CZECH TECHNICAL UNIVERSITY IN PRAGUE

FACULTY OF MECHANICAL ENGINEERING

MATHEMATICS II
EXERCISES

March 10, 2026

Experimental version of the collection of exercises for the academic year 2025/2026

Based on: *E. Brožíková, M. Kittlerová, F. Mráz: Sbírka příkladů z Matematiky II (2016)*

Contents

CHAPTER I.

Definite integral

I.1 Existence of definite integrals

• Determine whether the definite integrals exist:

***Example* 1.** $\displaystyle\int_0^1 f(x)\,dx = \int_0^1 \frac{x+3}{x^2+1}\,dx$

Solution : Yes, it exists, because the function $f(x) = \dfrac{x+3}{x^2+1}$ is continuous on the interval $\langle 0;1\rangle$. ■

***Example* 2.** $\displaystyle\int_1^{10} f(x)\,dx = \int_1^{10} \frac{x^2+3}{x^3-3x^2-4x}\,dx$

Solution : It does not exist, because the function $f(x) = \dfrac{x^2+3}{x^3-3x^2-4x} = \dfrac{x^2+3}{x(x+1)(x-4)}$ is not continuous at $x=4$ $\Big(x \in (1;10)\Big)$ and $\lim\limits_{x\to 4\pm} f(x) = \pm\infty$ (f is not bounded). ■

***Example* 3.** $\displaystyle\int_{-1}^1 f(x)\,dx = \int_{-1}^1 \frac{\mathrm{e}^{2x}-1}{x}\,dx$

Solution : The integral exists. Although the function $f(x) = \dfrac{\mathrm{e}^{2x}-1}{x}$ is not continuous at the point $x=0$ $\Big(x \in (-1;1)\Big)$, we have $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\mathrm{e}^{2x}-1}{x} = 2$ (f is bounded). ■

4. $\displaystyle\int_0^1 \frac{\sin x^2}{x}\,dx$ $\left[\text{yes}\,,\ \lim\limits_{x\to 0}\dfrac{\sin x^2}{x} = 0\right]$

5. $\displaystyle\int_0^2 x\mathrm{e}^{x^2}\,dx$ [yes]

6. $\displaystyle\int_0^{\pi/2} \frac{1}{1-2\cos x}\,dx$ $\left[\text{no}\,,\ \lim\limits_{x\to \frac{\pi}{3}\pm}\dfrac{1}{1-2\cos x} = \pm\infty\right]$

I.2 Newton-Leibniz formula

• Compute the integrals using the Newton-Leibniz formula:

***Example* 7.** $\displaystyle\int_0^{\pi/4} \frac{1+\sin^2 x}{\cos^2 x}\,dx$

Solution : $\displaystyle\int_0^{\pi/4} \frac{1+\sin^2 x}{\cos^2 x}\,dx = \int_0^{\pi/4} \frac{1+1-\cos^2 x}{\cos^2 x}\,dx = \int_0^{\pi/4} \left(\frac{2}{\cos^2 x} - 1\right)\,dx =$

$= \Big[2\,\tan\, x - x\Big]_0^{\pi/4} = 2\tan\,\tfrac{\pi}{4} - \tfrac{\pi}{4} = 2 - \tfrac{\pi}{4}.$ ■

***Example* 8.** $\displaystyle\int_3^8 \sqrt{1+x}\,dx$

Solution : $\displaystyle\int_3^8 (1+x)^{1/2}\,dx = \frac{2}{3}\Big[(1+x)^{3/2}\Big]_3^8 = \frac{2}{3}\cdot(9\sqrt{9}-4\sqrt{4}) = \frac{38}{3}.$ ■

***Example* 9.** $\displaystyle\int_0^2 \frac{x-3}{x^2+4}\,dx$

Solution : $\displaystyle\int_0^2 \frac{x-3}{x^2+4}\,dx = \frac{1}{2}\int_0^2 \frac{2x}{x^2+4}\,dx - 3\int_0^2 \frac{1}{x^2+4}\,dx =$

$\displaystyle= \frac{1}{2}\Big[\ln(x^2+4)\Big]_0^2 - \frac{3}{2}\Big[\arctan\frac{x}{2}\Big]_0^2 = \frac{1}{2}\cdot(\ln 8 - \ln 4) - \frac{3}{2}\cdot\arctan 1 = \frac{1}{2}\cdot\ln 2 - \frac{3\pi}{8}.$ ■

***Example* 10.** $\displaystyle\int_2^5 \frac{5x+1}{x^2+x-2}\,dx$

Solution : First decompose the integrand into partial fractions and compute the integral :

$\displaystyle\frac{5x+1}{x^2+x-1} = \frac{A}{x-1}+\frac{B}{x+2}\,,\quad 5x+1 = A(x+2)+B(x-1)$

$x=1:\ 6=3A \implies A=2\,;\qquad x=-2:\ -9=-3B \implies B=3$

$\displaystyle\int_2^5 \frac{5x+1}{x^2+x-2}\,dx = \int_2^5\left(\frac{2}{x-1}+\frac{3}{x+2}\right)dx = \Big[2\ln|x-1|+3\ln|x+2|\Big]_2^5 =$

$\displaystyle= 2\ln 4+3\ln 7-2\ln 1-3\ln 4 = 3\ln 7-\ln 4 = \ln\frac{343}{4}.$

CAUTION ! The same integral on another interval $\langle a;b\rangle$ may not exist if $\langle a;b\rangle$ contains at least one of the numbers $x=-2$ or $x=1$. ■

***Example* 11.** $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^2\frac{x}{2}\,dx$

Solution : Here we use the fact that the function $f(x)=\cos^2\frac{x}{2}$ is even, i.e.

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

$\displaystyle\int_{-\pi/2}^{\pi/2} \cos^2\frac{x}{2}\,dx = 2\int_0^{\pi/2}\cos^2\frac{x}{2}\,dx = 2\int_0^{\pi/2}\frac{1+\cos x}{2}\,dx = \Big[x+\sin x\Big]_0^{\pi/2} = \frac{\pi}{2}+1.$ ■

***Example* 12.** $\displaystyle\int_{-\pi/2}^{\pi/2} x^2\sin x\,dx$

Solution : Now we use the fact that the function $f(x)=x^2\sin x$ is odd, i.e. $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$ ■

***Example* 13.** $\displaystyle\int_0^{\pi}\Big|\frac{1}{2}-\cos x\Big|\,dx$

Solution : Remove the absolute value: $\displaystyle\frac{1}{2}-\cos x\ge 0$ for $x\in\Big\langle\frac{\pi}{3},\pi\Big\rangle$

and $\displaystyle\frac{1}{2}-\cos x<0$ for $x\in\big\langle 0,\frac{\pi}{3}\big)$, thus for the given integral we have:

$\displaystyle\int_0^{\pi}\Big|\frac{1}{2}-\cos x\Big|\,dx = \int_0^{\pi/3}-\Big(\frac{1}{2}-\cos x\Big)\,dx + \int_{\pi/3}^{\pi}\Big(\frac{1}{2}-\cos x\Big)\,dx =$

$\displaystyle= \Big[\sin x-\tfrac{1}{2}x\Big]_0^{\pi/3} + \Big[\tfrac{1}{2}x-\sin x\Big]_{\pi/3}^{\pi} = \tfrac{\sqrt{3}}{2}-\tfrac{\pi}{6}+\tfrac{\pi}{2}-\tfrac{\pi}{6}+\tfrac{\sqrt{3}}{2} = \sqrt{3}+\tfrac{\pi}{6}.$ ■

14.* $\displaystyle\int_{-4}^{4} \frac{1}{\sqrt{x^2+9}}\,dx$ $\big[2\ln 3\big]$ **15.** $\displaystyle\int_{0}^{\pi} \frac{\sin x}{2+\cos x}\,dx$ $\big[\ln 3\big]$

16. $\displaystyle\int_{0}^{1} \frac{\mathrm{e}^x}{\mathrm{e}^x+e}\,dx$ $\left[\ln\dfrac{2\mathrm{e}}{1+\mathrm{e}}\right]$ **17.** $\displaystyle\int_{-2}^{-1} \frac{2}{x^2-x}\,dx$ $\left[2\ln\dfrac{4}{3}\right]$

I.3 Integration by parts

• Compute the integrals using the integration by parts method:

***Example* 18.** $\displaystyle\int_{-\pi/2}^{\pi/2} |x|\cos x\,dx$

Solution : The integrand is even. Thus $\displaystyle\int_{-\pi/2}^{\pi/2} |x|\cos x\,dx = 2\int_{0}^{\pi/2} |x|\cos x\,dx = 2\int_{0}^{\pi/2} x\cos x\,dx =$

$$= \left|\begin{array}{ll} u=x, & v'=\cos x \\ u'=1, & v=\sin x \end{array}\right| = 2\Big[x\sin x\Big]_0^{\pi/2} - 2\int_0^{\pi/2}\sin x\,dx = 2\cdot\frac{\pi}{2} + 2\Big[\cos x\Big]_0^{\pi/2} = \pi - 2.$$

■

***Example* 19.** $\displaystyle\int_{0}^{e-1} \ln(x+1)\,dx$

Solution : $\displaystyle\int_{0}^{e-1} \ln(x+1)\,dx = \left|\begin{array}{ll} u=\ln(x+1), & v'=1 \\ u'=\dfrac{1}{x+1}, & v=x \end{array}\right| = \Big[x\ln(x+1)\Big]_0^{e-1} - \int_0^{e-1}\frac{x}{x+1}\,dx =$

$$= (e-1)\ln e - \int_0^{e-1}\frac{x+1-1}{x+1}\,dx = e-1-\Big[x-\ln(x+1)\Big]_0^{e-1} = e-1-(e-1-\ln e) = 1.$$

■

***Example* 20.** $I = \displaystyle\int_{0}^{\pi/2} \mathrm{e}^{2x}\sin x\,dx$

Solution : $\displaystyle\int_{0}^{\pi/2} \mathrm{e}^{2x}\sin x\,dx = \left|\begin{array}{ll} u=\mathrm{e}^{2x}, & v'=\sin x \\ u'=2\mathrm{e}^{2x}, & v=-\cos x \end{array}\right| - \Big[\mathrm{e}^{2x}\cos x\Big]_0^{\pi/2} + 2\int_0^{\pi/2}\mathrm{e}^{2x}\cos x\,dx =$

$$= \left|\begin{array}{ll} u=\mathrm{e}^{2x}, & v'=\cos x \\ u'=2\mathrm{e}^{2x}, & v=\sin x \end{array}\right| = 1 + 2\Big[\mathrm{e}^{2x}\sin x\Big]_0^{\pi/2} - 4\int_0^{\pi/2}\mathrm{e}^{2x}\sin x\,dx.$$

We obtained the equation $I = 1 + 2\mathrm{e}^{\pi} - 4I$, from which $5I = 1 + 2e^{\pi}$.

The result of the given example is $I = \dfrac{1}{5}(1+2\mathrm{e}^{\pi})$. ■

Example* 21. $I_n = \displaystyle\int_{0}^{\pi/2} \sin^n x\,dx$

Solution :

⋆ for $n=0$ we have $I_0 = \displaystyle\int_0^{\pi/2} dx = \frac{\pi}{2}$;

⋆ for $n=1$ we have $I_1 = \displaystyle\int_0^{\pi/2}\sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2} = 1$;

⋆ for $n=2$ we have $I_2 = \displaystyle\int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}\frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\Big[x - \frac{\sin 2x}{2}\Big]_0^{\pi/2} = \frac{\pi}{4}$;

⋆ for $n\geq 3$ we have $I_n = \displaystyle\int_0^{\pi/2}\sin^{n-2}x\cdot\sin^2 x\,dx = \int_0^{\pi/2}\sin^{n-2}x(1-\cos^2 x)\,dx =$

$$= \int_0^{\pi/2} \sin^{n-2} x\,dx - \int_0^{\pi/2} \sin^{n-2} x \cdot \cos x \cdot \cos x\,dx = \left| \begin{array}{ll} u = \cos x, & v' = \sin^{n-2} \cdot \cos x \\ u' = -\sin x, & v = \dfrac{1}{n-1} \sin^{n-1} x \end{array} \right| =$$

$$= I_{n-2} - \Big[\frac{\cos x \cdot \sin^{n-1} x}{n-1}\Big]_0^{\pi/2} - \frac{1}{n-1} \int_0^{\pi/2} \sin^n x\,dx.$$

We obtain the equation $I_n = I_{n-2} - 0 - \dfrac{1}{n-1} I_n$, from which we compute $I_n = \dfrac{n-1}{n} I_{n-2}$.

We perform a discussion:

$$\text{For} \quad \begin{cases} n = 2k \text{ we have } I_n = \dfrac{n-1}{n} I_{n-2} = \dfrac{2k-1}{2k} \cdot \dfrac{2k-3}{2k-2} \cdot \ldots \cdot \dfrac{1}{2} I_0 = \dfrac{(2k-1)!!}{(2k)!!} \cdot \dfrac{\pi}{2}; \\ n = 2k+1 \text{ we have } I_n = \dfrac{n-1}{n} I_{n-2} = \dfrac{2k}{2k+1} \cdot \dfrac{2k-2}{2k-1} \cdot \ldots \cdot \dfrac{2}{3} I_1 = \dfrac{(2k)!!}{(2k+1)!!} \cdot 1. \end{cases}$$

Thus we have derived the so-called **Wallis formulas**.

The same result also holds for $\displaystyle\int_0^{\pi/2} \cos^n x\,dx$, since $\displaystyle\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$. ■

22.* $\displaystyle\int_0^{\pi/2} \sin^7 x\,dx$ $\left[\dfrac{16}{35}\right]$ **23.*** $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^8 x\,dx$ $\left[\dfrac{35\pi}{128}\right]$

24.* $\displaystyle\int_0^{\pi} \cos^6 x\,dx$ $\left[\dfrac{5\pi}{16}\right]$ **25.*** $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^9 x\,dx$ $\Big[0\Big]$

26. $\displaystyle\int_0^{\sqrt{3}} x \cdot \arctan x\,dx$ $\left[\dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{2}\right]$ **27.** $\displaystyle\int_{1/e}^{e} |\ln x|\,dx$ $\left[2 - \dfrac{2}{e}\right]$

28. $\displaystyle\int_0^1 y \cdot \ln(x+y)\,dx,\ (y > 0)$ $\Big[y(y+1)\ln(y+1) - y^2 \ln y - y\Big]$

I.4 Substitution method

• Compute the integrals using the substitution method:

***Example* 29.** $\displaystyle\int_1^{\mathrm{e}^3} \frac{1}{x\sqrt{1+\ln x}}\,dx$

Solution : $\displaystyle\int_1^{\mathrm{e}^3} \frac{1}{x\sqrt{1+\ln x}}\,dx = \left| \begin{array}{lll} 1+\ln x = t & x_1 = 1 & \Longrightarrow t_1 = 1 \\ \dfrac{dx}{x} = dt & x_2 = \mathrm{e}^3 & \Longrightarrow t_2 = 4 \end{array} \right| =$

$$= \int_1^4 \frac{dt}{\sqrt{t}}\,dt = 2\Big[\sqrt{t}\Big]_1^4 = 2 \cdot (2-1) = 2.$$ ■

***Example* 30.** $\displaystyle\int_{1/\pi}^{2/\pi} \frac{1}{x^2} \cdot \sin\frac{1}{x}\,dx$

Solution : $\displaystyle\int_{1/\pi}^{2/\pi} \frac{1}{x^2} \cdot \sin\frac{1}{x}\,dx = \left| \begin{array}{lll} \dfrac{1}{x} = t & x_1 = \dfrac{1}{\pi} & \Longrightarrow t_1 = \pi \\ -\dfrac{dx}{x^2} = dt & x_2 = \dfrac{2}{\pi} & \Longrightarrow t_2 = \dfrac{\pi}{2} \end{array} \right| =$

$$= -\int_{\pi}^{\pi/2} \sin t\,dt = \int_{\pi/2}^{\pi} \sin t\,dt = \Big[-\cos t\Big]_{\pi/2}^{\pi} = 1\,.$$ ■

***Example* 31.** $\displaystyle\int_0^1 \frac{x}{1+x^4}\,dx$

Solution : $\displaystyle\int_0^1 \frac{x}{1+x^4}\,dx = \left|\begin{array}{ll} x^2=t & x_1=0 \implies t_1=0 \\ 2x\,dx=dt & x_2=1 \implies t_2=1 \end{array}\right| =$

$$= \frac{1}{2}\int_0^1 \frac{d\,t}{1+t^2} = \frac{1}{2}\Big[\arctan\, t\Big]_0^1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}.$$ ■

***Example* 32.** $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

Solution : $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx = \left|\begin{array}{ll} x=2\sin t & x_1=0 \implies t_1=0 \\ dx=2\cos t\,dt & x_2=2 \implies t_2=\pi/2 \end{array}\right| =$

$$= \int_0^{\pi/2} \sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt = 4\int_0^{\pi/2}\cos^2 t\,dt = 4\int_0^{\pi/2}\frac{1+\cos 2t}{2}\,dt =$$

$$= 2\Big[t+\frac{\sin 2t}{2}\Big]_0^{\pi/2} = 2\cdot\frac{\pi}{2} = \pi\,.$$ ■

***Example* 33.** $\displaystyle\int_1^2 \frac{x}{(x^2+4)^2}\,dx$

Solution : $\displaystyle\int_1^2 \frac{x}{(x^2+4)^2}\,dx = \left|\begin{array}{ll} x^2+4=t & x_1=1 \implies t_1=5 \\ 2x\,dx=dt & x_2=2 \implies t_2=8 \end{array}\right| =$

$$= \frac{1}{2}\int_5^8 \frac{dt}{t^2} = -\frac{1}{2}\cdot\Big[\frac{1}{t}\Big]_5^8 = -\frac{1}{2}\cdot\Big(\frac{1}{8}-\frac{1}{5}\Big) = \frac{3}{80}.$$ ■

34. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin^2 x+3}}\,dx$ $\Big[\ln\sqrt{3}\,\Big]$

35. $\displaystyle\int_{-2}^{-1} \frac{dx}{x^2+4x+5}$ $\left[\dfrac{\pi}{4}\right]$

36. $\displaystyle\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$ $\left[\dfrac{\pi^2}{72}\right]$

37. $\displaystyle\int_0^{\pi/4} \sin^5 x\cdot\cos x\,dx$ $\left[\dfrac{1}{48}\right]$

38. $\displaystyle\int_0^{\pi/3} \sin^3 x\,dx$ $\left[\dfrac{5}{24}\right]$

39. $\displaystyle\int_0^{1/2} r\sqrt{1-4r^2}\,dr$ $\left[\dfrac{1}{12}\right]$

40. $\displaystyle\int_0^4 \frac{1}{\sqrt{2x+1}+1}\,dx$ $\big[2(2-\ln 2)\big]$

41. $\displaystyle\int_0^{\pi/2} \frac{dx}{2+\cos x}$ $\left[\dfrac{\sqrt{3}\pi}{9}\right]$

CHAPTER II.

