \\ -4 & -2 & -7 \end{vmatrix} = -152 \neq 0.$

b) By the angle φ of two straight lines we understand the angle $0\le\varphi\le\dfrac{\pi}{2}$ of the corresponding directional vectors. We compute

$$\cos\varphi = \frac{|\vec{s}_p \cdot \vec{s}_q|}{\|\vec{s}_p\| \cdot \|\vec{s}_q\|} = \frac{|-7+4-3|}{\sqrt{1+4+1}\cdot\sqrt{49+4+9}} = \frac{6}{\sqrt{6}\cdot\sqrt{62}} = \sqrt{\frac{3}{31}}.$$

c) We find the vector $\vec{s}$ perpendicular to both the directional vectors $\vec{s}_p$ and $\vec{s}_q$

$$\vec{s} = \vec{s}_p \times \vec{s}_q = \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ 1 & 2 & -1 \\ -7 & 2 & 3 \end{vmatrix} = (8;4;16) \quad \Rightarrow \quad \vec{s} = (2;1;4).$$

A concrete point $C = [x;y;z]$ of the common perpendicular of the skew lines p and q is determined by means of the conditions that it intersects the straight lines p and q:

$$\left\{ \begin{array}{l} \overrightarrow{AC} \cdot (\vec{s}_p \times \vec{s}) = 0, \text{ where } \overrightarrow{AC} = (x-7;y-3;z-9) \\ \overrightarrow{BC} \cdot (\vec{s}_q \times \vec{s}) = 0, \text{ where } \overrightarrow{BC} = (x-3;y-1;z-2) \end{array} \right. \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{l} \begin{vmatrix} x-7 & y-3 & z-9 \\ 1 & 2 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 0 \\ \\ \begin{vmatrix} x-3 & y-1 & z-2 \\ -7 & 2 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 0 \end{array} \right. \Rightarrow$$

$$\left\{ \begin{array}{l} 9(x-7) - 6(y-3) - 3(z-9) = 0 \\ 5(x-3) + 34(y-1) - 11(z-2) = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} 3x - 2y - z - 6 = 0 \\ 5x + 34y - 11z - 17 = 0 \end{array} \right. .$$

We choose the point C in the plane $(xz) \Rightarrow C = [x;0;z]$, then for x and z we obtain the system

$$\left\{ \begin{array}{rl} 3x - z = & 6 \\ 5x - 11z = & 17 \end{array} \right. \quad \Rightarrow \quad x = \frac{7}{4},\ z = -\frac{3}{4} \quad \Rightarrow \quad C = \left[\frac{7}{4};0;-\frac{3}{4}\right].$$

The parametric equations of the common perpendicular of the skew lines p and q are:

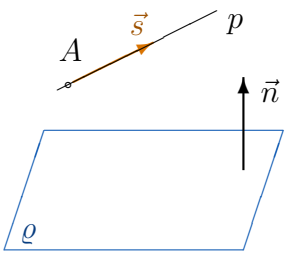
$$\left\{ \begin{array}{rcl} x &=& \frac{7}{4} + 2t \\ y &=& t \\ z &=& -\frac{3}{4} + 4t \end{array} \right. , \qquad t \in \mathbb{R}.$$

■

***Example* 142**: Determine the mutual position of the plane $\varrho:\ 2x - y + 3z + 7 = 0$ and the straight line

a) $p:\ \left\{ \begin{array}{rcl} x &=& 1 + t \\ y &=& 2 - 4t \\ z &=& -2t \end{array} \right. ;$ b) $q:\ \left\{ \begin{array}{rcl} x &=& 2 - u \\ y &=& 3 - 3u \\ z &=& 2 + 2u \end{array} \right. , \ t,\ u \in \mathbb{R}.$

Solution:



From the equation of the straight line p we read the point A and the directional vector $\vec{s}$. The mutual position is investigated by means of the mutual position of the directional vector $\vec{s}$ of the straight line p and the normal vector $\vec{n}$ of the plane ϱ.

If $\vec{s}\cdot\vec{n}=0$ $\begin{cases} A\in\varrho \quad\Leftrightarrow\quad p\in\varrho \text{ (the straight line lies in the plane),}\\ A\notin\varrho \quad\Leftrightarrow\quad p\parallel\varrho \\ \text{(the straight line has no common point with the plane).}\end{cases}$

If $\vec{s}\cdot\vec{n}\neq 0 \quad\Leftrightarrow\quad$ the straight line p intersects the plane.

a) $p:\ A=[1;2;0],\ \vec{s}_p=(1;-4;-2);\qquad \varrho:\ \vec{n}=(2;-1;3);$

$$\vec{s}_p\cdot\vec{n}=(1;-4;-2)\cdot(2;-1;3)=2+4-6=0,$$
$$A\in\varrho\,?\ :2\cdot 1-2+7\neq 0 \quad\Rightarrow\quad A\notin\varrho \quad\Rightarrow\quad p\parallel\varrho \quad\Rightarrow$$

the straight line p is parallel with the plane ϱ;

b) $q:\ B=[2;3;2],\ \vec{s}_q=(-1;-3;2)\qquad \varrho:\ \vec{n}=(2;-1;3)$

$$\vec{s}_q\cdot\vec{n}=(-1;-3;2)\cdot(2;-1;3)=-2+3+6\neq 0\Rightarrow$$

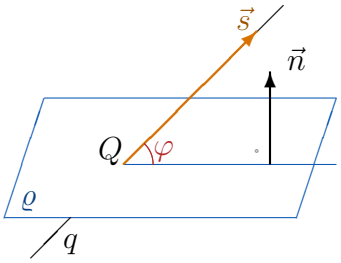
the straight line q intersects the plane ϱ. ■

***Example* 143**: Use the example **142** b), find the point of intersection of the straight line p with the plane ϱ and compute the angle of the straight line q with the plane ϱ.