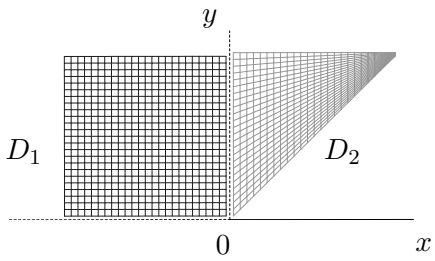
Differential calculus of functions of several variables

II.1 Domain of the function $z = f(x, y)$

• Determine the domains of the functions and sketch them:

***Example* 42**. $z = \dfrac{\ln(x^2 y)}{\sqrt{y - x}}$

Solution : The variables x and y must satisfy the following conditions simultaneously:



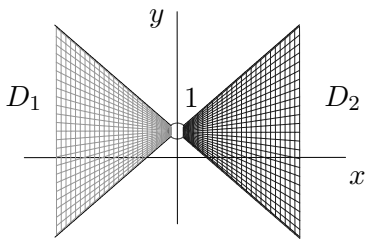
$$\begin{aligned} x^2 y > 0 \quad &\Longrightarrow \quad y > 0, \quad x \neq 0 \\ y - x > 0 \quad &\Longrightarrow \quad y > x \end{aligned}$$

$$D_1 = \big\{[x; y] \in \mathbb{E}_2 : x < 0, y > 0\big\},$$
$$D_2 = \big\{[x; y] \in \mathbb{E}_2 : x > 0, y > x\big\}.$$

For the domain of the given function we get $D = D_1 \cup D_2$ ■

***Example* 43**. $z = \arcsin \dfrac{y - 1}{x}$

Solution :



The variables x and y must satisfy the following conditions simultaneously:

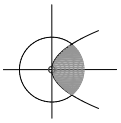
$$-1 \le \frac{y - 1}{x} \le 1 \quad \text{and} \quad x \neq 0 \ .$$

Thus: $\begin{aligned} x > 0: \quad & -x \le y - 1 \le x \quad \Longrightarrow \quad -x + 1 \le y \le x + 1 \\ x < 0: \quad & -x \ge y - 1 \ge x \quad \Longrightarrow \quad -x + 1 \ge y \ge x + 1 \end{aligned}$

$$D_1 = \big\{[x; y] \in \mathbb{E}_2 : x < 0, \quad x + 1 \le y \le 1 - x\big\},$$
$$D_2 = \big\{[x; y] \in \mathbb{E}_2 : x > 0, \quad 1 - x \le y \le x + 1\big\}.$$

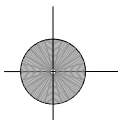
For the domain of the given function we get $D = D_1 \cup D_2$. ■

44. $z = \dfrac{\sqrt{x - y^2}}{\ln(1 - x^2 - y^2)}$



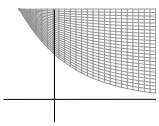
$\big[D = \{[x; y] \in \mathbb{E}_2 : 0 < x^2 + y^2 < 1, \ x \ge y^2\}\big]$

45. $z = \sqrt{\ln \dfrac{16}{x^2 + y^2}}$

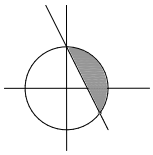


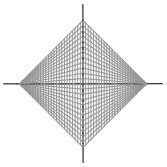
$\big[D = \{[x; y] \in \mathbb{E}_2 : 0 < x^2 + y^2 \le 16\}\big]$

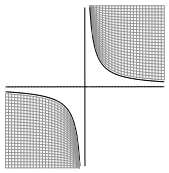
46. $z = 3 - 7 \ln(x + \ln y)$



$\big[D = \{[x; y] \in \mathbb{E}_2 : y > \mathrm{e}^{-x}\}\big]$

47. $z = \sqrt{2x + y - 4} + \sqrt{16 - x^2 - y^2}$  $\big[D = \{[x; y] \in \mathbb{E}_2 : x^2 + y^2 \leq 16;\ y \geq 4 - 2x\}\big]$

48. $z = \dfrac{3}{\sqrt{1 - |x| - |y|}}$  $\big[D = \{[x; y] \in \mathbb{E}_2 : |x| + |y| < 1\}\big]$

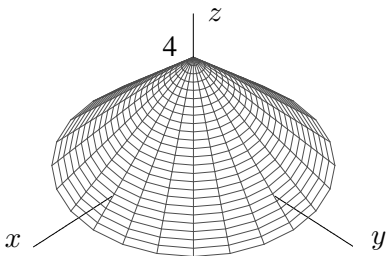
49. $z = \sqrt{xy - 4}$  $\big[D = \{[x; y] \in \mathbb{E}_2 : xy \geq 4\}\big]$

• Determine and sketch the domain, write the graph of the function, and sketch in $\mathbb{E}_3$ the surface that is the graph of the given function:

***Example* 50**. $f(x, y) = 4 - \sqrt{x^2 + y^2}$

Solution :

$x^2 + y^2 \geq 0 \quad \Rightarrow \quad D(f) = \big\{[x; y] \in \mathbb{E}_2; x^2 + y^2 \geq 0\big\} \quad \Rightarrow \quad D(f) = \mathbb{E}_2$



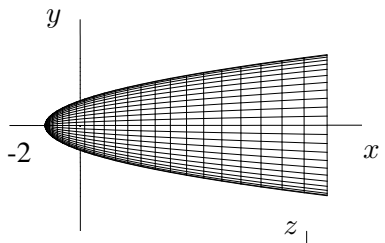
The graph of the function is the set

$G(f) = \Big\{[x; y; z] \in \mathbb{E}_3; [x; y] \in D(f),\ z = 4 - \sqrt{x^2 + y^2}\Big\}$.

■

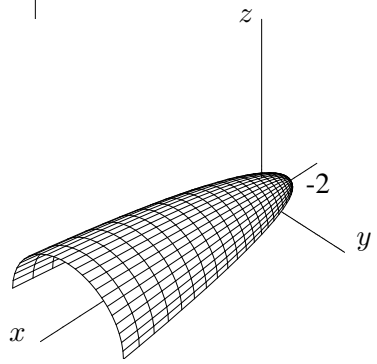
***Example* 51**. $f(x, y) = \sqrt{x - y^2 + 2}$

Solution :



$x - y^2 + 2 \geq 0 \quad \Rightarrow$
$D(f) = \big\{[x; y] \in \mathbb{E}_2; x - y^2 + 2 \geq 0\big\}$.

The boundary of $D(f)$ is the parabola $x + 2 = y^2$ with axis along the x-axis and vertex at $[-2; 0]$.

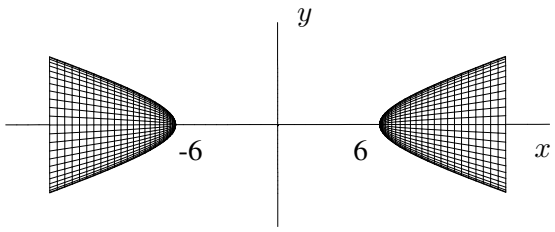


The graph of the function is the set
$G(f) = \big\{[x; y; z] \in \mathbb{E}_3; [x; y] \in D(f) \subset \mathbb{E}_2,\ z = \sqrt{x - y^2 + 2}\big\}$,
which is the "upper" half of the paraboloid of revolution with equation $x + 2 = y^2 + z^2$, whose axis of rotation is the x-axis, and vertex at $[-2; 0; 0]$.

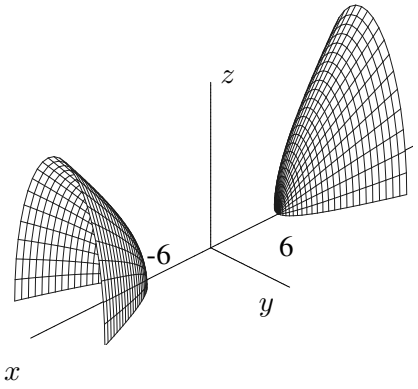
■

***Example* 52.** $f(x,y)=\sqrt{x^2-9y^2-36}$

Solution :



$x^2-9y^2-36\geq 0 \quad \Rightarrow$
$D(f)=\left\{[x;y]\in\mathbb{E}_2; x^2-9y^2-36\geq 0\right\}$,
the boundary is the hyperbola $\dfrac{x^2}{36}-\dfrac{y^2}{4}=1$
with vertices at $[-6;0]$ and $[6;0]$



The graph of the function is the set
$G(f)=\left\{[x;y;z]\in\mathbb{E}_3; [x;y]\in D(f)\subset\mathbb{E}_2,\ z=\sqrt{x^2-9y^2-36}\right\}$
and this is the "upper" half of an elliptic two-sheeted hyperboloid with vertices at $[-6;0;0]$ and $[6;0;0]$

■

II.2 Limit and continuity of a function

• Investigate the limits of the function at the point:

***Example* 53.** $\lim\limits_{[x;y]\to[0;0]}\dfrac{\sin(x^2+y^2)}{x^2+y^2}$

Solution : We use the fact that $\lim\limits_{[x;y]\to[0;0]}(x^2+y^2)=0$ and make the substitution $t=x^2+y^2$.

Then $\lim\limits_{[x;y]\to[0;0]}\dfrac{\sin(x^2+y^2)}{x^2+y^2}=\lim\limits_{t\to 0}\dfrac{\sin t}{t}=1$ ■

Example* 54. $\lim\limits_{\substack{[x;y]\to[1;1]\\ [x;y]\in M}}\dfrac{x^3-y^3}{x^4-y^4},\ [x;y]\in M,$ where $M=\left\{[x;y]\in\mathbb{E}_2\ ;\ x-y\neq 0\right\}$

Solution : This is the indeterminate form "$\frac{0}{0}$", but an elementary simplification suggests itself, which we perform

$$\lim_{\substack{[x;y]\to[1;1]\\ [x;y]\in M}}\frac{x^3-y^3}{x^4-y^4}=\lim_{\substack{[x;y]\to[1;1]\\ [x;y]\in M}}\frac{(x-y)(x^2+xy+y^2)}{(x-y)(x+y)(x^2+y^2)}=\lim_{\substack{[x;y]\to[1;1]\\ [x;y]\in M}}\frac{x^2+xy+y^2}{(x+y)(x^2+y^2)}=\frac{3}{4}.$$

■

55. $\lim\limits_{[x;y]\to[0;0]}\dfrac{e^{2(x^2+y^2)}-1}{x^2+y^2}$ [2] **56.** $\lim\limits_{[x;y]\to[0;0]}\dfrac{\tan(x^2+y^4)}{3(x^2+y^4)}$ $\left[\dfrac{1}{3}\right]$

***Example* 57**. Investigate the continuity of the function $f(x,y) = \begin{cases} \dfrac{x^2y^2}{\sqrt{x^2y^2+1}-1}, & [x;y] \neq [0;0] \\ 2, & [x;y] = [0;0] \end{cases}$ at the point $[0;0]$.

Solution : The function $f(x,y)$ is continuous at the point $[x_0;y_0]$ if and only if $\lim\limits_{[x;y]\to[x_0;y_0]} f(x,y) = f(x_0,y_0)$.

In our case $\lim\limits_{[x;y]\to[0;0]} \dfrac{x^2y^2}{\sqrt{x^2y^2+1}-1} = \text{"}\frac{0}{0}\text{"} = \left| \begin{array}{c} \text{substitution} \\ x^2y^2 = t \end{array} \right| = \lim\limits_{t\to 0} \dfrac{t}{\sqrt{t+1}-1} \overset{\ell'\text{H}}{=}$

$= \lim\limits_{t\to 0} \dfrac{1}{\frac{1}{2\sqrt{t+1}}} = 2 = f(0,0) \Longrightarrow$ the given function $f(x,y)$ is continuous at the point $[0;0]$. ■

• Determine the sets on which the given functions are defined:

58. $f(x,y) = \dfrac{x^2+x-12}{x^2+y^4+1}$ $[\mathbb{E}_2]$

59. $f(x,y,z) = \mathrm{e}^{z^2+x} \cdot \sin(x+y)$ $[\mathbb{E}_3]$

60. $f(x,y) = \dfrac{1}{x^2-2y}$ $\left[D(f) = \left\{[x,y] \in \mathbb{E}_2 : y \neq \dfrac{x^2}{2}\right\}\right]$

61. $f(x,y) = \dfrac{x^4-y^4}{x^2+y^2}$

$\left[D(f) = \mathbb{E}_2 \backslash [0;0]$, it can be continuously extended by defining $f(0,0) = \lim\limits_{[x;y]\to[0;0]} \dfrac{x^2y^2}{x^2+y^2} = 0\right]$

62. $f(x,y,z) = \dfrac{1}{\ln\sqrt{(x^2+y^2+z^2)}}$ $\left[D(f) = \{[x;y;z] \in \mathbb{E}_3 : x^2+y^2+z^2 \neq 1\} \backslash \{[0;0;0]\}\right]$

63. $f(x,y,z) = \dfrac{\sin\left(x^2+y^2+z^2\right)^2}{x^2+y^2+z^2}$ $\left[D(f) = \mathbb{E}_3 \backslash [0;0;0]$, it can be continuously extended by defining $f(0,0,0) = 0\right]$

64. $f(x,y,z) = \dfrac{1}{|xy|+|z|}$ $\left[D(f) = \{[x;y;z] \in \mathbb{E}_3 : xy \neq 0 \wedge z \neq 0\right]$

65. $f(x,y,z) = \dfrac{y+4}{x^2y-xy+4x^2-4x}$ $\left[D(f) = \{[x;y;z] \in \mathbb{E}_3 : x \neq 0, x \neq 1, y \neq -4\}\right]$

II.3 Partial derivatives of a function

Notation for the function $z = f(x, y)$:

first partial derivatives: $\dfrac{\partial f}{\partial x}(x, y) = f_x(x, y) = z_x = \dfrac{\partial z}{\partial x}$

$\dfrac{\partial f}{\partial y}(x, y) = f_y(x, y) = z_y = \dfrac{\partial z}{\partial y}$

first partial derivatives at the point $[a; b]$:

$$\left.\frac{\partial z}{\partial x}\right|_{[a;b]} = f_x(a, b); \quad \left.\frac{\partial z}{\partial y}\right|_{[a;b]} = f_y(a, b)$$

second partial derivatives: $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2} = f_{xx}; \quad \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2} = f_{yy}$

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y\,\partial x} = (f_x)_y; \quad \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x\,\partial y} = (f_y)_x$

• Find the first-order partial derivatives of the given functions with respect to their variables.

***Example* 66.** $f(x, y) = (2x - 3y)^4$

Solution : $\dfrac{\partial f}{\partial x} = 4(2x - 3y)^3 \cdot 2 \quad , \quad \dfrac{\partial f}{\partial y} = 4(2x - 3y)^3 \cdot (-3)$ ■

***Example* 67.** $f(x, y) = 5x^4y^2 + \dfrac{x}{y} + 2x^2 - 3y$

Solution : $\dfrac{\partial f}{\partial x} = 20x^3y^2 + \dfrac{1}{y} + 4x \quad , \quad \dfrac{\partial f}{\partial y} = 10x^4y - \dfrac{x}{y^2} - 3$ ■

***Example* 68.** $f(x, y) = y^{x^2+3}, \quad y > 0$

Solution : $\dfrac{\partial f}{\partial x} = y^{x^2+3} \cdot \ln y \cdot 2x \quad , \quad \dfrac{\partial f}{\partial y} = (x^2 + 3) \cdot y^{x^2+2}$ ■

***Example* 69.** $f(x, y) = \dfrac{y}{\sqrt{x^2 - y^2}}, \quad$ for $|x| > |y|$

Solution : $\dfrac{\partial f}{\partial x} = y \cdot \dfrac{-2x}{2(x^2 - y^2)^{3/2}} = \dfrac{-xy}{(x^2 - y^2)\sqrt{x^2 - y^2}}$

$$\frac{\partial f}{\partial y} = \frac{\sqrt{x^2 - y^2} - y \cdot \dfrac{-2y}{2\sqrt{x^2 - y^2}}}{x^2 - y^2} = \frac{x^2 - y^2 + y^2}{(x^2 - y^2)\sqrt{x^2 - y^2}} = \frac{x^2}{(x^2 - y^2)\sqrt{x^2 - y^2}}.$$

■

***Example* 70.** $f(x, y, z) = (x^y)^z$

Solution : $f(x, y, z) = x^{yz}, \quad \dfrac{\partial f}{\partial x} = yzx^{yz-1}, \quad \dfrac{\partial f}{\partial y} = x^{yz} \cdot \ln x \cdot z, \quad \dfrac{\partial f}{\partial z} = x^{yz} \cdot \ln x \cdot y\,.$

■

• Compute the partial derivatives of the given function with respect to all variables:

71. $f(x, y) = \ln(3x - y + 2)$ $\qquad \left[f_x = \dfrac{3}{3x - y + 2},\ f_y = \dfrac{-1}{3x - y + 2}\right]$

72. $f(x,y)=\dfrac{3x-2y}{y}$ $\left[f_x=\dfrac{3}{y},\ f_y=-\dfrac{3x}{y^2}\right]$

73. $f(x,y)=\ln\left(xy^2\right)$ $\left[f_x=\dfrac{1}{x},\ f_y=\dfrac{2}{y}\right]$

74. $f(x,y)=\cos x+\dfrac{1}{2}\sin y-3$ $\left[f_x=-\sin x,\ f_y=\dfrac{1}{2}\ \cos y\right]$

75. $f(x,y)=\dfrac{y}{\sqrt{x}}-x\ \sqrt{y}$ $\left[f_x=\dfrac{-y}{2\cdot\sqrt{x^3}}-\sqrt{y},\ f_y=\dfrac{1}{\sqrt{x}}-\dfrac{x}{2\cdot\sqrt{y}}\right]$

76. $f(x,y)=(x^2+y)\ \mathrm{e}^{-2x}$ $\left[f_x=2x\,\mathrm{e}^{-2x}-2(x^2+y)\,\mathrm{e}^{-2\mathrm{x}},\ f_y=\mathrm{e}^{-2x}\right]$

77. $f(x,y)=\dfrac{xy^2}{2}+\arctan\left(\dfrac{y}{x}\right)$ $\left[f_x=\dfrac{y^2}{2}-\dfrac{y}{x^2+y^2},\ f_y=xy+\dfrac{x}{x^2+y^2}\right]$

78. $f(x,y)=\ln(xy)-\sqrt{x^2+y^2-20}$ $\left[f_x=\dfrac{1}{x}-\dfrac{x}{\sqrt{x^2+y^2-20}},\ f_y=\dfrac{1}{y}-\dfrac{y}{\sqrt{x^2+y^2-20}}\right]$

79. $f(x,y)=x^3+\dfrac{y^3}{3}-\dfrac{1}{6}\,x^2y^4-15x$ $\left[f_x=3x^2-\dfrac{1}{3}\,xy^4-15,\ f_y=y^2-\dfrac{2}{3}\,x^2y^3\right]$

80. $f(x,y)=\dfrac{y^2}{x^2+y^2}$ $\left[f_x=\dfrac{-2xy^2}{(x^2+y^2)^2},\ f_y=\dfrac{2x^2y}{(x^2+y^2)^2}\right]$

81. $f(\varphi,\psi)=\sin\varphi\cdot\cos\psi$ $\left[f_\varphi=\cos\varphi\ \cos\psi,\ f_\psi=-\sin\varphi\ \sin\psi\right]$

82. $g(u,v)=v\cdot\tan\left(u^2v^3\right)$ $\left[g_u=\dfrac{2u\,v^4}{\cos^2(u^2v^3)},\ g_v=\tan\left(u^2\,v^3\right)+\dfrac{3u^2\,v^3}{\cos^2(u^2v^3)}\right]$

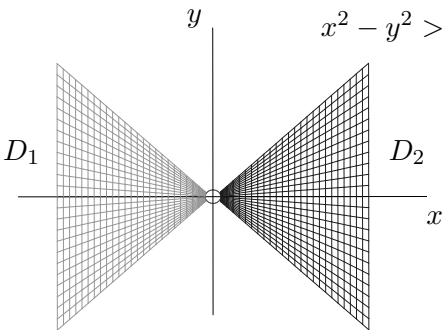
83. $f(t,u,v)=\ln(tu)-\mathrm{e}^{uv}+\cos(tv)$ $\left[f_t=\dfrac{1}{t}-v\ \sin(tv),\ f_u=\dfrac{1}{u}-v\,\mathrm{e}^{uv},\ f_v=-u\,\mathrm{e}^{uv}-t\ \sin(tv)\right]$

***Example* 84.** Determine the domain of differentiability of the function $f(x,y)=x\sqrt{x^2-y^2}$.

Solution : $\dfrac{\partial f}{\partial x}=\sqrt{x^2-y^2}+\dfrac{x^2}{\sqrt{x^2-y^2}},\qquad \dfrac{\partial f}{\partial y}=\dfrac{-xy}{\sqrt{x^2-y^2}}\ .$

From theory we know that continuity of the partial derivatives is sufficient for differentiability of functions. In this example this is the condition:

$x^2-y^2>0\implies |y|<|x|.$



$D_1=\{[x;y]\in\mathbb{E}_2: x<0, y\in(x;-x)\},$
$D_2=\{[x;y]\in\mathbb{E}_2: x>0, y\in(-x;x)\}\ .$

■

***Example* 85.** Given the function $f(x,y)=\ln(|x|+y)+\dfrac{1}{\sqrt{9-x^2-y^2}}$. Determine

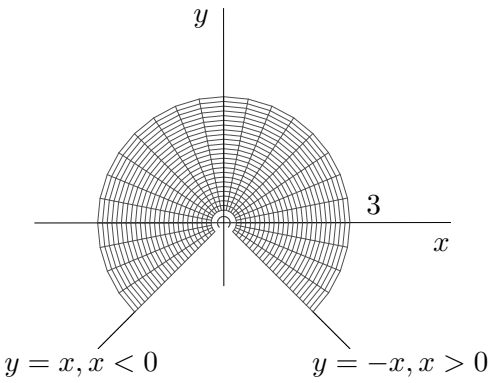
a) the conditions for the domain and illustrate the domain graphically,

b) the value $f(A)$, where $A=[-1;2]$, c) $\dfrac{\partial f}{\partial x}(A)$.

Solution :

a) conditions for the domain:

$$|x| + y > 0 \Longrightarrow y > -|x|$$
$$9 - x^2 - y^2 > 0 \Longrightarrow x^2 + y^2 < 9$$



b) $f(-1,2) = \ln(|-1|+2) + \dfrac{1}{\sqrt{9-1-4}} = \ln 3 + \dfrac{1}{2}$

c) $\dfrac{\partial f}{\partial x}(A) = \Big(\dfrac{(-x)'}{|x|+y} - \dfrac{1}{2}\cdot\dfrac{-2x}{\sqrt{(9-x^2-y^2)^3}}\Big)\Big|_A = -\dfrac{11}{24}.$ ■

86. Prove that the function $z = f(x,y) = y^2\sin(x^2-y^2)$ satisfies the differential equation $y^2 z_x + xyz_y = 2xz$ for all $[x;y] \in \mathbb{E}_2$.

[*Hint :* It is enough to compute z_x, z_y and substitute into the equation]

• Compute the partial derivatives of the given function at the point A :

87. $z = \sqrt{x^2-y^2}, \quad A = [2;0]$ $\qquad [z_x(A) = 1,\ z_y(A) = 0]$

88. $z = \dfrac{y}{x}, \quad A = [3;2]$ $\qquad [z_x(A) = -2/9,\ z_y(A) = 1/3]$

89. $f = x^2 e^y \sin z, \quad A = [1;0;\pi/6]$ $\qquad [f_x(A) = 1,\ f_y(A) = 1/2,\ f_z(A) = \sqrt{3}/2]$

90. $f = \ln(x^2 - y + 3z), \quad A = [2;1;1]$ $\qquad [f_x(A) = 2/3,\ f_y(A) = -1/6,\ f_z(A) = 1/2]$

II.4 Total differential and tangent plane

Notation for the function $z = f(x,y)$:
the (total) differential of the function f at the point $A = [x_0;y_0]$:

$$\mathrm{d}f[A] = \frac{\partial f}{\partial x}(A)\cdot(x-x_0) + \frac{\partial f}{\partial y}(A)\cdot(y-y_0)$$

Let $dx = x - x_0$, $dy = y - y_0$. Then

$$\mathrm{d}f[A] = \frac{\partial f}{\partial x}(A)\cdot dx + \frac{\partial f}{\partial y}(A)\cdot dy$$

***Example* 91**. Given the function $f(x,y) = \dfrac{y}{x} - \dfrac{x}{y}$.

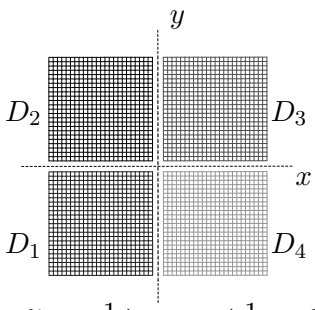
a) Determine and sketch the regions in which the function is differentiable.

b) Write the differential of the function at the point $A = [x_0;y_0]$.

Solution : A sufficient condition for differentiability is continuity of the partial derivatives $\Rightarrow$ continuity of the functions $\dfrac{\partial f}{\partial x} = -\dfrac{y}{x^2} - \dfrac{1}{y}$ $\dfrac{\partial f}{\partial y} = \dfrac{1}{x} + \dfrac{x}{y^2}$ $\Rightarrow$ $x \neq 0 \ \wedge\ y \neq 0$.

We obtain the following sets :

$$D_1 = \{[x;y] \in \mathbb{E}_2 : x < 0, y < 0\},$$
$$D_2 = \{[x;y] \in \mathbb{E}_2 : x < 0, y > 0\},$$
$$D_3 = \{[x;y] \in \mathbb{E}_2 : x > 0, y > 0\}.$$
$$D_4 = \{[x;y] \in \mathbb{E}_2 : x > 0, y < 0\},$$



$$\mathrm{d}f[A] = \frac{\partial f}{\partial x}(A)\,dx + \frac{\partial f}{\partial y}(A)\,dy \quad \text{i.e.} \quad \mathrm{d}f[A] = \left(-\frac{y_0}{x_0^2} - \frac{1}{y_0}\right)dx + \left(\frac{1}{x_0} + \frac{x_0}{y_0^2}\right)dy$$

■

***Example* 92**. Determine the total differential and the approximate increment of the function $z = \dfrac{y}{x}$ at the point $A = [2;1]$ for $\triangle x = 0.1$ and $\triangle y = 0.2$. Compare them.

Solution : The total differential at the point A is $\mathrm{d}z[A] = \dfrac{\partial z}{\partial x}(A)\,dx + \dfrac{\partial z}{\partial y}(A)\,dy.$

At the same time $\dfrac{\partial z}{\partial x} = -\dfrac{y}{x^2} \Rightarrow \dfrac{\partial z}{\partial x}(A) = -\dfrac{1}{4}$ $\qquad$ $\dfrac{\partial z}{\partial y} = \dfrac{1}{x} \Rightarrow \dfrac{\partial z}{\partial y}(A) = \dfrac{1}{2}$

$$\mathrm{d}z[A] = -\frac{1}{4}\,dx + \frac{1}{2}\,dy.$$

If we set $dx = \triangle x = 0.1$ and $dy = \triangle y = 0.2$, then we obtain the required differential of the function f at the point A for the given increments $\triangle x, \triangle y$:

$$\mathrm{d}z[A] = -\frac{1}{4}\cdot 0.1 + \frac{1}{2}\cdot 0.2 = 0.075,$$

while the exact increment is $\triangle z = z(x_0 + \triangle x, y_0 + \triangle y) - z(x_0, y_0) =$
$= z(2.1, 1.2) - z(2,1) = 0.071.$

■

***Example* 93**. Using the total differential, compute approximately the increment of the function $z = \arctan\dfrac{y}{x}$ when x changes from $x_0 = 1$ to $x_1 = 1.2$ and y from $y_0 = -3$ to $y_1 = -3.1$.

Solution : We approximately replace the increment by the differential, i.e.

$$\triangle z \doteq \mathrm{d}z[A] = \frac{\partial z}{\partial x}(A)\cdot dx + \frac{\partial z}{\partial y}(A)\cdot dy, \text{ where } A = [1;-3],\ dx = 0.2,\ dy = -0.1.$$

We compute

$$\frac{\partial z}{\partial x}(A) = \left(\frac{1}{1+(\frac{y}{x})^2}\cdot\left(-\frac{y}{x^2}\right)\right)\Bigg|_A = \frac{3}{10}\,, \quad \frac{\partial z}{\partial y}(A) = \left(\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x}\right)\Bigg|_A = \frac{1}{10}.$$

Then $\mathrm{d}z[A] = \dfrac{3}{10}\cdot 0.2 + \dfrac{1}{10}\cdot(-0.1) = 0.06 - 0.01 = 0.05.$ ■

***Example* 94**. Compute approximately the value of the expression $\ln(\sqrt{9.03} - \sqrt{0.99} - 1)$ using the total differential of a suitably chosen function.

Solution : Let $z(x,y) = \ln(\sqrt{x} - \sqrt{y} - 1)$, $x_0 = 9$, $y_0 = 1$, then $dx = 0.03$, $dy = -0.01$. We use the relation

$$\triangle z = z(x + x_0, y + y_0) - z(x_0, y_0) \doteq \frac{\partial z}{\partial x}(x_0, y_0)\cdot dx + \frac{\partial z}{\partial y}(x_0, y_0)\cdot dy,$$

from which we get

$$z(x + x_0, y + y_0) \doteq z(x_0, y_0) + \frac{\partial z}{\partial x}(x_0, y_0)\cdot dx + \frac{\partial z}{\partial y}(x_0, y_0)\cdot dy.$$

Let us prepare :
$z(x_0,y_0)=\ln(\sqrt{9}-\sqrt{1}-1)=0,\quad \dfrac{\partial z}{\partial x}(A)=\left(\dfrac{1}{\sqrt{x}-\sqrt{y}-1}\cdot\dfrac{1}{2\sqrt{x}}\right)\Big|_{[9;1]}=\dfrac{1}{6},$
$\dfrac{\partial z}{\partial y}(A)=\left(\dfrac{1}{\sqrt{x}-\sqrt{y}-1}\cdot\dfrac{-1}{2\sqrt{y}}\right)\Big|_{[9;1]}=-\dfrac{1}{2}$. Now we substitute and compute the required value $\quad \ln(\sqrt{9.03}-\sqrt{0.99}-1)\doteq\dfrac{1}{6}\cdot 0.03-\dfrac{1}{2}\cdot(-0.01)=0.01\,.$ ■

***Example* 95**. Compute the approximate value of the expression $0,98^{3,04}$ using the total differential of a suitably chosen function.

Solution : $z(x,y)=x^y,\ x_0=1,\ y_0=3,$ i.e. $A=[1;3],\ dx=-0.02,\ dy=0.04,$
$\dfrac{\partial z}{\partial x}=y\cdot x^{y-1},\quad \dfrac{\partial z}{\partial y}=x^y\ln x,\quad$ thus

$$z(0.98,3.04)\doteq z(A)+\mathrm{d}z[A]=z(A)+\frac{\partial z}{\partial x}(A)\cdot dx+\frac{\partial z}{\partial y}(A)\cdot dy=$$
$$=1+3\cdot(-0.02)+0=0.94.$$ ■

***Example* 96**. Find the equation of the tangent plane τ and the normal n to the graph of the function $z=2x^2-4y^2$ at the point $T=[2;1;?]$.
Compute approximately the value of the function at the point $[2.2;1.3]$.

Solution : The tangent plane τ has equation

$$\tau:\quad z-z_0=\frac{\partial z}{\partial x}(A)\cdot(x-x_0)+\frac{\partial z}{\partial y}(A)\cdot(y-y_0),$$

where $A=[x_0;y_0]=[2;1],$ and $z_0=z(x_0,y_0)$. In this case $z_0=2\cdot 4-4\cdot 1=4\quad\Longrightarrow$
$T=[2;1;4],\quad \dfrac{\partial z}{\partial x}(A)=4x\Big|_A=8,\quad \dfrac{\partial z}{\partial y}(A)=-8y\Big|_A=-8\quad\Longrightarrow$

$$\tau:\quad z-4=8(x-2)-8(y-1)\Longrightarrow 8x-8y-z-4=0.$$

The normal n is the line passing through the point T, whose direction vector is the normal vector of the plane τ. Thus $\quad n:\quad [x;y;z]=[2;1;4]+t(8;-8;-1),\ t\in\mathbb{R}.$

We compute the function value at the point $[2.2;1.3]$ by substituting the coordinates of the point into the equation of the tangent plane $z=4+8(x-2)-8(y-1)\quad\Rightarrow$
$$z(2.2,1.3)=4+8(2.2-2)-8(1.3-1)=4+1.6-2.4=3.2$$ ■

***Example* 97**. Find the equation of the tangent plane τ to the surface $z=x^2+xy-y^2+x+3$ parallel to the given plane $\varrho: 5x-3y-z=0$.

Solution : We must find the point A at which $\dfrac{\partial z}{\partial x}=5\ ,\ \dfrac{\partial z}{\partial y}=-3$. From this we obtain the system of equations $\begin{cases}2x+y+1=5,\\ x-2y=-3.\end{cases}$ Hence $x_0=1, y_0=2, z_0=z(x_0,y_0)=3$, so
$T=[1;2;3]$. The equation of the tangent plane $\tau:\quad 5x-3y-z+d=0,\quad T\in\tau$
$\Longrightarrow\quad \tau: 5x-3y-z+4=0.$ ■

• Compute approximate values of the given expressions using the total differential :

98. $\sqrt[3]{7.95}\cdot\sqrt{8.96}$ [5.9742] **99**. $\dfrac{\sqrt[4]{0.97}}{1.02^3\cdot\sqrt[3]{0.99}}$ [0.936]

100. $\sqrt{4.04}\cdot\ln 1.02\cdot\arctan 0.9$ [0.0314]

• Find the equation of the tangent plane τ and the equation of the normal n to the surface $z=f(x,y)$ at the point T :

101. $z=4\sqrt{x^2+y^2},\quad T=[3;4;?]$ $\quad\big[12x+16y-5z=0;\ [x;y;z]=[3;4;20]+t(12;16;-5),\ t\in\mathbb{R}\big]$

102. $z=xy,\quad T=[0;0;?]$ $\quad\big[z=0;\ [x;y;z]=t(0;0;1),\ t\in\mathbb{R}\big]$

103. $z=x^2\cdot\cos\dfrac{1}{y},\quad T=[1;\dfrac{2}{\pi};?]$ $\quad\Big[z=\dfrac{\pi^2}{4}\Big(y-\dfrac{2}{\pi}\Big);\ [x;y;z]=\Big[1;\dfrac{2}{\pi};0\Big]+t\Big(0;\dfrac{\pi^2}{4};-1\Big),\ t\in\mathbb{R}\Big]$

104. $z=\dfrac{1}{x}\arcsin y,\quad T=\Big[\dfrac{1}{2};\dfrac{\sqrt{2}}{2};?\Big]$ $\quad\left[\begin{array}{r}\pi\,x-2\sqrt{2}\,y+z-\pi+2=0;\\ [x;y;z]=\Big[\dfrac{1}{2};\dfrac{\sqrt{2}}{2};\dfrac{\pi}{2}\Big]+t(\pi;-2\sqrt{2};1),\ t\in\mathbb{R}\end{array}\right]$

• Find the equation of the tangent plane τ to the surface $z=z(x,y)$ parallel to the plane ϱ :

105. $z=2x^2-y^2,\quad \varrho: 8x-6y-z-15=0$ $\quad\big[\tau: 8x-6y-z+1=0\big]$

106. $z=\ln(x^2+2y^2),\quad \varrho: 2x-z+5=0$ $\quad\big[\tau: 2x-z-2=0\big]$

107. $z=x^2-y^2+6xy+2x,\quad \varrho: 4x+6y-z=0$ $\quad\big[\tau: 4x+6y-z-1=0\big]$

II.5 Derivatives and differentials of higher orders

***Example* 108**. Find all second-order partial derivatives of the function

$$f(x,y)=xy^3-y\cdot\mathrm{e}^{x+y^2}.$$

Solution : $\dfrac{\partial f}{\partial x}=y^3-y\cdot\mathrm{e}^{x+y^2},$

$\dfrac{\partial f}{\partial y}=3xy^2-\mathrm{e}^{x+y^2}-y\cdot\mathrm{e}^{x+y^2}\cdot 2y=3xy^2-\mathrm{e}^{x+y^2}(1+2y^2),$

$\dfrac{\partial^2 f}{\partial x^2}=\dfrac{\partial(\frac{\partial f}{\partial x})}{\partial x}=-y\cdot\mathrm{e}^{x+y^2},$

$\dfrac{\partial^2 f}{\partial y^2}=\dfrac{\partial(\frac{\partial f}{\partial y})}{\partial y}=6xy-2y\mathrm{e}^{x+y^2}(1+2y^2)-e^{x+y^2}4y=6xy-e^{x+y^2}(6y+4y^3),$

$\dfrac{\partial^2 f}{\partial y\,\partial x}=\dfrac{\partial(\frac{\partial f}{\partial x})}{\partial y}=3y^2-e^{x+y^2}-ye^{x+y^2}\cdot 2y=3y^2-e^{x+y^2}(1+2y^2),$

$\dfrac{\partial^2 f}{\partial x\,\partial y}=\dfrac{\partial(\frac{\partial f}{\partial y})}{\partial x}=3y^2-e^{x+y^2}(1+2y^2).$

We see that for the given function f it holds that $\dfrac{\partial^2 f}{\partial x\,\partial y}=\dfrac{\partial^2 f}{\partial y\,\partial x}$ at all points $[x;y]\in\mathbb{E}_2$.

■

***Example* 109**. Show that the function $u=u(x,t)=\arctan(2x-t)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial t\,\partial x}=0\ \text{ in }\ \mathbb{E}_2.$$

Solution : $\dfrac{\partial u}{\partial x}=\dfrac{2}{1+(2x-t)^2},\qquad \dfrac{\partial^2 u}{\partial x^2}=\dfrac{-2\cdot 2(2x-t)\cdot 2}{\big(1+(2x-t)^2\big)^2}=\dfrac{-8(2x-t)}{\big(1+(2x-t)^2\big)^2},$

$$\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial(\frac{\partial u}{\partial x})}{\partial t} = \frac{-2}{\big(1+(2x-t)^2\big)^2} \cdot 2(2x-t) \cdot (-1) = \frac{4(2x-t)}{\big(1+(2x-t)^2\big)^2}.$$

After substitution it is clear that the equation holds at all points $[x;t] \in \mathbb{E}_2$.

■

***Example* 110**.* Given the function $f(x,y)$

$$f(x,y) = \begin{cases} xy\dfrac{x^2-2y^2}{x^2+y^2} & \text{for } [x;y] \neq [0;0] \\ 0 & \text{for } [x;y] = [0;0]. \end{cases}$$

Prove that $\dfrac{\partial^2 f}{\partial x\, \partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y\, \partial x}(0,0).$

Solution : It is easy to verify that the function f is continuous at the point $[0;0]$:

$\lim\limits_{[x;y]\to[0;0]} f(x,y) = 0 = f(0,0)$. The derivatives $f_x(0,y)$, $f_y(x,0)$, $f_{xy}(0,0)$, $f_{yx}(0,0)$ are computed using the corresponding definitions :

$$\frac{\partial f}{\partial x}(0,y) = \lim_{h\to 0} \frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0} \frac{hy\frac{h^2-2y^2}{h^2+y^2}-0}{h} = -2y,$$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{k\to 0} \frac{f(x,k)-f(x,0)}{k} = \lim_{k\to 0} \frac{xk\frac{x^2-2k^2}{x^2+k^2}-0}{k} = x,$$

$$\frac{\partial^2 f}{\partial y\, \partial x}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0} \frac{-2k}{k} = -2,$$

$$\frac{\partial^2 f}{\partial x\, \partial y}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1,$$

$$\frac{\partial^2 f}{\partial x\, \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y\, \partial x}(0,0).$$

■

Let the scalar field $\varphi(x,y,z)$ and the vector field $\vec{f} = (U;V;W)$ be given, both having continuous second-order partial derivatives in the region $D \subset \mathbb{E}_3$.