Solution: Into the equation of the plane $\varrho:\ 2x-y+3z+7=0$ we substitute from the parametric equations of the straight line

$$q:\ \begin{cases} x &=& 2 &-& t\\ y &=& 3 &-& 3t\\ z &=& 2 &+& 2t\end{cases}$$ and we compute the parameter t of the point of intersection Q:

$$2(2-t)-(3-3t)+3(2+2t)+7=0 \quad\Rightarrow\quad 7t+14=0 \quad\Rightarrow\quad t=-2 \quad\Rightarrow$$
$$\Rightarrow\quad Q=[4;9;-2].$$



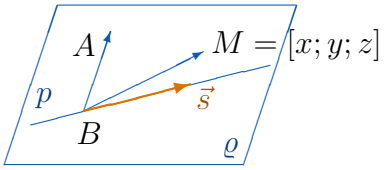
The angle of a straight line with a plane is the angle $\varphi\in\left\langle 0;\frac{\pi}{2}\right\rangle$, which is determined by means of the scalar product of the normal vector $\vec{n}$ of the plane ϱ and of the directional vector $\vec{s}$ of the straight line q:

$$\sin\varphi=\cos\left(\frac{\pi}{2}-\varphi\right)=\frac{|\vec{s}\cdot\vec{n}|}{\|\vec{s}\|\cdot\|\vec{n}\|}=\frac{|(-1;-3;2)\cdot(2;-1;3)|}{\sqrt{1+9+4}\sqrt{4+1+9}}=\frac{|-2+3+6|}{14}=\frac{1}{2},$$ the angle is $\varphi=\frac{\pi}{6}$. ■

***Example* 144**: Write the equation of the plane passing through the point $A=[5;-1;0]$ and through the straight line

$$p:\ \begin{cases} x &=& 3 &+& t\\ y &=& 2 &-& 3t\\ z &=& -1 &+& 4t\end{cases},\ t\in\mathbb{R}.$$

Solution:



From the parametric equations of the straight line we determine the point $B = [3; 2; -1]$ and the directional vector $\vec{s} = (1; -3; 4)$. Let us denote $M = [x; y; z]$ an arbitrary point of the plane sought.

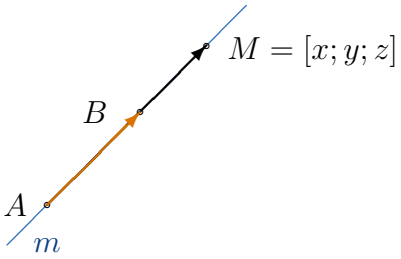
The vectors $\overrightarrow{BA}, \overrightarrow{BM}$ and $\vec{s}$ are coplanar and therefore $\overrightarrow{BM} \cdot (\overrightarrow{BA} \times \vec{s}) = 0$. In coordinates:

$$\begin{vmatrix} x-3 & y-2 & z+1 \\ 2 & -3 & 1 \\ 1 & -3 & 4 \end{vmatrix} = 0 \quad \Rightarrow \quad -9\,(x-3) - 7\,(y-2) - 3\,(z+1) = 0 \quad \Rightarrow$$

$$\varrho: \; 9\,x + 7\,y + 3\,z - 38 = 0. \qquad \blacksquare$$

***Example* 145**: The straight line p passes through the points $A = [1; 2; -3], B = [3; 0; 2]$. Write the equation of the straight line p in canonical and in parametric form.

Solution: We have already worked several times with the parametric form of the equation of a straight line, but not with the canonical one. Therefore we first derive the canonical form of the equation of a straight line in the general case.



Let us denote $\overrightarrow{AB} = \vec{s} = (s_1; s_2; s_3)$, $A = [x_0; y_0; z_0]$,

$$M = [x; y; z], \overrightarrow{AM} = (x - x_0; y - y_0; z - z_0).$$

The vectors $\overrightarrow{AM}$ and $\vec{s}$ are collinear, and that means that $\overrightarrow{AM}$ is a multiple of the vector $\vec{s}$, that is, their coordinates satisfy the conditions

$$\frac{x - x_0}{s_1} = \frac{y - y_0}{s_2} = \frac{z - z_0}{s_3}.$$

For $s_1 s_2 s_3 \neq 0$, these are the **symmetric equations** of the line (also called its canonical equations). We can understand them as a system of three equations

$$\frac{x - x_0}{s_1} = \frac{y - y_0}{s_2}, \qquad \frac{y - y_0}{s_2} = \frac{z - z_0}{s_3}, \qquad \frac{x - x_0}{s_1} = \frac{z - z_0}{s_3},$$

of which each describes a plane: the first is parallel with the axis z, the second is parallel with the axis x and the third with the axis y. All the planes belong to one pencil of planes whose axis is the straight line p.

Now we compute everything: $\vec{s} = \overrightarrow{AB} = (2; -2; 5)$, the symmetric form of the equation of the straight line p is $\dfrac{x-1}{2} = \dfrac{y-2}{-2} = \dfrac{z+3}{5}$, the parametric form is $\left\{ \begin{array}{rcrcl} x &=& 1 &+& 2\,t \\ y &=& 2 &-& 2\,t \\ z &=& -3 &+& 5\,t \end{array} \right. , \; t \in \mathbb{R}.$ $\blacksquare$

***Example* 146**: Write the equations of the straight lines p and q in canonical form:

$$p: \left\{ \begin{array}{r} 2\,x - 3\,y - 3\,z - 9 = 0 \\ x - 2\,y + z + 3 = 0 \end{array} \right. \qquad q: \left\{ \begin{array}{rcrcl} x &=& 2 &+& 3\,t \\ y &=& & - & t \\ z &=& -3 &+& 2\,t \end{array} \right. , \; t \in \mathbb{R}.$$

Solution:

The straight line p: On the straight line p we determine the point $A = [x; y; 0]$. We have chosen the point of intersection of the straight line p with the plane (xy). Its coordinates are computed from the system

$$\Rightarrow \quad \left\{ \begin{array}{r} 2\,x - 3\,y - 3\,z - 9 = 0 \\ x - 2\,y + z + 3 = 0 \\ z = 0 \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{r} 2\,x - 3\,y = 9 \\ x - 2\,y = -3 \end{array} \right. \quad \Rightarrow$$

$$x = \frac{\begin{vmatrix} 9 & -3 \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix}} = \frac{-27}{-1} = 27, \quad y = \frac{\begin{vmatrix} 2 & 9 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix}} = \frac{-15}{-1} = 15.$$

The directional vector $\vec{s}$ is computed from the vector product of the normal vectors of the planes:

$$\vec{s} = \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ 2 & -3 & -3 \\ 1 & -2 & 1 \end{vmatrix} = (-9; -5; -1) \quad \Rightarrow \quad \vec{s} = (9; 5; 1).$$

The canonical form of the equation of the straight line p : $\dfrac{x-27}{9} = \dfrac{y-15}{5} = \dfrac{z}{1}$.