Denote:

$$\nabla = \Big(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\Big), \quad \operatorname{grad}\varphi = \nabla\varphi = \Big(\frac{\partial\varphi}{\partial x}; \frac{\partial\varphi}{\partial y}; \frac{\partial\varphi}{\partial z}\Big),$$

$$\operatorname{div}\vec{f} = \nabla\cdot\vec{f} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}, \quad \operatorname{curl}\vec{f} = \nabla\times\vec{f} = \Big(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}; \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x}; \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\Big)$$

***Example* 111**. Given the scalar field $\varphi(x,y,z) = xy^2 + z^3 - xyz + 3$.
Compute $\operatorname{grad}\varphi$, $\operatorname{curl}\operatorname{grad}\varphi$.

Solution : The function $\varphi(x,y,z) = xy^2 + z^3 - xyz + 3$ is differentiable in the region $\mathbb{E}_3$. Then

$$\operatorname{grad}\varphi = \Big(\frac{\partial\varphi}{\partial x}; \frac{\partial\varphi}{\partial y}; \frac{\partial\varphi}{\partial z}\Big) = (y^2 - yz;\ 2xy - xz;\ 3z^2 - xy)$$

$\operatorname{curl}\operatorname{grad}\varphi =$

$$= \Big(\frac{\partial(3z^2-xy)}{\partial y} - \frac{\partial(2xy-xz)}{\partial z}; \frac{\partial(y^2-yz)}{\partial z} - \frac{\partial(3z^2-xy)}{\partial x}; \frac{\partial(2xy-xz)}{\partial x} - \frac{\partial(y^2-yz)}{\partial y}\Big) =$$

$$= \big(-x-(-x);\ -y+y;\ 2y-z-(2y-z)\big) = \vec{0}$$

NOTE: The second-order partial derivatives of the function φ are continuous in $\mathbb{E}_3$. ■

***Example* 112.** Given the vector field $\vec{f} = (U; V; W) = (xy; x^2 - z^2; \frac{y}{x+z})$.
Compute $\operatorname{div} \vec{f}$, $\operatorname{curl} \vec{f}$, $\operatorname{div}\operatorname{curl} \vec{f}$.

Solution : The vector field $\vec{f} = (U; V; W) = (xy; x^2 - z^2; \frac{y}{x+z})$ is defined on the set $D(f) = \{[x; y; z] \in \mathbb{E}_3 : x + z \neq 0\}$.

Partial derivatives:

$$\frac{\partial U}{\partial x} = y, \quad \frac{\partial V}{\partial x} = 2x, \quad \frac{\partial W}{\partial x} = -\frac{y}{(x+z)^2}$$
$$\frac{\partial U}{\partial y} = x, \quad \frac{\partial V}{\partial y} = 0, \quad \frac{\partial W}{\partial y} = \frac{1}{x+z}$$
$$\frac{\partial U}{\partial z} = 0, \quad \frac{\partial V}{\partial z} = -2z, \quad \frac{\partial W}{\partial z} = -\frac{y}{(x+z)^2}$$

$$\operatorname{div} \vec{f} = y + 0 - \frac{y}{(x+z)^2}, \quad \operatorname{curl} \vec{f} = \left(\frac{1}{x+z} + 2z;\ \frac{y}{(x+z)^2} - 0;\ 2x - x\right),$$

$$\operatorname{div}\operatorname{curl} \vec{f} = \operatorname{div}\left(\frac{1}{x+z} + 2z;\ \frac{y}{(x+z)^2};\ x\right) = -\frac{1}{(x+z)^2} + \frac{1}{(x+z)^2} + 0 = 0$$

NOTE: The coordinate functions U, V, W have continuous second-order partial derivatives in D. ■

113.* Let the scalar field $\varphi(x, y, z)$ in the region $D \subset \mathbb{E}_3$ have continuous partial derivatives of order 2. Prove that $\operatorname{curl}\operatorname{grad} \varphi = \vec{0}$ in D.

114.* Let the vector field $\vec{f}(U, V, W)$ in the region $D \subset \mathbb{E}_3$ have continuous partial derivatives of order 2. Prove that $\operatorname{div}\operatorname{curl} \vec{f} = 0$ in D.

• Find the differentials of the stated order :

***Example* 115**.* $z = \sin(2x + y)$, $\quad \mathrm{d}^2 z = ?$

Solution : The differential of order n is $\mathrm{d}^n f = \left(\frac{\partial}{\partial x} \cdot dx + \frac{\partial}{\partial y} \cdot dy\right)^n f$, then

$$\mathrm{d}^2 z = \left(dx\,\frac{\partial}{\partial x} + dy\,\frac{\partial}{\partial y}\right)^2 z = \frac{\partial^2 z}{\partial x^2}\,(dx)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\,dx\,dy + \frac{\partial^2 z}{\partial y^2}\,(dy)^2 =$$
$$= -4\sin(2x+y)\,(dx)^2 - 4\sin(2x+y)\,dx\,dy - \sin(2x+y)\,(dy)^2 =$$
$$= -\sin(2x+y)\,(2dx + dy)^2.$$ ■

***Example* 116**.* $z = x^3 - y^3 - xy + y^2$, $\quad \mathrm{d}^3 z = ?$

Solution :
$$\mathrm{d}^3 z = \frac{\partial^3 z}{\partial x^3}\,(dx)^3 + 3\frac{\partial^3 z}{\partial x^2\,\partial y}\,(dx)^2 dy + 3\frac{\partial^3 z}{\partial x \partial y^2}\,dx\,(dy)^2 + \frac{\partial^3 z}{\partial y^3}\,(dy)^3$$
$$z_x = 3x^2 - y, \qquad z_y = -3y^2 - x + 2y,$$
$$z_{xx} = 6x, \qquad z_{yy} = -6y + 2, \qquad z_{xy} = -1,$$
$$z_{xxx} = 6, \qquad z_{yyy} = -6, \qquad z_{xxy} = z_{xyy} = 0,$$
$$\mathrm{d}^3 z = 6(dx)^3 - 6(dy)^3.$$ ■

***Example* 117**.* $u = \mathrm{e}^{2x-3y}$, $\quad \mathrm{d}^2 u[A] = ?$, $\mathrm{d}^3 u[A] = ?$, $\mathrm{d}^n u[A] = ?$, $A = [0; 0]$

Solution : $\mathrm{d}^2 u[A] = \left(\mathrm{e}^{2x-3y}(2dx - 3dy)^2\right)\Big|_A = (2dx - 3dy)^2,$
$\mathrm{d}^3 u[A] = \left(\mathrm{e}^{2x-3y}(2dx - 3dy)^3\right)\Big|_A = (2dx - 3dy)^3,$

$$\mathrm{d}^n u[A] = \left(\mathrm{e}^{2x-3y}(2dx-3dy)^n\right)\Big|_A = (2dx-3dy)^n. \qquad \blacksquare$$

Differentials can be used in the important **Taylor theorem** : *Let $f(x,y)$ be a function $(n+1)$ times differentiable at every interior point of the rectangle* M *centered at the point $A=[x_0;y_0]$.*
Then for every point $[x;y]\in \mathrm{M}$ there exists a point $[\xi;\eta]\in \mathrm{M}$ such that

$$f(x,y) = f(A) + \mathrm{d}f[A] + \frac{d^2 f(A)}{2!} + \cdots + \frac{d^n f(A)}{n!} + R_{n+1},$$

where $\mathrm{d}f[A] = df(x_0,y_0) = \dfrac{\partial f}{\partial x}(A)\cdot(x-x_0) + \dfrac{\partial f}{\partial y}(A)\cdot(y-y_0),$

$$\mathrm{d}^2 f(A) = \frac{\partial^2 f}{\partial x^2}(A)\cdot(x-x_0)^2 + 2\cdot\frac{\partial^2 f}{\partial x\,\partial y}(A)\cdot(x-x_0)\cdot(y-y_0) + \frac{\partial^2 f}{\partial y^2}(A)\cdot(y-y_0)^2,$$

$$\vdots$$

$$\mathrm{d}^n f(A) = \sum_{k=0}^{n}\binom{n}{k}\frac{\partial^n f}{\partial x^k\,\partial y^{n-k}}(A)\cdot(x-x_0)^k\cdot(y-y_0)^{n-k},$$

$$R_{n+1} = \frac{1}{(n+1)!}\,d^{n+1}f(\xi,\eta).$$

Example* 118. Write the Taylor expansion of the function $f(x,y) = x^3 - 3xy^2 + y^2 + 4x - 5y$ in a neighborhood of the point $A=[2;-1]$ and use the result to compute an approximate value of the function f at the point $[2.1;-1.1]$.

Solution : $f(A)=16,\quad dx = x-x_0 = x-2,\quad dy = y-y_0 = y+1,$

$$\left.\begin{aligned} f_x(A) &= (3x^2-3y^2+4)\Big|_A = 13\\ f_y(A) &= (-6xy+2y-5)\Big|_A = 5\end{aligned}\right\} \Longrightarrow \mathrm{d}z(A) = 13\,dx + 5\,dy = 13(x-2)+5(y+1),$$

$$\left.\begin{aligned} f_{xx}(A) &= (6x)\Big|_A = 12\\ f_{yy}(A) &= (-6x+2)\Big|_A = -10\\ f_{xy}(A) &= (-6y)\Big|_A = 6\end{aligned}\right\} \Longrightarrow \begin{aligned}&\mathrm{d}^2 z(A) = 12\,(dx)^2 + 12\,dx\,dy - 10\,(dy)^2 =\\ &= 12(x-2)^2 + 12(x-2)(y+1) - 10(y+1)^2\,,\end{aligned}$$

$$\left.\begin{aligned} f_{xxx}(A) &= 6\\ f_{yyx}(A) &= -6\\ f_{yxx}(A) &= f_{yyy}(A) = 0\end{aligned}\right\} \Longrightarrow \mathrm{d}^3 z(A) = 6\,(dx)^3 - 18\,dx\,(dy)^2 = 6(x-2)^3 - 18(x-2)(y+1)^2,$$

$$f(x,y) = 16 + 13(x-2) + 5(y+1) + 6(x-2)^2 + 6(x-2)(y+1) - 5(y+1)^2 + (x-2)^3 - 3(x-2)(y+1)^2,$$

$R_4 = 0$, because derivatives of order 4 and higher are zero

$$\begin{aligned} f(2.1,-1.1) &= 16 + 13\cdot 0.1 + 5\,(-0.1) + 6\cdot 0.1^2 - 6\cdot 0.1^2 - 5\cdot 0.1^2 + 0.1^3 - 3\cdot 0.1^3 =\\ &= 17.3 - 0.552 = 16.748.\end{aligned} \qquad \blacksquare$$

Example* 119. Write the fourth-degree Taylor expansion of the function $f(x,y) = \cos(x^2+y^2)$ in a neighborhood of the point $[0;0]$.

Solution : We use the Taylor formula for $\cos z \doteq 1 - \dfrac{z^2}{2} + \dfrac{z^4}{4!}$, into which we substitute

$z = x^2+y^2$: $\qquad \cos(x^2+y^2) \doteq 1 - \dfrac{(x^2+y^2)^2}{2!} + \dfrac{(x^2+y^2)^4}{4!};$

$$T_4(x,y) = 1 - \frac{1}{2}(x^4 + 2x^2y^2 + y^4).$$ ■

• Find the second-order partial derivatives of the given function:

120. $\phi(s,t) = \ln(s^3 + t)$ $\left[\phi_{ss} = \dfrac{3s(2t - s^3)}{(s^3+t)^2},\ \phi_{tt} = \dfrac{-1}{(s^3+t)^2},\ \phi_{st} = \phi_{ts} = \dfrac{-3s^2}{(s^3+t)^2}\right]$

121. $\phi(x,t) = \dfrac{\cos x^2}{t}$ $\left[\phi_{xx} = \dfrac{-1}{t}(4x^2\cos x^2 + 2\sin x^2),\ \phi_{tt} = \dfrac{2}{t^3}\cos x^2,\ \phi_{xt} = \phi_{tx} = \dfrac{2x}{t^2}\sin x^2\right]$

122. $f(x,y) = \mathrm{e}^{ax+by}$ $\left[f_{xx} = a^2\mathrm{e}^{ax+by},\ f_{yy} = b^2\mathrm{e}^{ax+by},\ f_{xy} = f_{yx} = ab\,\mathrm{e}^{ax+by}\right]$

123. Verify that the function $u(x,t) = \sin(x - ct)$ and the function $u(x,t) = \sin(\omega ct)\cdot\sin(\omega x)$ satisfy the differential equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\dfrac{\partial^2 u}{\partial x^2}$ (the so-called wave equation).

124. Verify that the function $u(x,y) = \mathrm{e}^x\,\sin y$ satisfies the differential equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (Laplace's equation).

• Expand the function $f(x,y)$ according to Taylor's theorem in a neighborhood of the point A:

125.* $f(x,y) = x^3 + 5x^2 - 6xy + 2y^2,\ A = [1;-2]$

$$\left[\begin{array}{r} f(x,y) = 26 + 25(x-1) - 14(y+2) + 8(x-1)^2 \\ +2(y+2)^2 - 6(x-1)(y+2) + (x-1)^3 \end{array}\right]$$

126.* $f(x,y) = x^2 + 3xy - y^3 + 1,\ A = [2;-1]$

$$\left[\begin{array}{r} f(x,y) = (x-2) + 3(y+1) + (x-2)^2 + \\ +3(y+1)^2 + 3(x-2)(y+1) - (y+1)^3 \end{array}\right]$$

II.6 Gradient. Directional derivative

• Determine the angle between the gradients of the given functions at the point A :

***Example* 127**. $f(x,y,z) = x^y + yz,\ g(x,y,z) = \sin(xz) + x + y - \dfrac{z}{y} - 1,\ A = [1;1;0]$

Solution : $\operatorname{grad} f = \left(\dfrac{\partial f}{\partial x};\dfrac{\partial f}{\partial y};\dfrac{\partial f}{\partial z}\right) = (yx^{y-1};x^y\ln x + z;y) \Longrightarrow \operatorname{grad} f(A) = (1;0;1)$

$$\operatorname{grad} g = \left(z\cos(xz) + 1;1 + \frac{z}{y^2};x\cos(xz) - \frac{1}{y}\right) \Longrightarrow \operatorname{grad} g(A) = (1;1;0)$$

Let $\varphi = \sphericalangle(\operatorname{grad} f(A), \operatorname{grad} g(A))$. Then

$$\cos\varphi = \frac{\operatorname{grad} f(A)\cdot\operatorname{grad} g(A)}{\|\operatorname{grad} f(A)\|\cdot\|\operatorname{grad} g(A)\|} = \frac{(1;0;1)\cdot(1;1;0)}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2} \quad\Longrightarrow\quad \varphi = \frac{\pi}{3}.$$ ■

***Example* 128**. $f(x,y) = \arctan\dfrac{x}{y},\ g(x,y) = y\sqrt{x},\ A = [1;1]$

Solution : $\operatorname{grad} f(A) = \nabla f(A) = \left(\dfrac{1}{2};-\dfrac{1}{2}\right),\qquad \operatorname{grad} g(A) = \left(\dfrac{1}{2};1\right),$

$$\cos\varphi = \frac{\operatorname{grad} f(A)\cdot\operatorname{grad} g(A)}{\|\operatorname{grad} f(A)\|\|\operatorname{grad} g(A)\|} = \frac{\left(\frac{1}{2};-\frac{1}{2}\right)\cdot\left(\frac{1}{2};1\right)}{\sqrt{\frac{1}{2}}\sqrt{\frac{5}{4}}} = -\frac{\sqrt{2}}{2\sqrt{5}}. \quad\Longrightarrow$$

$$\varphi = \arccos\left(-\frac{\sqrt{10}}{10}\right).$$ ■

***Example* 129**. Determine at which points of $D(f) \equiv \mathbb{E}_3$ the gradient of the function $f(x,y,z) = x^2+y^2+z^2-2xyz$ is equal to the zero vector.

Solution : $\operatorname{grad} f = (2x-2yz; 2y-2xz; 2z-2xy) = (0;0;0) \implies \begin{array}{l} x-yz=0 \\ y-xz=0 \\ z-xy=0 \end{array}$

It is clear that one of the points is the point $A_1 = [0;0;0]$. We compute the others from the system :

$$\left.\begin{array}{c} x = yz \\ y - yz^2 = 0 \Longrightarrow z = \pm 1 \\ z - y^2 z = 0 \Longrightarrow y = \pm 1 \end{array}\right\} \Longrightarrow \begin{array}{ll} A_2 = [1;1;1], & A_3 = [-1;1;-1], \\ A_4 = [-1;-1;1], & A_5 = [1;-1;-1]. \end{array}$$

■

***Example* 130**. Determine at which points of $D(f) \equiv \mathbb{E}_2$ the gradient of the function $f(x,y) = (x^2+y^2)^{3/2}$ has magnitude 9.

Solution :

$$\|\operatorname{grad} f(x,y)\| = \|(3x\sqrt{x^2+y^2}, 3y\sqrt{x^2+y^2})\| = \sqrt{9x^2(x^2+y^2)+9y^2(x^2+y^2)} = 9,$$

$$9x^2(x^2+y^2)+9y^2(x^2+y^2) = 81,$$

$$(x^2+y^2)^2 = 9 \Longrightarrow x^2+y^2 = 3.$$

Therefore the required points lie on the circle with center $[0;0]$ and radius $\sqrt{3}$. ■

• Compute the directional derivative of the function f in the direction $\vec{s}$ at the point A :

***Example* 131**. $f(x,y) = 2x^4 + xy + y^3, \quad \vec{s} = (3;-4), \quad A = [1;2]$

Solution :

$$\frac{\partial f}{\partial \vec{s}}(A) = \operatorname{grad} f(A) \cdot \frac{\vec{s}}{\|\vec{s}\|}, \quad \operatorname{grad} f = (8x^3+y, x+3y^2),$$

$$\operatorname{grad} f(A) = (10;13), \quad \frac{\partial f}{\partial \vec{s}}(A) = (10;13)\cdot\frac{(3;-4)}{\sqrt{9+16}} = \frac{30-52}{5} = -\frac{22}{5}.$$ ■

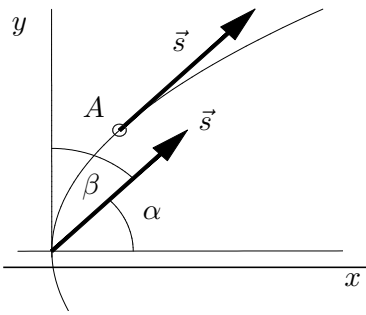
***Example* 132**. $f(x,y,z) = x^2+2y^2-z^2, \quad A = [-3;2;4]$; the direction $\vec{s}$ is the vector $\overrightarrow{AB}$, where $B = [-2;4;2]$.

Solution : $\vec{s} = \overrightarrow{AB} = (1;2;-2), \quad \dfrac{\partial f}{\partial \vec{s}}(A) = (2x;4y;-2z)\Big|_A \cdot \dfrac{\vec{s}}{\|\vec{s}\|} = (-6;8;-8)\cdot\dfrac{(1;2;-2)}{\sqrt{1+4+4}} =$

$$= \frac{1}{3}(-6+16+16) = \frac{26}{3}.$$ ■

***Example* 133**. Determine the derivative of the function $z = x^2 + \ln(x+y^2)$ at the point $A = [3; 2\sqrt{3}]$ in the direction of the tangent to the parabola $y^2 = 4x$. Consider the vector forming an acute angle with the vector $\widehat{\imath}$.

Solution : We obtain the coordinates of the direction vector of the tangent from its slope



$$y = 2\sqrt{x}, \quad y' = \frac{1}{\sqrt{x}},$$

$$\mathrm{k}_A = y'(A) = \frac{1}{\sqrt{3}} = \tan\alpha \quad \Longrightarrow \quad \alpha = \frac{\pi}{6},$$

$$\vec{s} = (\cos\alpha; \sin\alpha) = \left(\frac{\sqrt{3}}{2}; \frac{1}{2}\right), \ \|\vec{s}\| = 1.$$

Now let us prepare $\operatorname{grad} z(A) = \left(2x + \frac{1}{x+y^2}; \frac{2y}{x+y^2}\right)\Big|_A = \left(\frac{91}{15}; \frac{4\sqrt{3}}{15}\right)$, so the required derivative will be

$$\frac{\partial z}{\partial \vec{s}}(A) = \operatorname{grad} z(A) \cdot \frac{\vec{s}}{\|\vec{s}\|} = \left(\frac{91}{15}; \frac{4\sqrt{3}}{15}\right) \cdot \left(\frac{\sqrt{3}}{2}; \frac{1}{2}\right) = \frac{95\sqrt{3}}{30}. \qquad \blacksquare$$

***Example* 134**. Determine in which direction the derivative of the function $f(x,y) = x^3y + \frac{x}{y^2} + 2y$ at the point $A = [-1;1]$ is maximal and compute this derivative.

Solution : From general theory we know that the derivative is maximal in the direction of the gradient.

$$\operatorname{grad} f(A) = \left(3x^2y + \frac{1}{y^2}; x^3 - \frac{2x}{y^3} + 2\right)\Big|_A = (4;3) \implies \vec{s} = (4;3).$$

$$\frac{\partial f}{\partial \vec{s}}(A) = \operatorname{grad} f(A) \cdot \frac{\vec{s}}{\|\vec{s}\|} = \frac{16+9}{5} = 5 \qquad \blacksquare$$

***Example* 135**. Given the function $z = \sqrt{2x+y}$, the point $A = [1;2]$, and the vector $\vec{s} = (-1;1)$. Determine
a) at which points the function z is differentiable, b) $\frac{\partial z}{\partial \vec{s}}(A)$,
c) the tangent plane to the graph of the function z at the point $T = [1;2;?]$.

Solution : a) $\frac{\partial z}{\partial x} = \frac{1}{\sqrt{2x+y}}$, $\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{2x+y}} \implies 2x+y > 0 \Longrightarrow y > -2x$,

b) $\frac{\partial z}{\partial \vec{s}}(A) = \left(\frac{1}{2}; \frac{1}{4}\right) \cdot \frac{(-1;1)}{\sqrt{2}} = \frac{-1}{4\sqrt{2}}$,

c) $T = [1;2;2]$, $\tau: \quad z - 2 = \frac{1}{2}(x-1) + \frac{1}{4}(y-2)$. $\blacksquare$

***Example* 136**. Determine the vector $\vec{s}$ in whose direction the rate of change of the function values of the function $f(x,y,z) = x^2 + y^2 + z^2 - 2xyz$ at the point $A = [1;-1;2]$ is maximal, and compute this maximal rate.

Solution : The function increases maximally, or decreases maximally, in the direction $\vec{s}$, or $(-\vec{s})$, where

$$\vec{s} = \operatorname{grad} f(A) = (6;-6;6) \implies \widehat{s} = \frac{\vec{s}}{\|\vec{s}\|} = \frac{(1;-1;1)}{\sqrt{3}},$$

$$\frac{\partial f}{\partial \vec{s}}(A) = (6;-6;6) \cdot \frac{(1;-1;1)}{\sqrt{3}} = \frac{18}{\sqrt{3}} = 6\sqrt{3} \quad \text{and} \quad \frac{\partial f}{\partial(-\vec{s})}(A) = -6\sqrt{3}. \qquad \blacksquare$$

***Example* 137**. Determine at which points the gradient of the function $f(x,y,z) = x^2 + y^2 + z^2 - 2xyz$ is perpendicular to the x-axis.

Solution : $\widehat{\imath} = (1,0,0)$ is the direction vector of the x-axis, $\operatorname{grad} f = (2x - 2yz; 2y - 2xz; 2z - 2xy)$,

$\widehat{\imath} \perp \operatorname{grad} f$ means $\widehat{\imath} \cdot \operatorname{grad} f = 0$. Hence $2x - 2yz = 0$

so the required points lie on the surface $x = yz$. $\blacksquare$

***Example* 138**. Determine the angle of the vectors $\operatorname{grad} f(A)$ and $\operatorname{grad} g(A)$, where $f(x,y,z) = x - 3y + \sqrt{3xy} + z^3$, $g(x,y,z) = z\sqrt{x^2+y^2} + xyz$, $A = [3;4;0]$.

Solution :
$$\operatorname{grad} f(A) = \left(1 + \frac{1}{2}\sqrt{\frac{3y}{x}}; -3 + \frac{1}{2}\sqrt{\frac{3x}{y}}; 3z^2\right)\Big|_A = \left(2; -\frac{9}{4}; 0\right)$$

$$\operatorname{grad} g(A) = \left(\frac{zx}{\sqrt{x^2+y^2}} + yz; \frac{zy}{\sqrt{x^2+y^2}} + xz; \sqrt{x^2+y^2} + xy\right)\Big|_A = (0;0;17)$$

$$\cos\varphi = \frac{(2;-\frac{9}{4};0)\cdot(0;0;17)}{\|(2;-\frac{9}{4};0)\|\,\|(0;0;17)\|} = 0 \implies \varphi = \frac{\pi}{2} \Longrightarrow \operatorname{grad} f(A) \perp \operatorname{grad} g(A)$$

$\blacksquare$

• Determine the gradients of the given functions

139. $f(x,y)=\dfrac{1}{\sqrt{(x^2+y^2)^3}}$ $\left[\operatorname{grad} f(x,y)=\left(\dfrac{-3x}{\sqrt{(x^2+y^2)^5}};\ \dfrac{-3y}{\sqrt{(x^2+y^2)^5}}\right)\right]$

140. $f(x,y)=\sin(x^2\,y)+\dfrac{x^2}{3}$ $\left[\operatorname{grad} f(x,y)=\left(2xy\,\cos(x^2y)+\dfrac{2x}{3};\ x^2\,\cos(x^2y)\right)\right]$

141. $f(x,y)=\ln\left(x+\sqrt{x^2+y^2}\right)$

$$\left[\operatorname{grad} f(x,y)=\left(\frac{1}{x+\sqrt{x^2+y^2}}\cdot\left(1+\frac{x}{\sqrt{x^2+y^2}}\right);\ \frac{1}{x+\sqrt{x^2+y^2}}\cdot\frac{y}{\sqrt{x^2+y^2}}\right)\right]$$

142. $f(x,y,z)=x^2\,yz+\ln y-15$ $\left[\operatorname{grad} f(x,y,z)=\left(2xyz;\ x^2z+\dfrac{1}{y};\ x^2\,y\right)\right]$

143. $f(u,v,t)=t\sqrt{u^2+v^2}$ $\left[\operatorname{grad} f(u,v,t)=\left(\dfrac{tu}{\sqrt{u^2+v^2}};\ \dfrac{tv}{\sqrt{u^2+v^2}};\ \sqrt{u^2+v^2}\right)\right]$

• Determine the gradients of the given functions at the point A

144. $f(x,y)=\operatorname{arccotan}(x-2y),\quad A=[4;1]$ $\left[\operatorname{grad} f(A)=\left(-\dfrac{1}{5};\ \dfrac{2}{5}\right)\right]$

145. $(x,y,z)=x\sqrt{yz},\quad A=[-2;1;4]$ $\left[\operatorname{grad} f(A)=\left(2;\ -2;\ -\dfrac{1}{2}\right)\right]$

146. $f(x,y,z)=\dfrac{x^2}{z}+\dfrac{z^2}{2y}-\dfrac{4}{x},\quad A=[1;2;-3]$ $\left[\operatorname{grad} f(A)=\left(\dfrac{10}{3};\ -\dfrac{9}{8};\ -\dfrac{29}{18}\right)\right]$

147. At which points is the gradient of the function $f(x,y)=\ln\left(x+\dfrac{1}{y}\right)$ the vector $\left(1;-\dfrac{16}{9}\right)$?

$\left[\text{at the points } \left[-\dfrac{1}{3};\dfrac{3}{4}\right] \text{ and } \left[\dfrac{7}{3};-\dfrac{3}{4}\right]\right]$

148. At which point is the gradient of the function $f(x,y,z)=x^2+y^2-2z^2+xy+3y+8z$

a) perpendicular to the z-axis; b) parallel to the z-axis; c) equal to the zero vector?

$$\left[\begin{array}{r}\text{a) at the points of the plane } z=2 \text{ i.e. } [x;y;2]\\ \text{b) at the points of the line } x=1,\ y=-2,\ z=t,\ t\in\mathbb{R}\\ \text{c) } [1;-2;2]\end{array}\right]$$

149. Find the angle φ of the gradients of the function $f(x,y)=\arcsin\dfrac{x-1}{y},\ y\neq 0$, at the points $A=[1;1], B=[3;4]$. $\left[\varphi=\arccos\dfrac{2}{\sqrt{5}}\right]$

• Given a function $f(x,y)$, a point A, and a vector $\vec{s}$. Determine at which points the function f is differentiable, compute $\dfrac{\partial f}{\partial \vec{s}}(A)$ and write the equation of the tangent plane to the graph of the function f at the point $T=[A;\ f(A)]$:

150. $f(x,y)=|x|+y,\ A=[1;0],\ \vec{s}=(-1;1)$ $\left[\{[x;y]\in\mathbb{E}_2: x\neq 0\},\ \dfrac{\partial f}{\partial \vec{s}}(A)=0,\ x+y-z=0\right]$

151. $f(x,y)=x^2+3xy+y^2,\ A=[1;0],\ \vec{s}=(1;2)$ $\left[\mathbb{E}_2,\ \dfrac{\partial f}{\partial \vec{s}}(A)=\dfrac{8}{\sqrt{5}},\ 2x+3y-z-1=0\right]$

152. Determine in which direction the derivative of the function $f(x,y)=\ln\dfrac{x+y}{x-y}$, at the point $A=[3;0]$ is maximal and compute this derivative. $\left[\vec{s}=\operatorname{grad} f(A)=\left(0;\dfrac{6}{9}\right),\ \dfrac{\partial f}{\partial \vec{s}}(A)=\dfrac{2}{3}\right]$

153. Determine the derivative of the function $f(x,y)=x^2-y^2$ at the point $A=[2;3]$ in the direction $\vec{s}$, forming the angle $\alpha=\dfrac{\pi}{3}$ with the vector $\vec{\imath}$. $\left(\alpha=\dfrac{\pi}{3}\text{ is the direction angle}\right)$ $\left[\dfrac{\partial f}{\partial \vec{s}}(A)=2-3\sqrt{3}\right]$

154.* Determine the derivative of the function $f(x,y,z)=x^2-3xy+y^2z-5z$ at the point $A=[1;-2;-1]$ in the direction $\vec{s}$, whose direction angles are $\alpha=\dfrac{\pi}{3},\beta=\dfrac{3\pi}{4},\gamma\in\left(0;\dfrac{\pi}{2}\right)$. $\left[\dfrac{\partial f}{\partial \vec{s}}(A)=\dfrac{7-\sqrt{2}}{2}\right]$

***Example* 155**. Given the function $z=f(x,y)=\sqrt{18-x^2-2y^2}$

a) Write a sufficient condition for differentiability of a function of n variables in an open set $M\subset\mathbb{E}_n$. Determine and sketch in $\mathbb{E}_2$ the set D in which the function $z=f(x,y)$ is differentiable. Justify!

b) Determine and sketch the graph of the function $z=\sqrt{18-x^2-2y^2}$.

c) Compute the derivative of the function f at the point $A=[1;-2]$ in the direction determined by the vector $\vec{s}=\overrightarrow{AB}$, where the point $B=[0;0]$. Describe the behavior of the function f at the point A in the given direction (the function increases or decreases, how quickly (estimate)).

d) Write the equation of the tangent plane and the parametric form of the normal to the graph of the function $z=f(x,y)$ at the point of contact $T=[1;-2;\ ?]$.

e) Write the equations of the level curves (i.e. $f(x,y)=k$) of this function for $k=0,k=3$. Sketch the level curves.

Solution:

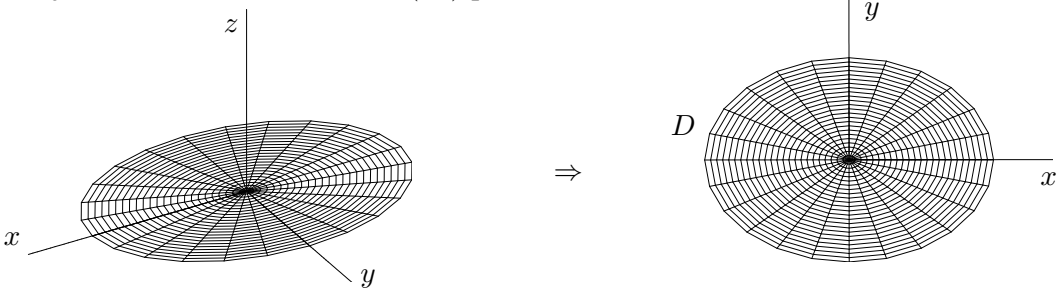
a) $\dfrac{\partial f}{\partial x}=\dfrac{1}{2\sqrt{18-x^2-2y^2}}\cdot(-2x),\quad \dfrac{\partial f}{\partial y}=\dfrac{1}{2\sqrt{18-x^2-2y^2}}\cdot(-4y)\quad\Rightarrow$

$18-x^2-2y^2>0\quad\Rightarrow\quad D=\left\{[x;y]\in\mathbb{E}_2:18-x^2-2y^2>0\right\}$,

i.e. the interior of the ellipse with center at the origin, semiaxes: $a=3\sqrt{2},\ b=3$,

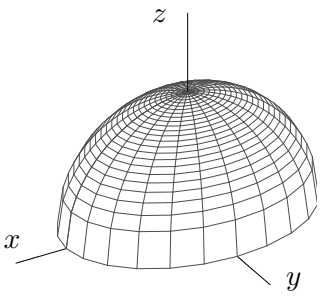
JUSTIFICATION: in the region D the given function has continuous partial derivatives

Projection of the surface onto the (xy)-plane



b)

The graph is the "upper" half (i.e. $z\geq 0$) of the ellipsoid $x^2+2y^2+z^2=18$



c) $\operatorname{grad} f(A)=(-1/3;4/3),\ \vec{s}=(-1;2),\ \dfrac{\partial f}{\partial \vec{s}}(A)=\dfrac{3\sqrt{5}}{5}$, the given function is at the point A increasing in the given direction, the tangent direction forms an angle of about 50° with the direction $\vec{s}$

d) $A = [1; -2] \in D(f)$, $T = [1; -2; ?]$ lies on the surface $\Rightarrow z_T = f(A) = 3$

The tangent plane τ has equation $z - z_0 = \frac{\partial z}{\partial x}(A) \cdot (x - x_0) + \frac{\partial z}{\partial y}(A) \cdot (y - y_0)$,

$\frac{\partial z}{\partial x}(A) = -\frac{1}{3}, \quad \frac{\partial z}{\partial y}(A) = \frac{4}{3} \quad \Rightarrow$

$\tau : z = 3 - \frac{1}{3}(x-1) + \frac{4}{3}(y+2)$, after rearrangement to general form $x - 4y + 3z = 18$,

normal $n : [x; y; z] = [1; -2; 3] + t\,(1; -4; 3)$, $t \in \mathbb{R}$

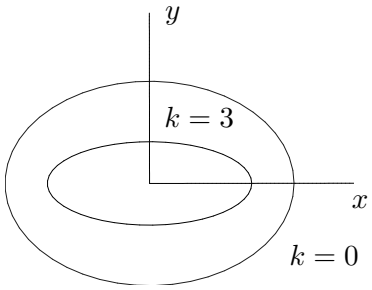
e)

The level curves are ellipses:

$x^2 + 2y^2 = k$

$x^2 + 2y^2 = 18$ (boundary of $D(f)$, for k=0),

$x^2 + 2y^2 = 9$ (for $k = 3$).



***Example* 156**. Given the function $z = f(x, y) = -\sqrt{5y - x^2}$.

a) Write (and justify) at which points $[x; y] \in \mathbb{E}_2$ the function $z = f(x, y)$ is differentiable. Sketch the set of these points.

b) Compute the first-order partial derivatives of the given function at the point $A = [4; 5]$. Describe the behavior of the given function at the point A (the function increases or decreases, in which direction, and estimate how quickly).

c) Determine the direction $\vec{s}$ in which the function f decreases most rapidly at the point A. Compute the derivative of the function f at the point A in this direction $\vec{s}$.

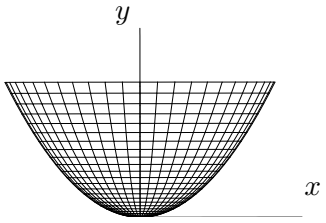
d) Write the differential of the function f at the point $A = [4; 5]$. Using it, compute the approximate value of the function f at the point $[4.3;\ 5.3\,]$.

e)Sketch the graph of the function $z = f(x, y)$.

Solution :

a) On the set $D = \{[x; y] \in \mathbb{E}_2 : 5y - x^2 > 0\}$ the given function has continuous partial derivatives.

D is the "interior" of the parabola with vertex at the origin and axis along the y-axis



b) $\frac{\partial f}{\partial x}(A) = \frac{4}{3}$, the given function is increasing at the point A in the positive direction of the x-axis, and does so at an angle of about $50° - 55°$,

$\frac{\partial f}{\partial y}(A) = -\frac{5}{6}$, the given function is decreasing at the point A in the positive direction of the y-axis, the angle between the tangent vector and the vector $\vec{j}$ is about $40°$.

c) $\vec{s} = -\mathrm{grad} f(A) = (-4/3; 5/6)$,

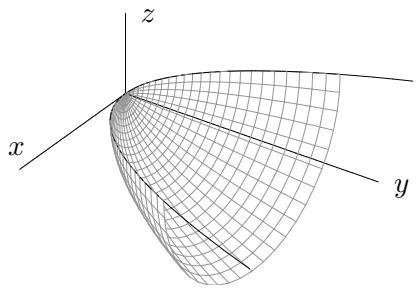
$\frac{\partial f}{\partial \vec{s}}(A) = \frac{\mathrm{grad} f(A) \cdot (-\mathrm{grad} f(A))}{\| - \mathrm{grad} f(A)\|} = -\|\mathrm{grad} f(A)\| = -\sqrt{89}/6 \doteq -1.6$,

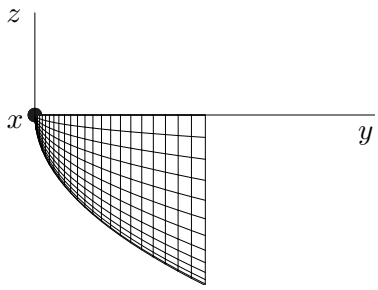
the function is decreasing at the point A in the direction of the vector $\vec{s}$ and the angle between the tangent and the vector $\vec{s}$ is about $60°$

d) $\mathrm{d}f[A] = \frac{4}{3}\,dx - \frac{5}{6}\,dy$, $f(4.3;\ 5.3\,) \doteq f(A) + \frac{4}{3} \cdot \frac{3}{10} - \frac{5}{6} \cdot \frac{3}{10} = -2.85$.

e) The graph is the "lower" half ($z \leq 0$) of the paraboloid of revolution $y = (x^2 + z^2)/5$,

the axis of rotation is along the y-axis.