The straight line q: For the symmetric form of the equation of the straight line q we read the point and the directional vector of the straight line from the parametric equations $X = [2; 0; 3] + t\,(3; -1; 2)$ or from each equation we express the parameter t and compare the resulting expressions:

$$\frac{x-2}{3} = t, \quad \frac{y}{-1} = t, \quad \frac{z-3}{2} = t \qquad \Rightarrow \qquad \frac{x-2}{3} = \frac{y}{-1} = \frac{z-3}{2},$$

which is the symmetric form of the equation of the straight line q sought. ■

147. Write the parametric equations of the straight lines

a) $\left\{ \begin{array}{r} 2\,x + 3\,y - z - 4 = 0 \\ 3\,x - 5\,y + 2\,z + 1 = 0 \end{array} \right.$; b) $\left\{ \begin{array}{r} x + 2\,y - z - 6 = 0 \\ 2\,x - y + z + 1 = 0 \end{array} \right.$.

$$\left[\text{a)} \left\{ \begin{array}{rcl} x = & 1 & + \ t \\ y = & & -7\,t \\ z = & -2 & - 19\,t \end{array} \right. ; \quad \text{b)} \left\{ \begin{array}{rcl} x = & 1 & - \ t \\ y = & 2 & + 3\,t \\ z = & -1 & + 5\,t \end{array} \right. , \ t \in \mathbb{R} \right]$$

148. Write the parametric equations of the median passing through the vertex C of the triangle ΔABC, where $A = [3; 6; -7],\ B = [-5; 2; 3],\ C = [4; -7; -2]$.

$$[x = 4 + 5\,t,\ y = -7 - 11\,t,\ z = -2]$$

149. Find the angle of the straight lines:

$$p: \left\{ \begin{array}{rcl} x = & 5 & + 2\,t \\ y = & 2 & - \ t \\ z = & 7 & + \ t \end{array} \right. , \ t \in \mathbb{R}, \qquad q: \left\{ \begin{array}{r} x + 3\,y + z + 2 = 0 \\ x - y - 3\,z - 2 = 0 \end{array} \right. .$$

$$[0,\ p \parallel q]$$

150. Find the distance of the point A from the straight line p:

a) $A = [1; -1; -2],\ p: \begin{cases} x = & -3 + 3t \\ y = & -2 + 2t \\ z = & 8 - 2t \end{cases},\ t \in \mathbb{R};$

b) $A = [5; -6; 8],\ p$ is the axis x.

[a) 7; b) 10]

151. Find out whether the straight lines p and q intersect. If they do, find the coordinates of the point of intersection, and if they do not intersect, find their shortest distance, where $p: X = [3; 4; -1] + t\,(1; 1; -1),\ t \in \mathbb{R}$ and q is the straight line passing through the points $B = [-6; -5; 1],\ C = [0; 7; -2]$.

[p, q are skew lines, $d = \sqrt{14}$]

152. Determine m so that the straight line $p: X = [-1; 2; -3] + t\,(3; m; -2),\ t \in \mathbb{R}$ is parallel with the plane $\varrho: x - 3y + 6z + 7 = 0$.

[$m = -3$]

153. Determine a, b so that the straight line $p: X = [2; -1; 5] + t\,(a; 4; -3),\ t \in \mathbb{R}$ is perpendicular to the plane $\varrho: 3x - 2y - bz + 15 = 0$. Find the point of intersection of line p and plane ϱ.

$\left[a = -6,\ b = -\frac{3}{2};\ [-4; 3; 2]\right]$

154. Through the point $A = [0; -3; 0]$ construct the plane perpendicular to the two planes $\alpha: x + 2y + 3z = 5,\ \beta: 3x - 5y + 4z = 12$.

[$23x + 5y - 11z + 15 = 0$]

155. Through the straight line $p: X = [1; -3; -2] + t\,(2; -1; 5),\ t \in \mathbb{R}$ construct the plane perpendicular to the plane $\alpha: x + y - 3z + 7 = 0$.

[$-2x + 11y + 3z + 41 = 0$]

156. Through the point $A = [4; 3; -1]$ construct the plane perpendicular to the straight line $p: X = [1; 2; 5] + t\,(1; 2; 3), t \in \mathbb{R}$.

[$x + 2y + 3z - 7 = 0$]

157. Find the angle of the straight line p determined by the points $A = [2; 0; 3],\ B = [2; 2; 2]$ with the plane $\alpha: 3z - y + 10 = 0$.

[$\varphi = \pi/4$]

158. Write the canonical equations of these straight lines

a) $\begin{cases} x - 2y + 3z - 4 = 0 \\ 3x + 2y - 5z - 4 = 0 \end{cases}$, b) $[x; y; z] = [3; 2; 1] + t \cdot (-1; 1; 4),\ t \in \mathbb{R}$.

$\left[\text{a)}\frac{x-2}{2} = \frac{y+1}{7} = \frac{z}{4};\quad \text{b)}\frac{x-3}{-1} = y - 2 = \frac{z-1}{4}\right]$

159. Write the parametric equations of the straight line which passes through the point $A = [1; -1; -3]$ and is parallel with the straight line $\begin{cases} \frac{x-1}{2} = \frac{y+2}{5} \\ z = 1 \end{cases}$.

$[X = [1; -1; -3] + t\,(2; 5; 0),\ t \in \mathbb{R}]$

160. Find the angle of the straight line $\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z-5}{2}$ with the plane (xz).

$\left[\sin\varphi = \frac{2}{3}\right]$

II.6 Quadric surfaces in $\mathbb{E}_3$

***Example* 161**: Write the equation of the sphere with the centre at the point $S = [1; 4; -7]$ and touching the plane $\alpha:\ 6\,x + 6\,y - 7\,z + 42 = 0$.

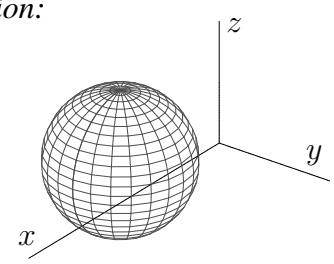
Solution: The radius of the sphere equals the distance of the point S from the plane α :

$$r = \frac{|6 \cdot 1 + 6 \cdot 4 - 7 \cdot (-7) + 42|}{\sqrt{36 + 36 + 49}} = \frac{121}{\sqrt{121}} = 11.$$

In this way the sphere is determined: $(x-1)^2 + (y-4)^2 + (z+7)^2 = 121$. ■

***Example* 162**: Find the coordinates of the centre and the radius of the sphere $x^2 + y^2 + z^2 - 6\,x + 8\,y + 2\,z + 10 = 0$.