157. a) Determine and sketch in $\mathbb{E}_2$ the regions in which the function $z = \ln(xy - 4)$ is differentiable and justify.
b) Determine the gradient of this function f at the point $A = [-2; -4]$. Describe what the computed vector represents.
c) Determine the magnitude of the derivative of the given function at the point A in the direction of the gradient.
d) Determine the vector $\vec{s}$ in whose direction the derivative of the given function at the point A is zero. Verify it by computation.
e) Write the equations of the level curves of this function for $k = 0$ and for $k = \ln 4$.

$$\Big[a) D_1 = \big\{[x; y] \in \mathbb{E}_2 : x < 0, y < 4/x\big\},\ D_2 = \big\{[x; y] \in \mathbb{E}_2 : x > 0, y > 4/x\big\},\Big]$$

$$\big[b) \mathrm{grad} f(A) = (-1; -1/2) \text{ gives the direction of maximal increase of the given function at the point } A\big]$$

$$\big[c) \vec{s} \cdot \mathrm{grad} f(A) = 0 \Rightarrow \vec{s} \perp \mathrm{grad} f(A), \text{e.g. } \vec{s} = (1/2; -1),\ d) y = 5/x,\ y = 8/x.\ \big]$$

158. Given the function $f(x, y) = x^2 - y^2 + 6xy + 2x$, the point $A = [-1; 2]$ and the direction $\vec{s} = (2; -2)$. a) Compute $\dfrac{\partial f}{\partial \vec{s}}(A)$. Is it the direction of maximal increase of the function f at the point A?
b) Determine all points at which $\mathrm{grad}\, f(x, y) = \vec{0}$.
c) Find the point of contact and the equation of the tangent plane τ to the graph of the function f parallel to the plane $\varrho:\ 4x + 6y - z + 3 = 0$.

$$\Big[a) \frac{\partial f}{\partial \vec{s}}(A) = 11\sqrt{2},\ \text{NO}\Big]$$

$$\big[b)\ x = -1/10,\ y = -3/10,\ c)\ T = [1; 0; 3],\ 4x + 6y - z - 1 = 0.\ \big]$$

159. Given the function $z = f(x, y) = y^2 \sin(x^2 - y^2)$.
a) Determine where the function is differentiable and compute $f_x,\ f_y$.
b) Write the equation of the tangent plane to the graph of the function f at the point of contact $[1;\ 1;\ ?]$.
c) Using the differential, compute an approximate value of the function f at the point $[0.9;\ 1.1]$.

$$[a) f_x = 2xy^2 \cos(x^2 - y^2),\ f_y = 2y \sin(x^2 - y^2) - 2y^3 \cos(x^2 - y^2)]$$

$$[\ b)\ z = 2x - 2y,\ c)\ f(0.9;\ 1.1\ ) \doteq -0.4.]$$

II.7 Derivative of a composite function

Let the differentiable functions $z = f(x, y)$, $x = x(u, v)$, $y = y(u, v)$ be given. Then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u},$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

• Compute the derivatives of the given differentiable functions.

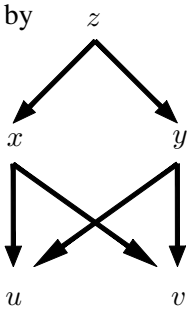
***Example* 160**. Let the differentiable functions $z = f(x, y)$, $x = u \cos v$, $y = u \sin v$ be given.

a) Compute the derivatives of the composite function $\dfrac{\partial z}{\partial u}$, $\dfrac{\partial z}{\partial v}$.

b) Verify them by direct computation for $z = f(x, y) = \mathrm{e}^x \ln y$.

Solution :

a) The dependence between the variables can be represented by an oriented graph, from which we build the formulas for the individual derivatives.



$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \cos v + \frac{\partial f}{\partial y} \cdot \sin v,$$
$$\frac{\partial z}{\partial v} = -\frac{\partial f}{\partial x} \cdot u \sin v + \frac{\partial f}{\partial y} \cdot u \cos v$$

b1) Substitute for x and y and obtain the function $z(u, v) = \mathrm{e}^{u \cos v} \ln(u \sin v)$, then

$$\frac{\partial z}{\partial u} = \mathrm{e}^{u \cos v} \cdot \cos v \cdot \ln(u \sin v) + \mathrm{e}^{u \cos v} \cdot \frac{1}{u \sin v} \cdot \sin v,$$
$$\frac{\partial z}{\partial v} = \mathrm{e}^{u \cos v} \cdot (-u \sin v) \cdot \ln(u \sin v) + \mathrm{e}^{u \cos v} \cdot \frac{1}{u \sin v} \cdot u \cos v\,.$$

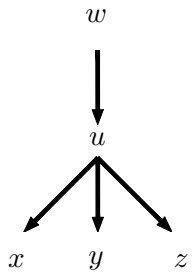
b2) Substitute into the formulas from a) for $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \cos v + \frac{\partial f}{\partial y} \cdot \sin v = e^x \ln y \cdot \cos v + \frac{e^x}{y} \sin v,$$
$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot u \sin v + \frac{\partial f}{\partial y} \cdot u \cos v = -e^x \ln y \cdot u \sin v + \frac{e^x}{y} u \cos v.$$

Substituting for x and y gives the same result as in b1). ■

***Example* 161**. $w = f(x^4 + y^4 - 2z^4)$. Compute the expression $V = \dfrac{1}{4x^3}\dfrac{\partial w}{\partial x} + \dfrac{1}{4y^3}\dfrac{\partial w}{\partial y} + \dfrac{1}{4z^3}\dfrac{\partial w}{\partial z}$.

Solution : Let $u = x^4 + y^4 - 2z^4 \quad \Rightarrow \quad w = f(u)$ be a function of one variable.



$$\frac{\partial w}{\partial x} = \frac{df}{du} \cdot \frac{\partial u}{\partial x} = f'(u) \cdot 4x^3,$$
$$\frac{\partial w}{\partial y} = \frac{df}{du} \cdot \frac{\partial u}{\partial y} = f'(u) \cdot 4y^3,$$
$$\frac{\partial w}{\partial z} = \frac{df}{du} \cdot \frac{\partial u}{\partial z} = f'(u) \cdot (-8z^3).$$

After substitution we easily compute that $V = 0$. ■

162. Verify that the function $y = f(x+at) + g(x-at)$ satisfies the partial differential equation $\dfrac{\partial^2 y}{\partial t^2} - a^2 \dfrac{\partial^2 y}{\partial x^2} = 0$, where the constant $a \in \mathbb{R}$. Assume that f and g have continuous second-order partial derivatives. [Hint: $u = x+at,\ v = x-at,\ y = f(u)+g(v)$]

163. Verify that the function $z = f\left(\dfrac{y}{x}\right)$ satisfies the equation $x \cdot \dfrac{\partial z}{\partial x} + y \cdot \dfrac{\partial z}{\partial y} = 0$. Assume that f is a differentiable function.

164. Let the functions $z = f(u,v),\ u = x^2 - y^2,\ v = \mathrm{e}^{xy}$ be given. Compute the differential expression $W = y \cdot \dfrac{\partial z}{\partial x} + x \cdot \dfrac{\partial z}{\partial y}$, where f is a differentiable function. $\left[W = (x^2+y^2)\,\mathrm{e}^{xy} \dfrac{\partial f}{\partial v}\right]$

165. Let the function $f(u,v) = x^y$ be given, where $x = u^2 + v^2,\ y = uv + v^2$. Compute $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ at the point A, whose coordinates are $u = 1,\ v = -1$. $\left[\dfrac{\partial f}{\partial u}(A) = \dfrac{\partial f}{\partial v}(A) = -\ln 2\right]$

II.8 Implicitly defined functions

***Example* 166**. Prove that the equation $x^3 + y^3 = 2x^2 + xy - 1$ implicitly defines a unique function $y = f(x)$ in a neighborhood of the point $A = [1;0]$. Determine $f'(1)$ and $f''(1)$.

Solution : Let $F(x,y) = x^3 + y^3 - 2x^2 - xy + 1$. This function is continuous and has continuous first- and second-order partial derivatives in $\mathbb{E}_2$, thus also in a neighborhood of the point A, verify this. For the existence and uniqueness of the implicit function $y = f(x)$ with continuous first and second derivatives it is now sufficient that $F(A) = F(1,0) = 0$ and $F_y(A) = (3y^2 - x)\Big|_A = -1 \neq 0$. Next we compute

$$y' = f'(x) = -\frac{F_x}{F_y} = -\frac{3x^2 - 4x - y}{3y^2 - x}, \qquad f'(1) = -\frac{F_x(A)}{F_y(A)} = -\frac{3-4}{-1} = -1,$$

$$y'' = f''(x) = \frac{d}{dx}\Big(f'(x)\Big) = -\frac{(6x - 4 - y')(3y^2 - x) - (3x^2 - 4x - y)(6y \cdot y' - 1)}{(3y^2 - x)^2}$$

$$f''(1) = -\frac{(6 - 4 + 1)(-1) - (3-4)(-1)}{1} = 4. \qquad \blacksquare$$

***Example* 167**. Write the equation of the tangent t and the normal n at the point $A = [1;1]$ to the curve defined implicitly by the equation $F(x,y) \equiv x^3y + y^3x + x^2y - 3 = 0$,

Solution : Since $F(1,1) = 0$ and $F_y(A) = \left(x^3 + 3y^2x + x^2\right)\Big|_{[1;1]} = 5 \neq 0$, the equation $F(x,y) = 0$ indeed defines the function $y = y(x)$, whose graph passes through the point A.

$$y'(1) = -\frac{F_x(A)}{F_y(A)} = \left(-\frac{3x^2y + y^3 + 2xy}{x^3 + 3y^2x + x^2}\right)\Big|_A = -\frac{6}{5}$$

$$t: \quad y - y_0 = y'(x_0)(x - x_0), \text{ where } A = [x_0;y_0], \quad y - 1 = -\frac{6}{5}(x-1) \Longrightarrow 6x + 5y - 11 = 0,$$

$$n: \quad y - y_0 = -\frac{1}{y'(x_0)}(x - x_0), \quad y - 1 = \frac{5}{6}(x-1) \Longrightarrow 5x - 6y + 1 = 0. \qquad \blacksquare$$

***Example* 168**. The equation $\ln\sqrt{x^2+y^2} = \arctan\dfrac{y}{x}$ implicitly defines the function $y = f(x)$ for $x \neq 0$. Determine y' and y''.

Solution : We may use the known formula $y' = -\dfrac{F_x}{F_y}, F_y \neq 0$ or we may differentiate the given equation directly and keep in mind that y depends on x, i.e. $y = y(x)$.

Let us differentiate directly :

$$\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{2x+2yy'}{2\sqrt{x^2+y^2}}=\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{y'x-y}{x^2}\Longrightarrow\frac{x+yy'}{x^2+y^2}=\frac{y'x-y}{x^2+y^2}\quad\Longrightarrow$$

$$x+yy'=y'x-y\Longrightarrow x+y=y'(x-y)\Longrightarrow\quad y'=\frac{x+y}{x-y},\quad x\neq y,\ x\neq 0.$$

We compute the second derivative similarly :

$$\Big(\frac{x+y}{x-y}\Big)'=\frac{(1+y')(x-y)-(x+y)(1-y')}{(x-y)^2}=\frac{x-y-x-y+y'(x-y+x+y)}{(x-y)^2}$$

$$y''=\frac{-2y+2xy'}{(x-y)^2}=\frac{-2y+2x\cdot\frac{x+y}{x-y}}{(x-y)^2}=\frac{2(x^2+y^2)}{(x-y)^3},\quad x\neq y,\ x\neq 0.$$ ■

***Example* 169**. Show that the equation $x^2+2xy+y^2-4x+2y-3=0$ implicitly defines a function $y=f(x)$ whose graph passes through the point $A=[0;1]$. Determine whether the function f is convex in a neighborhood of the point $x_0=0$. Write the equation of the tangent to the graph of the function at the point A.

Solution : $F(x,y)=x^2+2xy+y^2-4x+2y-3$ is differentiable in $\mathbb{E}_2$ (thus also in a neighborhood of the point A) and has continuous second-order partial derivatives in $\mathbb{E}_2$. Furthermore:

$$F(A)=F(0,1)=0,\quad F_y(A)=(2x+2y+2)\Big|_A=4\neq 0,$$

$$y'=-\frac{F_x}{F_y}=-\frac{2x+2y-4}{2x+2y+2}=-\frac{x+y-2}{x+y+1}\Longrightarrow y'(0)=\frac{1}{2},$$

$$y''=-\frac{(1+y')(x+y+1)-(x+y-2)(1+y')}{(x+y+1)^2}=-\frac{3(1+y')}{(x+y+1)^2}\quad\Longrightarrow$$

$y''(0)=-\dfrac{3(1+\frac{1}{2})}{(0+1+1)^2}=-\dfrac{9}{8}<0$. The function $y=f(x)$ is concave in a neighborhood of the point x_0, since $y''(0)<0$. The tangent at the point A has equation $y-1=\dfrac{1}{2}x$. ■

***Example* 170**. Prove that the relation $F(x,y,z)=z^3+3x^2z-2xy=0$ implicitly defines a unique function $z=f(x,y)$ in a neighborhood of the point $A=[-1;-2;1]$. Determine $\operatorname{grad} f(-1,-2)$.

Solution : The function F has continuous partial derivatives in a neighborhood of the point A,

$$F(A)=F(-1,-2,1)=0,\quad F_z(A)=(3z^2+3x^2)\Big|_A=6\neq 0.$$

Thus the existence and uniqueness of the function $z=f(x,y)$ are proved in a neighborhood of the point A, which

has continuous first-order partial derivatives in a neighborhood of the point $[-1;-2]$.

Now $\operatorname{grad} f(-1,-2)=\left(\dfrac{\partial f}{\partial x}(A),\dfrac{\partial f}{\partial y}(A)\right)$, where

$$\frac{\partial f}{\partial x}=-\frac{F_x}{F_z}=-\frac{6xz-2y}{3z^2+3x^2}\Longrightarrow\frac{\partial f}{\partial x}(-1,-2)=\frac{1}{3},$$

$$\frac{\partial f}{\partial y}=-\frac{F_y}{F_z}=-\frac{-2x}{3z^2+3x^2}\Longrightarrow\frac{\partial f}{\partial y}(-1,-2)=-\frac{1}{3},$$

$$\operatorname{grad} f(-1,-2)=\Big(\frac{1}{3};-\frac{1}{3}\Big)=\frac{1}{3}(1;-1).$$ ■

***Example* 171**. In a neighborhood of the point $[2;-2;1]$ the function $z=f(x,y)$ is given in implicit form by the equation $\ln z+x^2yz+8=0$. Determine $\dfrac{\partial z}{\partial\vec{s}}(A)$, where $\vec{s}=\overrightarrow{AB}$, $A=[2;-2]$, $B=[3;-3]$.

Solution : Let $F(x,y,z)=\ln z+x^2yz+8$. The functions $F_x=2xyz$, $F_y=x^2z$, $F_z=\dfrac{1}{z}+x^2y$ are continuous in a neighborhood of the point $[2;-2;1]$.

At points where $F(x,y,z)=0$ and $F_z(x,y,z)\neq 0$ we have:

$$\frac{\partial f}{\partial x}=-\frac{F_x}{F_z}=-\frac{2xyz}{\frac{1}{z}+x^2y}\Longrightarrow\frac{\partial f}{\partial x}(A)=-\frac{8}{7},\ \frac{\partial f}{\partial y}=-\frac{F_y}{F_z}=-\frac{x^2z}{\frac{1}{z}+x^2y}\Longrightarrow\frac{\partial f}{\partial y}(A)=\frac{4}{7},$$

$$\frac{\partial f}{\partial \vec{s}}(A)=\operatorname{grad} f(A)\cdot\frac{\vec{s}}{\|\vec{s}\|}=\left(-\frac{8}{7};\frac{4}{7}\right)\cdot\frac{(1;-1)}{\sqrt{2}}=\frac{-12}{7\sqrt{2}}=\frac{-6\sqrt{2}}{7}.$$

■

Let the surface $z=z(x,y)$ and the point of contact $T=[A;f(A)]$ be given, where $A=[x_0;y_0]$, then

the tangent plane $\tau:\ z-z_0=\dfrac{\partial z}{\partial x}(x_0,y_0)\,(x-x_0)+\dfrac{\partial z}{\partial y}(x_0,y_0)\,(y-y_0)$

the normal $n:\ X=T+t\,\vec{n},\ t\in\mathbb{R},\qquad$ where $\vec{n}=\left(\dfrac{\partial z}{\partial x}(x_0,y_0),\ \dfrac{\partial z}{\partial y}(x_0,y_0),\ -1\right)$

***Example* 172**. Write the equation of the tangent plane τ and the normal n to the surface $z=z(x,y)$ defined implicitly by the equation $F(x,y,z)\equiv x^2+y^2+z^2-25=0$ at the point $A=[3;0;4]$.

Solution : We know that a normal vector to the surface has the expression

$$\vec{n}=\left(\frac{\partial z}{\partial x};\frac{\partial z}{\partial y};-1\right)=\left(-\frac{F_x}{F_z};-\frac{F_y}{F_z};-1\right)\sim\vec{n}=(F_x;F_y;F_z).$$

If the surface is expressed in implicit form, then the latter notation is more convenient. Thus $\vec{n}=(2x;2y;2z)\sim(x;y;z)\big|_A=(3;0;4)$,

$$\tau:3(x-3)+0\cdot y+4(z-4)=0\Longrightarrow 3x+4z-25=0,$$
$$n:[x;y;z]=[3;0;4]+t(3;0;4),\ t\in\mathbb{R}.$$

■

***Example* 173**. Write the equation of the tangent plane to the surface defined implicitly by the equation $F(x,y,z)\equiv x(y+z)+z^2-5=0$ parallel to the plane $\varrho:3x-3y+6z=2$.

Solution : $\vec{n}=(3;-3;6)\sim(1;-1;2)$

$$\vec{n}=(F_x;F_y;F_z)=(y+z;x;x+2z)=k(1;-1;2),\ k\in\mathbb{R}\setminus\{0\}$$

$$\left.\begin{aligned}y+z&=k\\x&=-k\\x+2z&=2k\end{aligned}\right\}$$ and from this system we must determine the coordinates of the point of contact A.

We get $x=-k,\ \ y=-\dfrac{k}{2},\ \ z=\dfrac{3}{2}\,k$. We know that the required point must lie on the given surface, so we substitute into $F(x,y,z)=0$ and determine the parameter k , and thus also the coordinates of the required point. We obtain successively

$$-k\left(-\frac{k}{2}+\frac{3}{2}\,k\right)+\left(\frac{3}{2}\,k\right)^2=5\ \Rightarrow\ -k^2+\frac{9}{4}\,k^2=5\ \Rightarrow\ k^2=4\ \Rightarrow\ k=\pm 2,$$
$$A_1=[-2;-1;3],A_2=[2;1;-3],$$
$$\tau_1:x-y+2z-5=0,\quad \tau_2:x-y+2z+5=0.$$

■

***Example* 174**.* Determine the equation of the tangent at the point $A=[-2;1;6]$ to the curve in $\mathbb{E}_3$, which is the intersection of two surfaces given implicitly by the equations $2x^2+3y^2+z^2=\ 47\ , x^2+2y^2=z$.