Solution:



We group the terms containing $x,\ y$ and z and complete them to squares:

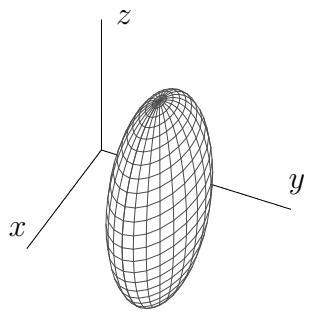
$$(x^2 - 6\,x + 9) + (y^2 + 8\,y + 16) + (z^2 + 2\,z + 1) = -10 + 9 + 16 + 1 \quad \Rightarrow$$
$$(x-3)^2 + (y+4)^2 + (z+1)^2 = 16 \quad \Rightarrow \quad S = [3; -4; -1],\ r = 4.$$ ■

***Example* 163**: Determine which quadric surfaces these are, name them and find the centre, the vertex, the axes according to what of it exists.

a) $9\,x^2 + 36\,y^2 + 4\,z^2 - 18\,x - 144\,y + 117 = 0$;
b) $9\,x^2 + 36\,y^2 - 4\,z^2 - 9 = 0$;
c) $x^2 + y^2 - 2\,x - 8\,z + 17 = 0$;
d) $x^2 + y^2 + 10\,x - 24 = 0$;
e) $x^2 - 4\,y^2 - 4\,z^2 - 2\,x - 16\,z - 19 = 0$.
f) $y^2 + 4\,y - 8\,z + 12 = 0$;
g) $x^2 + 4\,y^2 = 9\,z^2$;
h) $x^2 - 4\,y^2 = z$;
i) $y^2 - z^2 = 0$;

Solution: We transform every equation of a quadric surface to the so called **canonical form**, from which we read everything needed:

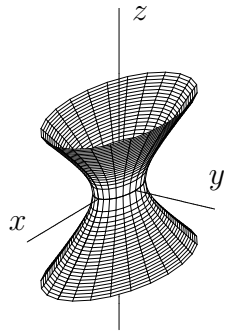
a)



$$9\,(x^2 - 2\,x \qquad) + 36\,(y^2 - 4\,y \qquad) + 4\,z^2 = -117 \quad \Rightarrow$$
$$9\,(x^2 - 2\,x + 1) + 36\,(y^2 - 4\,y + 4) + 4\,z^2 = -117 + 9 + 144,$$
$$9\,(x-1)^2 + 36\,(y-2)^2 + 4\,z^2 = 36 \quad \Rightarrow$$
$$\Rightarrow \quad \frac{(x-1)^2}{4} + \frac{(y-2)^2}{1} + \frac{z^2}{9} = 1.$$

The resulting surface is an ellipsoid with the centre at the point $S = [1; 2; 0]$ and with the semi–axes $a = 2,\ b = 1,\ c = 3$.

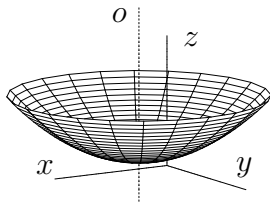
b)



$$9\,x^2 + 36\,y^2 - 4\,z^2 = 9 \quad \Rightarrow \quad x^2 + 4\,y^2 - \frac{4\,z^2}{9} = 1.$$

The surface is a one sheet elliptic hyperboloid with the centre at the point $[0;0;0]$ and with the semi–axes $a = 1,\ b = \frac{1}{2},\ c = \frac{3}{2}$.

c)



$$(x^2 - 2\,x + 1) + y^2 = 8\,z - 17 + 1 \quad \Rightarrow$$
$$\Rightarrow \quad 8\,(z - 2) = (x - 1)^2 + y^2.$$

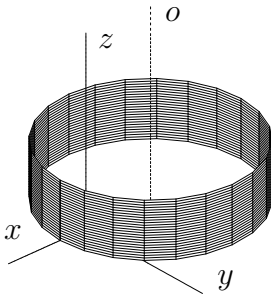
The surface is a paraboloid of revolution with the vertex at the point $[1;0;2]$ and with the axis of revolution which is parallel with the axis z and passes through the point $[1;0;0]$.

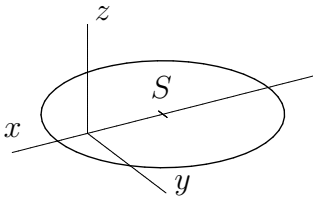
d)

$$(x^2 + 10\,x + 25) + y^2 = 24 + 25 \quad \Rightarrow \quad (x + 5)^2 + y^2 = 49.$$

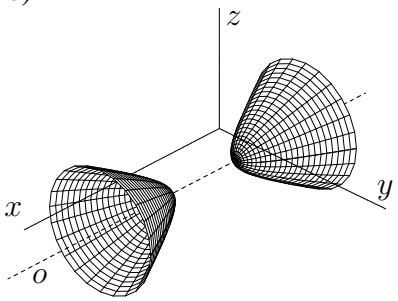
In the space $\mathbb{E}_3$ this means a cylindrical surface of revolution with the axis of revolution parallel with the axis z.

In the plane (xy) this equation expresses the circle with the centre at the point $[-5;0;0]$ and with the radius $r = 7$.





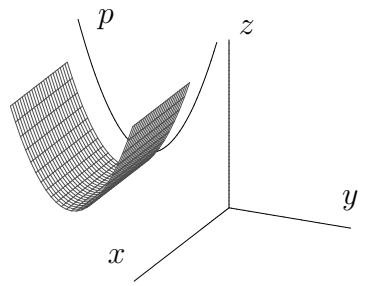
e)



$$(x^2 - 2\,x + 1) - 4\,y^2 - 4\,(z^2 + 4\,z + 4) = 19 + 1 - 16,$$
$$(x - 1)^2 - 4\,y^2 - 4\,(z + 2)^2 = 4 \quad \Rightarrow$$
$$\Rightarrow \quad \frac{(x - 1)^2}{4} - y^2 - (z + 2)^2 = 1.$$

We have obtained the equation of a two sheet hyperboloid of revolution with the centre at the point $[1;0;-2]$ and with the axis of revolution parallel with the axis x.

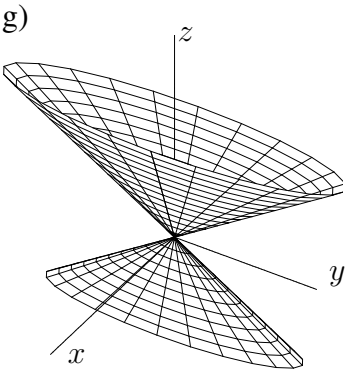
f)



$$y^2 + 4\,y + 4 = 8\,z - 12 + 4 \quad \Rightarrow$$
$$\Rightarrow \quad (y+2)^2 = 8\,(z-1).$$

In the plane (yz) this equation expresses a parabola p, but in $\mathbb{E}_3$ it is a cylindrical surface whose generating lines are parallel with the axis x and whose directrix is the parabola p.