Solution : We determine the tangent plane at the point A to the first surface $\tau_1:-4x+3y+6z-47=0$ and the tangent plane to the second surface $\tau_2:-4x+4y-z-6=0$. The direction vector of the required tangent is $\vec{s}\ =\vec{n}_1\times\vec{n}_2=(-27;-28;-4)\sim(27;28;4)$ and the equation of the tangent is

$$[x;y;z]=[-2;1;6]+t(27;28;4),\ t\in\mathbb{R}.$$

We give another possible procedure. With the given expression of the curve as the intersection of two surfaces we assume that the only independent variable is x. Then y and z will be functions of the variable x, i.e. $y(x), z(x)$. The required tangent vector $\vec{s}$ will be $(1, y'(x), z'(x))$.

Therefore we differentiate the given system with respect to x :

$$\begin{matrix} 4x+6yy'+2zz'=0 \\ 2x+4yy'=z' \end{matrix} \quad \text{or equivalently} \quad \begin{matrix} 2x+3yy'+zz'=0 \\ 2x+4yy'=z' \end{matrix} .$$

We are interested in the tangent vector at the given point, so we substitute the coordinates of the point A and obtain successively the mutually equivalent systems

$$\left.\begin{matrix} -4+3y'+6z'=0 \\ -4+4y'-z'=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 3y'+6z'=4 \\ 4y'-z'=4 \end{matrix}\right\} \Rightarrow \begin{matrix} y'=\dfrac{\begin{vmatrix} 4 & 6 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 6 \\ 4 & -1 \end{vmatrix}} = \dfrac{-28}{-27} = \dfrac{28}{27}, \\ z'=\dfrac{\begin{vmatrix} 3 & 4 \\ 4 & 4 \end{vmatrix}}{-27} = \dfrac{-4}{-27} = \dfrac{4}{27} \end{matrix} .$$

The required tangent vector is $\vec{s} = \left(1; \frac{28}{27}; \frac{4}{27}\right) \sim (27; 28; 4)$. The parametric form of the tangent is thus again $[x; y; z] = [-2; 1; 6] + t(27; 28; 4),\ t \in \mathbb{R}$. ■

***Example* 175**. a) Find the unit vector $\widehat{n}$ of the normal at the point $A = [1; -1; 1]$ to the surface expressed implicitly in the form $F(x, y, z) \equiv x^2 + y^2 + z^2 - 3 = 0$.

b) Compute $\dfrac{\partial g}{\partial \widehat{n}}(A)$, where $g(x, y, z) = xy^2z^3$.

Solution : a) $\vec{n} = (F_x; F_y; F_z) = (2x; 2y; 2z) \sim (x; y; z)$, $\widehat{n}(A) = \dfrac{\vec{n}(A)}{\|\vec{n}(A)\|} = \dfrac{1}{\sqrt{3}}(1; -1; 1)$

b) $\dfrac{\partial g}{\partial \widehat{n}}(A) = \operatorname{grad} g(A) \cdot \widehat{n}(A) = (1; -2; 3) \cdot \dfrac{(1; -1; 1)}{\sqrt{3}} = 2\sqrt{3}.$ ■

• Write the equation of the tangent t and the normal n to the curve defined implicitly by the equation $F(x, y) = 0$ at the point A :

176. $F(x, y) \equiv \arcsin x + xy^2 = 0,\ A = [0; 2]$ $[t:\ x = 0, n:\ y = 2]$

177. $F(x, y) \equiv x^2y + xy^2 - axy - a^3 = 0,\ A = [a; a]$ $[t:\ y = -x + 2a, n:\ y = x]$

178. Prove that the equation $\ln(x + y) + 2x + y = 0$ defines the function $y = f(x)$ satisfying $f(-1) = 2$. Write the equation of the tangent to the curve $y = f(x)$ at the point $A = [-1; 2]$.
$[t : y - 2 = -\frac{3}{2}(x + 1)]$

• A function of one variable $y = f(x)$ is defined implicitly by the equation $F(x, y) = 0$.

a) Compute the first-order partial derivatives of the function F.

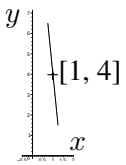
b) Verify that the equation $F(x, y) = 0$ implicitly determines in a neighborhood of the point $[x_0; y_0]$ the function $y = f(x)$, which has a continuous derivative $f'(x)$ in a neighborhood of the point x_0.

c) Determine the value of the derivative $f'(x_0)$ and write the equation of the tangent and the normal to the graph of the function f at the point of contact $[x_0; y_0]$. Describe the behavior of this function at the point x_0, i.e. increasing or decreasing.

d) Use the tangent equation to compute approximately the value $y = f(x)$ at the given point x_1. Sketch the tangent.

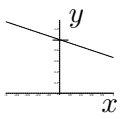
179. $F(x,y) = x^2y - x^3 - 2\cdot\sqrt{y} + 1,\ [x_0;y_0] = [1;4],\ x_1 = 0.8$

$$\left[\begin{array}{r} a)\ F_x = 2xy - 3x^2,\ F_y = x^2 - \dfrac{1}{\sqrt{y}} \\ c)\ f'(1) = -10,\ t:\ y = 4 - 10(x-1),\ n:\ y = 4 + \dfrac{1}{10}(x-1), \\ \text{the function } f \text{ is at the point } x_0 = 1 \text{ decreasing} \\ d)\ f(0.8) \doteq 6 \end{array}\right]$$



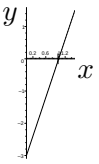
180. $F(x,y) = y\,\mathrm{e}^x + y^2 - 2x^2y - 2,\ [x_0;y_0] = [0;1],\ x_1 = -0.3$

$$\left[\begin{array}{r} a)\ F_x = y\,\mathrm{e}^x - 4xy,\ F_y = \mathrm{e}^x + 2y - 2x^2 \\ c)\ f'(0) = -1/3,\ t:\ y = 1 - x/3,\ n:\ y = 1 + 3x, \\ \text{the function } f \text{ is at the point } x_0 = 0 \text{ decreasing} \\ d)\ f(-0.3) \doteq 1.1 \end{array}\right]$$



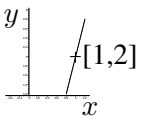
181. $F(x,y)) = \ln(x-y) + 2x + y^2 - 2 = 0,\ [x_0;y_0] = [1;0],\ x_1 = 1.1$

$$\left[\begin{array}{r} a)\ F_x = \dfrac{1}{x-y} + 2,\ F_y = -\dfrac{1}{x-y} + 2y \\ c)\ f'(1) = 3,\ t:\ 3x - y = 3,\ n:\ x + 3y = 1, \\ \text{the function } f \text{ is at the point } x_0 = 1 \text{ increasing} \\ d)\ f(1.1) \doteq 0.3 \end{array}\right]$$



182. $F(x,y) = xy\mathrm{e}^{x-y} - \dfrac{2}{\mathrm{e}} = 0,\ [x_0;y_0] = [1;2],\ x_1 = 1.1$

$$\left[\begin{array}{r} a)\ F_x = (y + xy)\,\mathrm{e}^{x-y},\ F_y(x,y) = (x - xy)\,\mathrm{e}^{x-y} \\ c)\ f'(1) = 4,\ t:\ y = 2 + 4(x-1),\ n:\ x + 4y = 9, \\ \text{the function } f \text{ is at the point } x_0 = 1 \text{ increasing} \\ d)\ f(1.1) \doteq 2.4 \end{array}\right]$$



• A function of one variable $y = f(x)$ is defined implicitly by the equation $F(x,y) = 0$.

a) Verify (all assumptions) that the equation $F(x,y) = 0$ implicitly determines the function $y = f(x)$, whose graph passes through the point $[x_0;y_0]$ and has continuous first and second derivatives in a neighborhood of the point x_0.

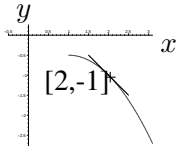
b) Determine the values of the derivatives $f'(x_0)$ and $f''(x_0)$.

c) Write the Taylor polynomial of degree 2, $T_2(x)$ of the function f centered at the point x_0. Using $T_2(x)$ compute approximately the value $f(x)$ for the given x_1 .

d) Sketch the graph of the function f in a neighborhood of the point $[x_0;y_0]$.

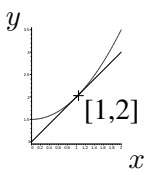
183. $F(x,y) = x^3 + 2x^2y + y^4 - 1 = 0,\ [x_0;y_0] = [2;-1],\ x_1 = 2.2$

$$\left[\begin{array}{r} b) f'(2) = f''(2) = -1, \\ c)\ T_2(x) = 1 - x - \dfrac{(x-2)^2}{2}, \\ f(2.2) \doteq -1.22 \end{array}\right]$$



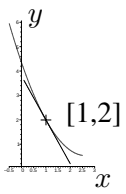
184. $F(x,y) = x^3 + \dfrac{y^2}{2} - x^2y - 1 = 0,\ [x_0;y_0] = [1;2],\ x_1 = 1.2$

$$\left[\begin{array}{r} b)\, f'(1) = f''(1) = 1, \\ c)\, T_2(x) = 1 + x + \dfrac{(x-1)^2}{2}, \\ f(1.2) \doteq 2.22 \end{array}\right]$$



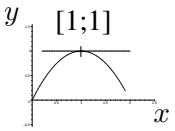
185. $F(x,y) = x^3 + xy^2 + xy - 7 = 0,\ [x_0;y_0] = [1;2],\ x_1 = 0.9$

$$\left[\begin{array}{r} b)\, f'(1) = -\dfrac{9}{5},\ f''(1) = \dfrac{138}{125}, \\ c)\, T_2(x) = 2 - \dfrac{9}{5}(x-1) - \dfrac{69}{125}(x-1)^2, \\ f(1.2) \doteq \dfrac{27319}{12500} \end{array}\right]$$



186. $F(x,y) = x^2 + 2y^2 - 2xy - y = 0,\ [x_0;y_0] = [1;1],\ x_1 = 1.2$

$$\left[\begin{array}{r} b)\, f'(1) = 0,\ f''(1) = -2, \\ \text{a local maximum occurs at the point} \\ c)\, T_2(x) = 1 - (x-1)^2,\ f(1.2) \doteq 0.96 \end{array}\right]$$



• Write the equation of the tangent plane τ and the normal n to the surface $F(x,y,z) = 0$ at the point A :

187. $x^2 + 2y^2 + 3z^2 - 21 = 0,\ A = [1;2;2]$
$[\tau:\ x + 4y + 6z - 21 = 0,\ n:\ [x;y;z] = (1;2;2) + t(1;4;6),\ t \in \mathbb{R}]$

188. $x^3 + y^3 + z^3 + xyz - 6 = 0,\ A = [1,2,-1]$
$[\tau:\ x + 11y + 5z - 18 = 0,\ n:\ [x;y;z] = (1;2;-1) + t(1;11;5),\ t \in \mathbb{R}]$

189. $xyz^2 - x - y - z = 0,\ A = [1,-1,-1]$
$[\tau:\ -2x + z + 3 = 0,\ n:\ [x;y;z] = (1;-1;-1) + t(-2;0;1),\ t \in \mathbb{R}]$

• Write the equation of such a tangent plane to the surface $F(x,y,z) = 0$, that is parallel to the plane ϱ.

190. $x^2 + y^2 + z^2 - 1 = 0,\ \varrho: x + 2y + z = 0$
$[x + 2y + z \pm \sqrt{6} = 0$, points of contact $T_{1,2} = [\pm\frac{1}{\sqrt{6}}; \pm\frac{2}{\sqrt{6}}; \pm\frac{1}{\sqrt{6}}]]$

191. $x^2 + 2y^2 + 3z^2 - 21 = 0,\ \varrho:\ x + 4y + 6z = 0$ $\quad [x + 4y + 6z \pm 21 = 0,\ T_{1,2} = [\pm1; \pm2; \pm2]]$

192. Let the function $z = f(x,y)$ be given implicitly by $\mathrm{e}^z - xyz = \mathrm{e}$. Determine $f_x(A), f_y(A), f_{xy}(A)$, where the point $A = [0;\mathrm{e};1]$ $\quad [f_x(A) = 1,\ f_y(A) = 0,\ f_{xy}(A) = 1/\mathrm{e}]$

193. Two surfaces are given in implicit form by the equations $x + 2y - \ln z + 4 = 0$ and $x^2 - xy - 8x + \dfrac{1}{2}z^2 + \dfrac{11}{2} = 0$. Determine the mutual position of the tangent planes to both surfaces at the common point $T = [2;-3;1]$. $\quad [x + 2y - z + 5 = 0$ is the common tangent plane$]$

• A function of two variables $z = f(x, y)$ is defined implicitly by the equation $F(x, y, z) = 0$.

a) Verify (all assumptions) that the equation $F(x, y, z) = 0$ implicitly determines the function $z = f(x, y)$, whose graph passes through the point $A = [x_0; y_0; z_0]$ and has continuous first-order partial derivatives in a neighborhood of the point $[x_0; y_0]$.

b) Compute the derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and determine $\operatorname{grad} f\,(x_0, y_0)$.

c) Write the equation of the tangent plane τ and the equation of the normal n to the graph of the function f at the point A (i.e. also to the surface described by the equation $F(x, y, z) = 0$).

d) Determine the differential of the function f at the point $[x_0; y_0]$. Compute approximately the value of the function f at the given point $[x_1; y_1]$.

e) Determine the derivative of the function f at the point $[x_0; y_0]$ in the direction $\vec{u}$. Write the direction $\vec{s}$, in which the function f at the point $[x_0; y_0]$ decreases most rapidly. Compute the derivative of the function f at the point $[x_0; y_0]$ in the direction $\vec{s}$.

194. $F(x, y, z) = 2x^3yz - x + y^3 + z^3 - 2 = 0, A = [1; 2; -1], [x_1; y_1] = [0.9; 2.1], \vec{u} = (2; 4)$

$$\left[\begin{array}{r} b)\ \frac{\partial f}{\partial x} = -\frac{6x^2yz - 1}{2x^3y + 3z^2},\ \frac{\partial f}{\partial y} = -\frac{2x^3z + 3y^2}{2x^3y + 3z^2}, \quad \operatorname{grad} f\,(1, 2) = \left(\frac{13}{7}; -\frac{10}{7}\right) \\ c)\ \tau:\ 13x - 10y - 7z = 0,\ n:\ [x; y; z] = [1; 2; -1] + t(13; -10; -7),\ t \in \mathbb{R} \\ d)\ \mathrm{d}f[A] = \frac{13}{7}(x - 1) - \frac{10}{7}(y - 2), \quad f(0.9, 2.1) \doteq -\frac{93}{70} \doteq -1.328 \\ e)\ \frac{\partial f}{\partial \vec{u}}(1, 2) = -\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}, \quad \vec{s} = (-13; 10), \quad \frac{\partial f}{\partial \vec{s}}(1, 2) = -\frac{\sqrt{269}}{7} \end{array}\right]$$

195. $F(x, y, z) = x^3 + 3xy^2z + 2yz^3 = 0, A = [1; -1; -1], [x_1; y_1] = [0.9; -1.1], \vec{u} = (1; -2)$

$$\left[\begin{array}{r} b)\ \frac{\partial f}{\partial x} = -\frac{3x^2 + 3y^2z}{3xy^2 + 6yz^2},\ \frac{\partial f}{\partial y} = -\frac{6xyz + 2z^3}{3xy^2 + 6yz^2}, \quad \operatorname{grad} f\,(1, -1) = \left(0; \frac{4}{3}\right) \\ c)\ \tau:\ 4y - 3z + 1 = 0,\ n:\ [x; y; z] = [1; -1; -1] + t(0; 4; -3),\ t \in \mathbb{R} \\ d)\ \mathrm{d}f[A] = \frac{4}{3}(y + 1), \quad f(0.9, -1.1) \doteq -1.13 \\ e)\ \frac{\partial f}{\partial \vec{u}}(1, -1) = -\frac{8}{3\sqrt{5}}, \quad \vec{s} = \left(0; -\frac{4}{3}\right), \quad \frac{\partial f}{\partial \vec{s}}(1; -1) = -\frac{4}{3} \end{array}\right]$$

196. $F(x, y, z) = z\mathrm{e}^{x+y} - xy + z^2 - 6 = 0,\ A = [0; 0; 2],\ [x_1; y_1] = [-0.2; 0.1],\ \vec{u} = (-1; 2)$

$$\left[\begin{array}{r} b)\ \frac{\partial f}{\partial x} = -\frac{z\mathrm{e}^{x+y} - y}{\mathrm{e}^{x+y} + 2z},\ \frac{\partial f}{\partial y} = -\frac{z\mathrm{e}^{x+y} - x}{\mathrm{e}^{x+y} + 2z}, \quad \operatorname{grad} f\,(0, 0) = \left(-\frac{2}{5}; -\frac{2}{5}\right) \\ c)\ \tau:\ 2x + 2y + 5z = 10,\ n:\ [x; y; z] = [0; 0; 2] + t(2; 2; 5),\ t \in \mathbb{R} \\ d)\ \mathrm{d}f[A] = -\frac{2}{5}(x - 0) - \frac{2}{5}(y - 0), \quad f(-0.2, 0.1) \doteq 2.04 \\ e)\ \frac{\partial f}{\partial \vec{u}}(0, 0) = -\frac{2}{5\sqrt{5}}, \quad \vec{s} = \left(\frac{2}{5}; \frac{2}{5}\right), \quad \frac{\partial f}{\partial \vec{s}}(0, 0) = -\frac{\sqrt{8}}{5} \end{array}\right]$$

II.9 Extrema of functions

Theorem (necessary condition for a local extremum). Let the function $f(x, y)$ be differentiable at the point A. If the function f has a local extremum at the point A, then $\operatorname{grad} f(A) = \vec{0}$.

Denote the leading principal minors of the matrix of second derivatives by

$$\Delta_1(A) = \frac{\partial^2 f}{\partial x^2}(A), \qquad \Delta_2(A) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(A), & \frac{\partial^2 f}{\partial x\,\partial y}(A) \\ \frac{\partial^2 f}{\partial y\,\partial x}(A), & \frac{\partial^2 f}{\partial y^2}(A) \end{vmatrix}$$

Theorem (sufficient conditions for a local extremum of a function of two variables).
Let the function $f(x, y)$ have continuous second-order partial derivatives at the point A.

Suppose that at the point A the necessary condition $\operatorname{grad} f(A) = \vec{0}$ is satisfied.

Then the following hold:
If $\Delta_1(A) > 0$ and $\Delta_2(A) > 0$, then the function f has a strict local minimum at the point A.
If $\Delta_1(A) < 0$ and $\Delta_2(A) > 0$, then the function f has a strict local maximum at the point A.
If $\Delta_2(A) < 0$, then the function f does not have a local extremum at the point A.

Procedure for computing local extrema

Step 1: Using the necessary condition, determine the so-called critical points, i.e.

a) points at which the function $f(x, y)$ is not differentiable,

b) points that are solutions of the system of equations $\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0$.

Step 2: At points **b)** proceed according to the sufficient condition.

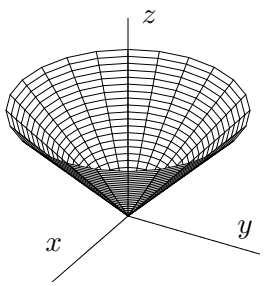
• Find the **local** extrema of the function:

***Example* 197**. $z(x, y) = \sqrt{x^2 + y^2}$

Solution : The given function is defined on $\mathbb{E}_2$. The partial derivatives

$z_x = \frac{x}{\sqrt{x^2+y^2}},\ z_y = \frac{y}{\sqrt{x^2+y^2}}$ do not exist at the point $A = [0; 0]$, which is the only critical point. Therefore there may be a local extremum at the point A, since the function f is not differentiable at the point A. However, we cannot decide using the above sufficient condition, so we use the definition of a local extremum. For every point in a punctured neighborhood $P(A)$ of the point A, it holds that $f(X) > f(A)$. Thus the function f has a strict local minimum at the point $A = [0; 0]$ with value $f(0, 0) = 0$.