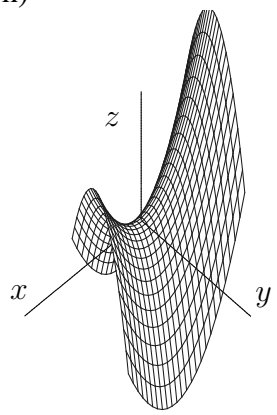
g)



$$x^2 + 4\,y^2 = 9\,z^2.$$

It is an elliptic conical surface with the vertex $V = [0;0;0]$ and with the axis coinciding with the axis z.

h)



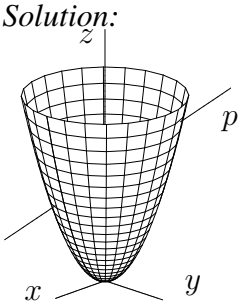
$$x^2 - 4\,y^2 = z.$$

The surface is a hyperbolic paraboloid whose saddle point is the origin $[0;0;0]$.

i) $y^2 - z^2 = 0 \quad \Rightarrow \quad (y-z)(z+y) = 0 \quad \Rightarrow$ the surface consists of two planes: $y = z$ and $y = -z$, which pass through the axis x. Hence the axis x is their line of intersection. ■

***Example* 164**: Find the points of intersection of the straight line $p : X = [-1;3;9] + t\,(1;-1;-1)$ with the surface $2\,z = x^2 + y^2$. What surface is it?

Solution:



The surface $2\,z = x^2 + y^2$ is a paraboloid of revolution with the vertex at the origin of the coordinate system and with the axis in the axis z. The points of intersection with the straight line p are obtained by substituting x, y, z from the equation of the line into the equation of the surface, computing the parameters of the points of intersection and then the points of intersection themselves:

$$2\,(9-t) = (-1+t)^2 + (3-t)^2 \quad \Rightarrow \quad 18 - 2\,t = 1 - 2\,t + t^2 + 9 - 6\,t + t^2 \quad \Rightarrow$$

$2\,t^2 - 6\,t - 8 = 0 \quad \Rightarrow \quad t^2 - 3\,t - 4 = 0 \quad \Rightarrow \quad (t-4)(t+1) = 0$, for $t_1 = 4$ we obtain the point of intersection $P_1 = [3;-1;5]$ and for $t_2 = -1$ the point of intersection $P_2 = [-2;4;10]$. ■

165. Write the equation of the sphere which has its centre at the point $= [3; 0; -2]$ and passes through the point $[3; 0; 0]$.

$[(x-3)^2 + y^2 + (z+2)^2 = 4]$

166. Write the equation of the cylindrical surface of revolution which has its axis in the axis x and the radius $r = 3$.

$[y^2 + z^2 = 9]$

167. At the points of intersection of the straight line $p : X = [1; 0; 1] + t\,(1; -1; 2)$ with the sphere $(x-2)^2 + (y+1)^2 + (z-3)^2 = 6$ construct the tangent planes to this surface.

$$\left[\begin{array}{rr} x - y + 2\,z - 3 = 0 & \text{the tangent planes are parallel,} \\ x - y + 2\,z - 15 = 0 & \text{because the straight line } p \text{ passes} \\ & \text{through the centre of the sphere} \end{array}\right]$$

168. Show that the plane $x = 2$ intersects the ellipsoid $\dfrac{x^2}{16} + \dfrac{y^2}{12} + \dfrac{z^2}{4} = 1$ in an ellipse and find its semi–axes and vertices.

$[3,\ \sqrt{3},\ [2; 3; 0],\ [2; -3; 0],\ [2; 0; \sqrt{3}],\ [2; 0; -\sqrt{3}]]$

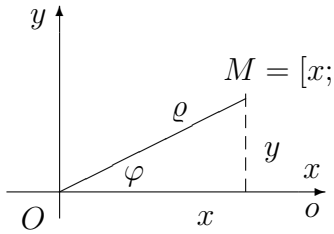
169. What is the section of the hyperbolic paraboloid $z = 4\,x^2 - y^2$ with the plane: a) $y = 6$; b) $x = 1$; c) $z = 1$; d) $z = 0$.

[a) parabola; b) parabola; c) hyperbola; d) two intersecting straight lines]

II.7* Some engineering curves

We restrict ourselves only to those curves which the students meet in the 1st and in the 2nd semester. We describe their construction, write their equation, add a figure and, where appropriate, state some of their special properties.

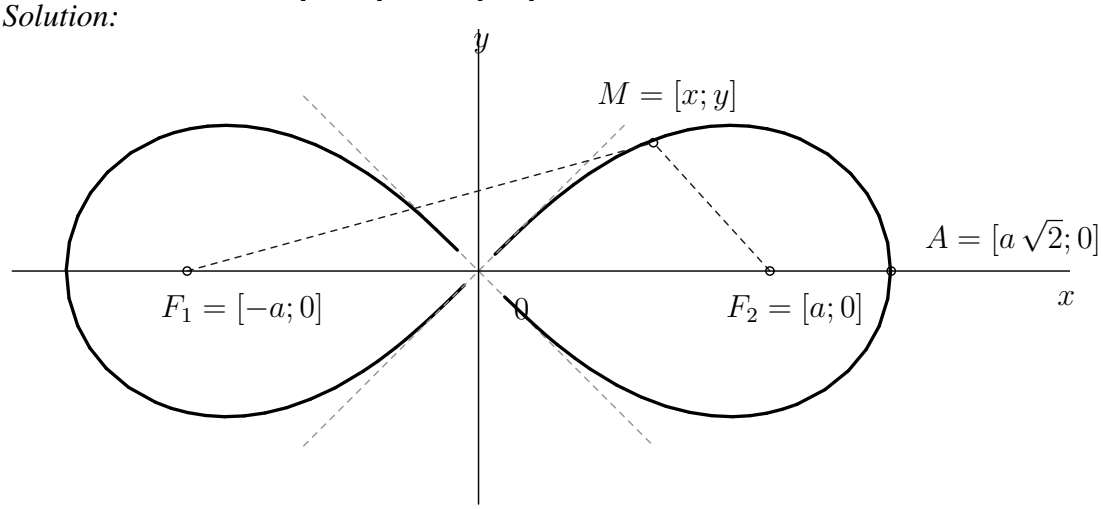
For the analytic expression of many plane curves it is convenient to use, besides the Cartesian coordinates $[x; y]$, the polar coordinates $[\varrho; \varphi]$ determined by the origin O (the so called pole) and the polar axis o, where ϱ is the distance of the point M from the origin and φ is the angle which the radius vector $\overrightarrow{OM}$ makes with the axis o.



$$\begin{cases} x = \varrho \cdot \cos\varphi \\ y = \varrho \cdot \sin\varphi \end{cases} \qquad \begin{cases} \varrho = +\sqrt{x^2 + y^2} \\ \tan\varphi = \dfrac{y}{x}, \quad 0 \le \varphi < 2\,\pi \end{cases}$$

***Example* 170**: **Bernoulli's lemniscate** is the locus of the points which have a constant product of the distances a^2 from two fixed points $F_1, F_2 (|F_1F_2| = 2\,a)$. Derive its equation for $F_1 = [-a; 0], F_2 = [a; 0]$.

Solution:



By the definition there holds

$$\begin{aligned} |F_1M| \cdot |F_2M| = a^2 \quad &\Longrightarrow \quad \sqrt{(x+a)^2 + y^2} \cdot \sqrt{(x-a)^2 + y^2} = a^2, \\ \Big((x+a)^2 + y^2\Big)\Big((x-a)^2 + y^2\Big) &= a^4, \\ (x^2 + 2\,a\,x + a^2 + y^2)(x^2 - 2\,a\,x + a^2 + y^2) &= a^4, \\ (x^2 + y^2)^2 + 2\,a^2(x^2 + y^2) + a^4 - 4\,a^2x^2 &= a^4, \\ (x^2 + y^2)^2 + 2\,a^2y^2 - 2\,a^2x^2 &= 0, \\ (x^2 + y^2)^2 &= 2\,a^2(x^2 - y^2) \end{aligned}$$

- the eq. of Bernoulli's lemniscate.

The equation in Cartesian coordinates is relatively complicated, therefore we pass on to polar coordinates:

$$(\varrho^2)^2 = 2\,a^2(\varrho^2\cos^2\varphi - \varrho^2\sin^2\varphi) \Rightarrow \varrho^4 = 2\,a^2\varrho^2(\cos^2\varphi - \sin^2\varphi) \Rightarrow \varrho^2 = 2\,a^2\cos 2\,\varphi \Rightarrow$$

$$\varrho = a\,\sqrt{2\,\cos 2\,\varphi}$$ - the equation of the lemniscate in polar coordinates.

From the condition $\cos 2\,\varphi \geq 0$ we obtain $\varphi \in \langle 0; \frac{\pi}{4}\rangle \cup \langle \frac{3\,\pi}{4}; \frac{5\,\pi}{4}\rangle \cup \langle \frac{7\,\pi}{4}; 2\,\pi)$.

Special properties:

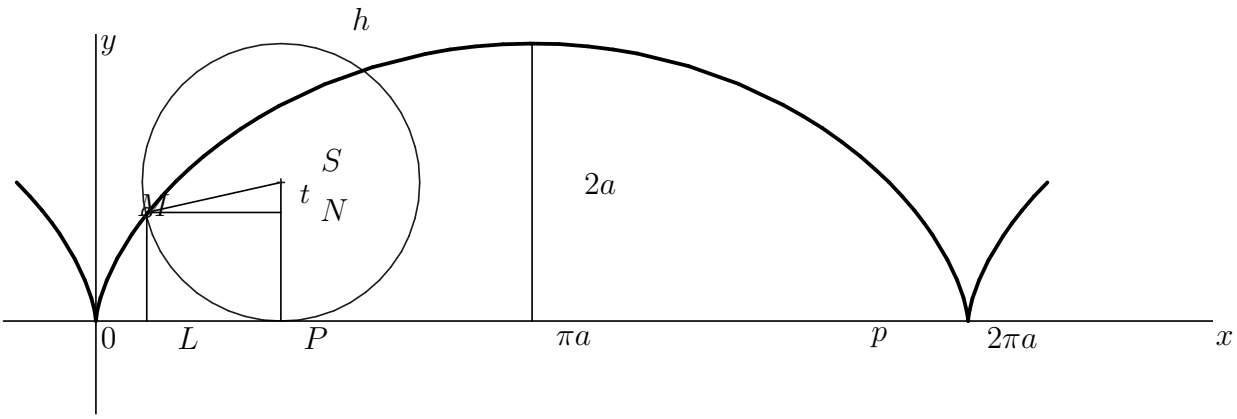
a) The tangents at the origin make with the axis x the angle $+\frac{\pi}{4}$ and $-\frac{\pi}{4}$, hence they are perpendicular to each other.

b) The points of intersection with the axis x are the points $A = [a\sqrt{2}; 0]$ and $B = [-a\sqrt{2}; 0]$ (see M II, ex. 326).

c) The area bounded by the lemniscate is $2\,a^2$, which is the area of the square with the side $a\sqrt{2}$ (the length $OA = a\sqrt{2}$). ■

***Example* 171**: Derive the parametric equations of the curve called the **simple cycloid**, described by a point M lying on a circle h of radius a which rolls without slipping along the straight line p.

Solution: We choose the axis x as the straight line p along which the circle h rolls , and let us denote t the angle of rotation of the circle.



$$x = OL = OP - LP, \qquad y = ML = SP - SN, \qquad SP = a,$$
$$OP = \overset{\frown}{MP} = at, \qquad LP = MN = a\,\sin t, \qquad SN = a\,\cos t.$$

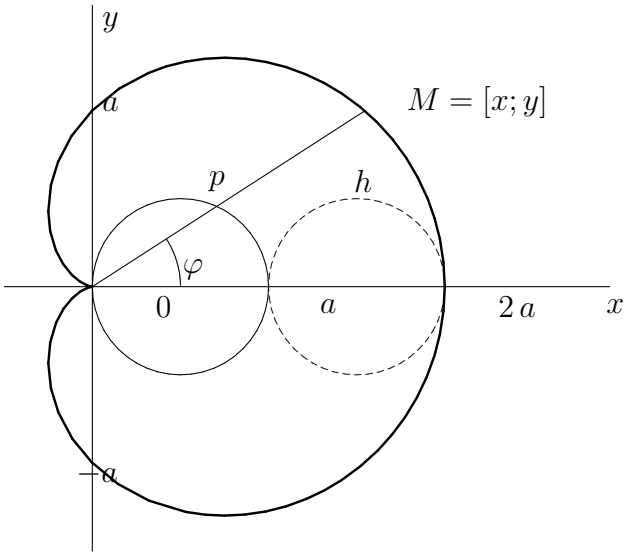
Substituting we obtain the equations of the cycloid:

$$\begin{cases} x = a\,t - a\,\sin t = a\,(t - \sin t) \\ y = a - a\,\cos t = a\,(1 - \cos t) \end{cases}.$$ One arch of the cycloid is obtained for $t \in \langle 0; 2\,\pi\rangle$.

Special property: the area bounded by the axis x and one arch of the cycloid is $3\pi a^2$, which is three times the area of the rolling circle. ■

***Example* 172**: Let us have two circles of the same radius having an external contact. The **cardioid** is the curve described by a point lying on the circle h rolling without slipping along the other, fixed circle p.

The derivation of the equation is beyond the scope of these lecture notes, therefore we write it directly in polar coordinates.



$\varrho = a\,(1 + \cos\varphi), \quad \varphi \in \langle 0; 2\,\pi)$, where a is the diameter of the given circles and φ is the angle which $\overrightarrow{OM}$ makes with the positive semi–axis x (the polar axis).

The points of intersection with the axis x are obtained by substituting $\varphi = 0$ or $\varphi = \pi$:

$$\varrho(0) = 2\,a, \quad \varrho(\pi) = 0 \quad \Longrightarrow \quad [2\,a; 0],\ [0; 0].$$

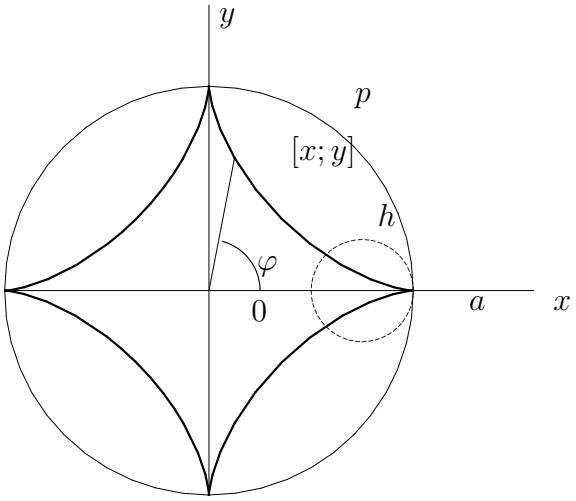
The points of intersection with the axis y are obtained for $\varphi = \dfrac{\pi}{2}$ and $\varphi = \dfrac{3\,\pi}{2}$

$$\varrho\left(\frac{\pi}{2}\right) = a, \quad \varrho\left(\frac{3\,\pi}{2}\right) = a \quad \Longrightarrow \quad [0; a],\ [0; -a].$$

The area of the cardioid is $\dfrac{3}{2}\,\pi\,a^2$. ■

***Example* 173**: Given a fixed circle p of radius a and a rolling circle h with an internal contact, of radius $a/4$. A point of the rolling circle h describes a curve which is called the **astroid**.

Again we shall not derive the equation and we write it in the usual form in polar coordinates:

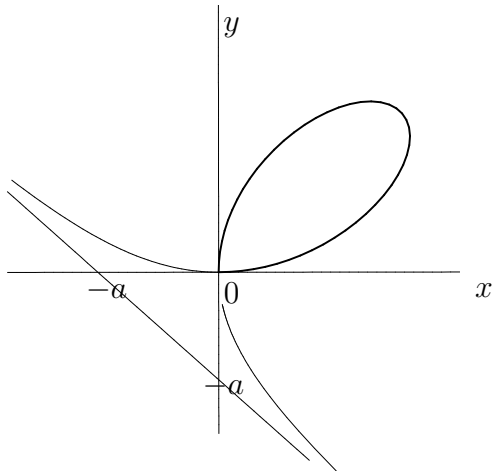


$$\begin{cases} x = a\,\cos^3\varphi \\ y = a\,\sin^3\varphi \end{cases}, \quad 0 \le \varphi < 2\,\pi.$$

If we raise both equations to the power $()^{\frac{2}{3}}$ and add them, we obtain $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, which is another frequently used form.

The area of the astroid is $\dfrac{3}{8}\,\pi\,a^2$. ■

***Example* 174**: The **folium of Descartes** is a curve of the 3rd degree with the equation $x^3 + y^3 - 3axy = 0$ in rectangular coordinates.



By means of the substitution $y = tx$ we obtain

$$x^3 + t^3x^3 - 3atx^2 = 0, \quad x \neq 0 \quad \Longrightarrow$$

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}$$

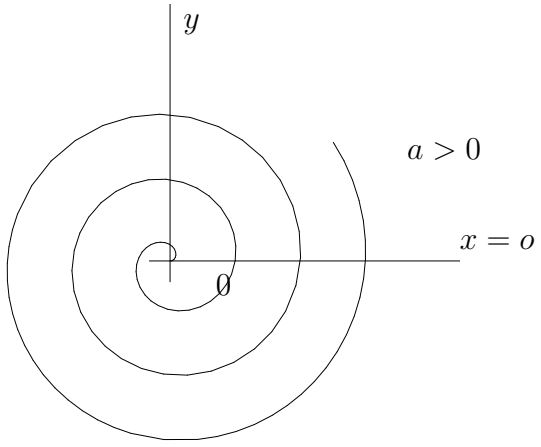
are the equations in parametric form.

In integral calculus the equation in polar coordinates is more useful for the loop of the curve

$$\varrho(\varphi) = \frac{3a\cos\varphi\sin\varphi}{\cos^3\varphi + \sin^3\varphi}, \quad 0 \leq \varphi \leq \frac{\pi}{2}.$$

The area bounded by the loop of the curve is $\frac{3}{2}a^2$. ■

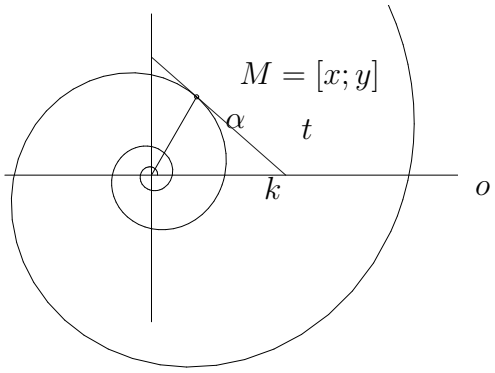
***Example* 175**: The **Archimedean spiral** is the trajectory of a point moving at constant speed along a straight line which rotates uniformly.



The equation of this curve in polar coordinates is $\varrho = a\,\varphi$. ■

REMARK: There exists a large number of spirals, of which the most interesting is the logarithmic spiral.

***Example* 176**: The **logarithmic spiral** is the curve satisfying the equation in polar coordinates $\varrho = k\,e^{a\,\varphi}, \quad k \neq 0,\ a \neq 0$ are given real constants.



This curve has the following important property: the tangent t and the radius vector $\overrightarrow{OM}$ of an arbitrary point M make a constant angle α.

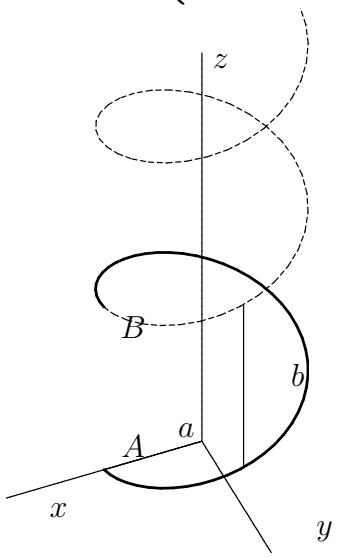
A related constant-angle curve in navigation is the **loxodrome** (or rhumb line) on a sphere. It meets the meridians at a constant angle, so it represents a course with constant compass bearing. The spherical loxodrome and the plane logarithmic spiral are different curves.

■

REMARK: So far all the curves have been plane ones. A space curve is given either as the curve of intersection of two surfaces or by its parametric equations.

***Example* 177**: The **helix on the cylindrical surface** $x^2 + y^2 = a^2$ has the equations

$$\begin{cases} x = a\,\cos t \\ y = a\,\sin t \\ z = b\,t \end{cases}$$, where for $t \in \langle 0; 2\,\pi \rangle$ we obtain one turn of the helix.

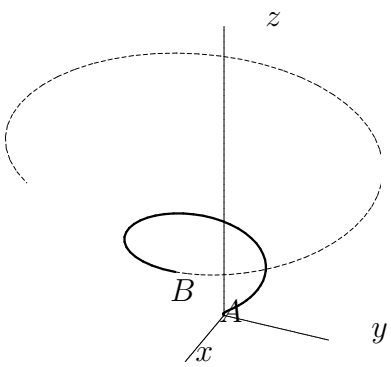


The initial point is the point A with the parameter $t = 0 \Longrightarrow A = [a; 0; 0]$ and the end point is $B = [a; 0; 2\,\pi\,b]$ for $t = 2\,\pi$.

The tangent to the helix makes a constant angle with the plane $z = 0$.

***Example* 178**: The **helix on the conical surface** $z = \sqrt{x^2 + y^2}$ has the equations

$$\begin{cases} x = t\,\cos t \\ y = t\,\sin t \\ z = t \end{cases}$$, for $t \in \langle 0; 2\,\pi \rangle$ we again obtain one turn of the helix.



The initial point is the point $A = [0;0;0]$ for $t = 0$ and the end point is $B = [2\,\pi;0;2\,\pi]$ for $t = 2\,\pi$. ■

***Example* 179**: Write the parametric equations of the curve of intersection of the surfaces:

a) $x^2 + y^2 = 4, \quad x - 2\,y + z = 0;$

b) $x^2 + y^2 + z^2 = 2\,a^2, \quad z = \sqrt{x^2 + y^2};$

c) $x^2 = 3\,y, \quad 2\,xy = 9\,z.$

Solution:

a) The first equation is the equation of a cylindrical surface of revolution with the axis of revolution in the axis z. The second equation $x - 2\,y + z = 0$ expresses a plane passing through the origin of the coordinate system.
The line of intersection of these two surfaces is an ellipse. We write the parametric equations of the ellipse using the polar system in the plane (xy) and from the equation of the plane we express the variable $z = 2\,y - x$:

$$\begin{cases} x = 2\,\cos t \\ y = 2\,\sin t \\ z = 4\,\sin t - 2\,\cos t \end{cases}, \quad t \in \langle 0; 2\,\pi).$$

b) The first equation $x^2 + y^2 + z^2 = 2\,a^2$ is the equation of a sphere with the centre $S = [0;0;0]$ and $r = a\,\sqrt{2}$. The second equation $z = \sqrt{x^2 + y^2}$ is the equation of a conical surface of revolution with the vertex at the point $[0;0;0]$ and with the axis of revolution in the axis z.
The surfaces intersect in a circle, which we determine from the system

$$\begin{matrix} x^2 + y^2 + z^2 = 2\,a^2 \\ z = \sqrt{x^2 + y^2} \end{matrix} \quad \Rightarrow \quad 2\,(x^2 + y^2) = 2\,a^2 \quad \Rightarrow \quad \begin{matrix} x^2 + y^2 = a^2 \\ z = \sqrt{x^2 + y^2} \end{matrix} \quad \Rightarrow \quad z = a.$$

The circle of intersection lies in the plane $z = a$. For the parametrization we again use polar coordinates and we obtain:

$$\begin{cases} x = a\,\cos t \\ y = a\,\sin t \\ z = a \end{cases}, \qquad t \in \langle 0; 2\,\pi).$$

c) The first equation is the equation of a parabolic cylindrical surface whose generating lines are parallel with the axis z. The second equation is the equation of a hyperbolic paraboloid.

As the variable x we choose the parameter t. From the first equation we express the variable y and from the second equation the variable z. We obtain the parametric equations of the curve of intersection

$$\begin{cases} x = t \\ y = \frac{1}{3}\,t^2 \\ z = \frac{2}{27}\,t^3 \end{cases}, \quad t \in \mathbb{R}\,.$$ ■

Recommended literature

[1] J. NEUSTUPA: **Matematika I** ČVUT, Praha 1998 (*The basic literature for the subject Mathematics I. Lecture notes intended for the students of the Faculty of Mechanical Engineering, CTU in Prague.)*

[2] B. BUDÍNSKÝ, J. CHARVÁT: **Matematika I** SNTL/Alfa, Praha 1987 (*A detailed and understandably written textbook.)*

[3] J. NEUSTUPA, S. KRAČMAR : **Sbírka příkladů z Matematiky I.** ČVUT, Praha 1998 (*An extensive collection of unsolved problems in Mathematics I.)*

[4] K. REKTORYS : **Přehled užité matematiky** SNTL Praha 1988 (*An extensive encyclopaedia of applied mathematics written for the needs of the technical sciences.)*