NOTE. The inequality $f(X) > f(A)$ holds in any punctured neighborhood $P(A)$. Therefore the function f has a strict global minimum at the point $A = [0; 0]$ as well.



The example has an illustrative geometric interpretation. It is enough to realize what surface is the graph of the function $z = \sqrt{x^2 + y^2}$.

■

***Example* 198.** $f(x,y) = x^3 - x^2 + \frac{1}{2}y^2 - xy - 6x$

Solution : The function $f(x,y)$ has continuous first- and second-order partial derivatives.

1) According to the necessary condition $f_x = 0, \quad f_y = 0$ we obtain the system:

$$\left.\begin{array}{l} f_x = 3x^2 - 2x - y - 6 \\ f_y = y - x \end{array}\right\} \implies \left.\begin{array}{r} 3x^2 - 2x - y - 6 = 0 \\ y - x = 0 \end{array}\right\} \text{ we substitute}$$

$$\left.\begin{array}{r} 3(x^2 - x - 2) = 0 \\ y = x \end{array}\right\} \implies \left.\begin{array}{r} (x-2)(x+1) = 0 \\ y = x \end{array}\right\} \implies$$

and we obtain two critical points $[2;2]$, $[-1;-1]$.

Let us prepare the second derivatives

$$\left.\begin{array}{l} f_{xx} = 6x - 2 \\ f_{xy} = -1 = f_{yx} \\ f_{yy} = 1 \end{array}\right\} \implies \Delta_1(x,y) = 6x - 2,\ \Delta_2(x,y) = \begin{vmatrix} 6x-2 & -1 \\ -1 & 1 \end{vmatrix} = 6x - 3$$

2) From the sufficient condition

	Δ_2	Δ_1	conclusion
$[2;2]$	9	10	strict local minimum, $f(2,2) = -10$
$[-1;-1]$	-9		not an extremum

■

***Example* 199.** $z(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$

Solution : Here we have a function of two variables in explicit form. The function z is differentiable on $\mathbb{E}_2$, therefore the necessary condition for the existence of local extrema is: $z_x = 0,\ z_y = 0$.

$$\begin{array}{lll} z_x = 6x^2 + y^2 + 10x & \implies & 6x^2 + y^2 + 10x = 0 \\ z_y = 2xy + 2y & \implies & y(x+1) = 0 \implies \text{either } y = 0 \text{ or } x = -1 \end{array}$$

$$y = 0: \quad 6x^2 + 10x = 0 \implies \left\{\begin{array}{lll} x_1 = 0 & \implies & A_1 = [0;0] \\ x_2 = -\frac{5}{3} & \implies & A_2 = \left[-\frac{5}{3};0\right] \end{array}\right.$$

$$x = -1: \quad y^2 + 6 - 10 = 0 \implies y^2 = 4 \implies \left\{\begin{array}{lll} y = 2 & \implies & A_3 = [-1;2] \\ y = -2 & \implies & A_4 = [-1;-2] \end{array}\right.$$

Now let us prepare the second-order derivatives:

	A_1	A_2	A_3	A_4
$z_{xx} = 12x + 10$	10	-10	-2	-2
$z_{yy} = 2x + 2$	2	$-4/3$	0	0
$z_{xy} = 2y = z_{yx}$	0	0	4	-4

If $\Delta_2(A_i) = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} \left\langle \begin{array}{l} > 0, \quad \text{then a local extremum exists at the point } A_i. \\ < 0, \quad \text{then no local extremum exists at the point } A_i\ . \end{array}\right.$

For A_1 : $\Delta_2(A_1) = \begin{vmatrix} 10 & 0 \\ 0 & 2 \end{vmatrix} > 0,\ \Delta_1(A_1) = z_{xx}(A_1) > 0,$ we obtain a local minimum

$$z(A_1) = 0.$$

For A_2 : $\Delta_2(A_2) = \begin{vmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{vmatrix} > 0,\ \Delta_1(A_2) = z_{xx}(A_2) < 0,$ there is a local maximum

$$z(A_2) = \frac{125}{27}.$$

For $A_3,\ A_4$ we have $\Delta_2(A_3) = \Delta_2(A_4) = -16 < 0$, and therefore at the points A_3 and A_4 local extrema do not exist.

■

***Example* 200**. The function $y = f(x)$ is expressed in implicit form by
$$F(x, y) = x^2 + 2xy - y^2 + 8 = 0.$$

Solution : We compute y' :
$$y' = -\frac{F_x}{F_y} = \frac{x+y}{y-x} \quad \text{for } x \neq y.$$
Set $y' = 0$, i.e. $x + y = 0 \implies y = -x$. Substitute into the equation to determine the coordinates of possible local extrema
$$x^2 - 2x^2 - x^2 + 8 = 0 \implies 2x^2 = 8 \implies x_{1,2} = \pm 2, \quad y_{1,2} = \mp 2.$$
We obtained two points $A = [2; -2]$ and $B = [-2; 2]$. Now we compute
$$y'' = \frac{(1+y')(y-x) - (x+y)(y'-1)}{(y-x)^2}, \quad \text{so}$$
$y''(A) = \dfrac{-4}{16} < 0 \implies$ $f(2) = -2$ is a local maximum and
$y''(B) = \dfrac{4}{16} > 0 \implies$ $f(-2) = 2$ is a local minimum.

■

***Example* 201**.* $f(x, y, z) = x^3 + y^2 + z^2 + 12xy + 2z$

Solution :
$$\Delta_3 = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \quad \Delta_1 = f_{xx}$$

According to **Sylvester's** theorem on quadratic forms:

If $\Delta_2(P) > 0$, $\begin{cases} \Delta_3(P) > 0, \Delta_1(P) > 0, \textit{then } f(P) \textit{ is a strict local minimum.} \\ \Delta_3(P) < 0, \Delta_1(P) < 0, \textit{ then } f(P) \textit{ is a strict local maximum.} \end{cases}$

If $\Delta_2(P) < 0$, *then at the point* P *no local extremum exists.*

The computation proceeds as follows:
$$\left.\begin{array}{lll} f_x = 3x^2 + 12y = 0 & \text{i.e.} & x^2 + 4y = 0, \\ f_y = 2y + 12x = 0 & \text{i.e.} & y = -6x, \\ f_z = 2z + 2 = 0, & \text{i.e.} & z = -1 \end{array}\right\}, \quad \begin{array}{l} \text{then} \quad x^2 - 24x = 0 \quad \Rightarrow \text{and} \\ x_1 = 0, \ x_2 = 24, \end{array}$$
$$A = [0; 0; -1], \quad B = [24; -144; -1]$$

$f_{xx} = 6x, \quad f_{yy} = 2, f_{zz} = 2, \quad f_{xy} = 12, \quad f_{xz} = 0, \quad f_{yz} = 0.$

Since $\Delta_2(A) = \begin{vmatrix} 0 & 12 \\ 12 & 2 \end{vmatrix} < 0$, at the point A no local extremum occurs.

Since $\Delta_2(B) = \begin{vmatrix} 144 & 12 \\ 12 & 2 \end{vmatrix} = 144 > 0, \quad \Delta_3(B) = \begin{vmatrix} 144 & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 288 > 0$ and

$\Delta_1(B) = 144 > 0$, the value $f(B) = f(24, -144, -1) = -6913$ is a local minimum.
We may also verify directly that the quadratic form $\mathrm{d}^2 f[B]$ is positive definite :

$$\mathrm{d}^2 f[B] = 144\,(dx)^2 + 24\,dx\,dy + 2\,(dy)^2 + 2\,(dz)^2 = (12\,dx + dy)^2 + (dy)^2 + 2\,(dz)^2 > 0$$

■

• Find local extrema of the given functions with the given constraint:

***Example* 202.** $z = 2(x^2 + y^2)$, if $x + y = 2$.

Solution : Geometrically, this means finding the extrema of the z-coordinate on the intersection curve of the **paraboloid of revolution** $z = 2(x^2 + y^2)$ with the **plane** $x + y = 2$.

From the constraint $x + y = 2$ express, for example, $y = 2 - x$ and substitute into the given function $z(x, y) = 2(x^2 + y^2)$. Thus we obtain $\tilde{z}(x) = z(x, 2 - x) = 2(x^2 + (2 - x)^2)$, so that $\tilde{z}(x) = 4(x^2 - 2x + 2)$.

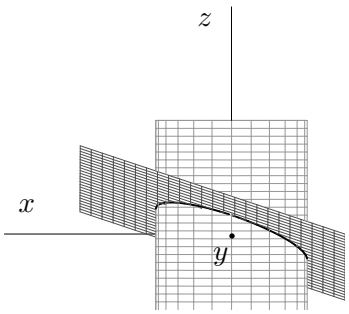
For the one-variable function $\tilde{z}(x)$ we seek a local extremum. Thus

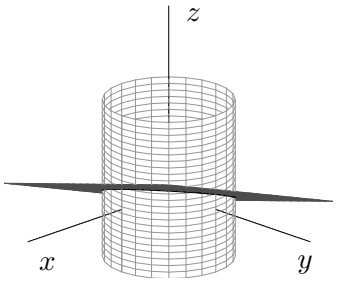
$\tilde{z}'(x) = 4(2x - 2) = 0$. Hence $x_1 = 1, y_1 = 2 - 1 = 1$ and $\tilde{z}''(x) = 8 > 0 \quad \Rightarrow$
$z(1, 1) = 4$ is a local minimum.

■

***Example* 203.** $z = \dfrac{x}{3} + \dfrac{y}{4}$, if $x^2 + y^2 = 1$.

Solution : Geometrically, this means finding the extrema of the z-coordinate on the intersection curve of the **plane** $z = \dfrac{x}{3} + \dfrac{y}{4}$ with the **circular cylinder** $x^2 + y^2 = 1$.

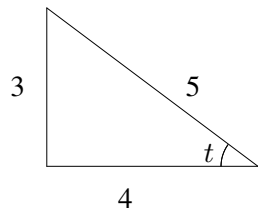




Since from the constraint $x^2 + y^2 = 1$ we cannot uniquely express either x or y, we switch to polar coordinates $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$, where $r = 1$, thus $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$.

Then $\tilde{z} = z(\cos t, \sin t) = \dfrac{\cos t}{3} + \dfrac{\sin t}{4}$ and further $\tilde{z}' = \dfrac{dz}{dt} = \dfrac{-\sin t}{3} + \dfrac{\cos t}{4} = 0$, so that

$$\tan\, t = \frac{3}{4}.$$



As we see in the figure, $\sin t = \pm\dfrac{3}{5}$, $\cos t = \pm\dfrac{4}{5}$.

For $\sin t_1 = \dfrac{3}{5}$, $\cos t_1 = \dfrac{4}{5}$ wehave $t_1 \in \left(0; \dfrac{\pi}{2}\right)$

and for $\sin t_2 = -\dfrac{3}{5}$, $\cos t_2 = -\dfrac{4}{5}$ wehave $t_2 \in \left(\pi; \dfrac{3\pi}{2}\right)$.

Further, $\tilde{z}'' = \dfrac{d^2\tilde{z}}{dt^2} = -\dfrac{\cos t}{3} - \dfrac{\sin t}{4} \quad \Longrightarrow \quad \tilde{z}''(t_1) < 0,\ \tilde{z}''(t_2) > 0.$

We have arrived at the local maximum $\tilde{z}(t_1) = z\left(\dfrac{4}{5}, \dfrac{3}{5}\right) = \dfrac{4}{15} + \dfrac{3}{20} = \dfrac{5}{12}$

and the local minimum $\tilde{z}(t_2) = z\left(-\dfrac{4}{5}, -\dfrac{3}{5}\right) = -\dfrac{5}{12}$.

■

***Example* 204.** $z = \sin^2 x + \sin^2 y$, if $y = x - \frac{\pi}{4}$.

Solution :
$\tilde{z}(x) = z(x, x - \pi/4) = \sin^2 x + \sin^2(x - \frac{\pi}{4}), \quad \tilde{z}'(x) = 2\sin x \cos x + 2\sin(x - \frac{\pi}{4})\cos(x - \frac{\pi}{4}) =$
$= \sin 2x + \sin(2x - \frac{\pi}{2}) = \sin 2x + \sin 2x \cos\frac{\pi}{2} - \cos 2x \sin\frac{\pi}{2} = \sin 2x - \cos 2x$

If we set $\tilde{z}'(x) = 0$, we obtain $\sin 2x = \cos 2x$ and further

$$2x = \frac{\pi}{4} + k\pi, \quad x = \frac{\pi}{8} + k\frac{\pi}{2}, \text{ so that } k_1 = 2n,\ k_2 = 2n+1.$$

We compute the second derivative: $\tilde{z}''(x) = 2\cos 2x + 2\sin 2x$,

$$\tilde{z}''\left(\frac{\pi}{8} + 2n \cdot \frac{\pi}{2}\right) = 2 \cdot \frac{\sqrt{2}}{2} + 2 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2} > 0,$$
$$\tilde{z}''\left(\frac{\pi}{8} + (2n+1) \cdot \frac{\pi}{2}\right) = -2 \cdot \frac{\sqrt{2}}{2} - 2 \cdot \frac{\sqrt{2}}{2} = -2\sqrt{2} < 0.$$

Thus $\tilde{z}\left(\frac{\pi}{8} + n\pi\right) = 2\sin^2\frac{\pi}{8} = 1 - \cos\frac{\pi}{4} = 1 - \frac{\sqrt{2}}{2}$ is a local minimum

and $\tilde{z}\left(\frac{\pi}{8} + (2n+1)\frac{\pi}{2}\right) = \sin^2\frac{5\pi}{8} + \sin^2\frac{\pi}{8} = \frac{1}{2}\left(1 - \cos\frac{5\pi}{4} + 1 - \cos\frac{\pi}{4}\right) =$

$$= \frac{1}{2}\left(2 + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 1 \text{ is a local maximum.}$$

■

Example* 205. $z = x^2 + 2y^2$, if $x^2 - 2x + 2y^2 + 4y = 0$.

Solution : Here we use the **Lagrange function**: *If the function* $z = f(x, y)$ *and the constraint* $g(x, y) = 0$ *are given, then the Lagrange function has the form*

$$L(x, y, \lambda) = f(x, y) + \lambda\, g(x, y).$$

Its stationary points are obtained by solving the system

$$\left.\begin{aligned} L_x &= 0 \\ L_y &= 0 \\ L_\lambda = g(x, y) &= 0. \end{aligned}\right\}$$

The function $z = f(x, y)$ can have extrema under the constraint $g(x, y) = 0$ only at points $[x; y]$ for which there exists $\lambda \in \mathbb{R}$ such that $[x, y, \lambda]$ is a critical point of the function $L(x, y, \lambda)$.

In our example we have $L(x, y, \lambda) = x^2 + 2y^2 + \lambda(x^2 - 2x + 2y^2 + 4y)$

$L_x = 2x + \lambda(2x - 2) = 0$ and hence $x = \frac{\lambda}{1+\lambda}$,

$L_y = 4y + \lambda(4y + 4) = 0$ and hence $y = \frac{-\lambda}{1+\lambda}$.

Therefore $x^2 - 2x + 2y^2 + 4y = 0$, so that $\frac{\lambda^2}{(1+\lambda)^2} - \frac{2\lambda}{1+\lambda} + \frac{2\lambda^2}{(1+\lambda)^2} - \frac{4\lambda}{1+\lambda} = 0$,

$\frac{3\lambda^2}{(1+\lambda)^2} = \frac{6\lambda}{1+\lambda}$, $\lambda \neq -1$ and subsequently $\frac{3\lambda^2}{1+\lambda} = 6\lambda$, $3\lambda^2 + 6\lambda = 0 \Longrightarrow \lambda \cdot (\lambda + 2) = 0$.

Here we have $\lambda_1 = -2$, so $x_1 = 2,\ y_1 = -2$, $A_1 = [2,-2]$ or
$\lambda_2 = 0$ so $x_2 = 0,\ y_2 = 0$, $A_2 = [0;0]$. Further,
$L_{xx} = 2 + 2\lambda,\ L_{yy} = 4 + 4\lambda,\ L_{xy} = 0$ and hence

$$\Delta(A_1) = \begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix} > 0,\ L_{xx}(A_1) < 0, \quad L(A_1) = 12 \text{ is a local maximum,}$$

$$\Delta(A_2) = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} > 0,\ L_{xx}(A_2) > 0, \quad L(A_2) = 0 \text{ is a local minimum.}$$

■

***Example* 206.** In the plane $x + 2y - z + 3 = 0$ find the point whose sum of squares of distances from the points $A = [1;1;1]$ and $B = [2;2;2]$ is minimal.

Solution : Denote the required point by $P = [x;y;z]$ and form the sum $|\overrightarrow{AP}|^2 + |\overrightarrow{BP}|^2$, which we write using the coordinates of the points

$$f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2 + (x-2)^2 + (y-2)^2 + (z-2)^2.$$

The point P must lie in the plane $x + 2y - z + 3 = 0$. Thus this plane equation represents the constraint, which must be satisfied.

If we express, for example, $z = x + 2y + 3$ and substitute into $f(x,y,z)$, then the function $f(x,y,x+2y+3)$ becomes a function of two variables x and y :

$$\tilde{f}(x,y) = f(x,y,x+2y+3) = (x-1)^2 + (y-1)^2 + (x+2y+2)^2 + (x-2)^2 + (y-2)^2 + (x+2y+1)^2,$$

$$\tilde{f}_x = 2(x-1) + 2(x+2x+2) + 2(x-2) + 2(x+2y+1) = 8(x+y),$$
$$\tilde{f}_y = 2(y-1) + 4(x+2y+2) + 2(y-2) + 4(x+2y+1) = 4(2x+5y) + 6.$$

Set $\left.\begin{matrix} \tilde{f}_x = 0, \text{ then} \\ \tilde{f}_y = 0, \text{ so} \end{matrix}\right.$ $\left.\begin{matrix} x + y = 0 \Longrightarrow y = -x, \\ 4x + 10y = -3 \Longrightarrow -6x = -3, \end{matrix}\right\}$ $x_1 = \frac{1}{2},\ y_1 = -\frac{1}{2}.$

We therefore have $z_1 = \frac{5}{2}$, $P = \left[\frac{1}{2}; -\frac{1}{2}; \frac{5}{2}\right]$.

Further, $\left.\begin{matrix} \tilde{f}_{xx}(P) = 8 \\ \tilde{f}_{xy}(P) = 8 \\ \tilde{f}_{yy}(P) = 20 \end{matrix}\right\}$ thus $\Delta_2(P) = \begin{vmatrix} 8 & 8 \\ 8 & 20 \end{vmatrix} > 0, \Delta(P)\tilde{f}_{xx} > 0$

and $f(P) = \frac{27}{2}$ is a local minimum.

■

Theorem (Sufficient condition for the existence of absolute extrema of a function on a given set). If the function $f(x,y)$ is continuous on a nonempty set M that is bounded and closed, then f attains a maximum and a minimum on the set M.

A special case is the so-called **constrained extremum**, namely when $M = \{[x;y] \in \mathbb{E}_2 : g(x,y) = 0\}$, where g is a given function. Whenever possible, we express one variable from the constraint $g(x,y) = 0$ and substitute it into the given function f. Thus we obtain the problem of determining extrema of a function of one variable.

Procedure for computing absolute extrema.

Recall that M^0 is the interior and ∂M is the boundary of the set M.

Step 1: Verify that the assumptions of the theorem just stated are satisfied, thereby justifying the existence of absolute extrema.

Step 2: Determine all critical points X of the function f in M^0.

Step 3: Determine all points $X \in \partial M$ at which the function f may attain absolute extrema.

Step 4: Determine the values of the function f at all computed points. The greatest value equals $\max_M f$ and the smallest value equals $\min_M f$.

• Determine the **absolute (global)** extrema of the function $z = f(x, y)$ on the set M :

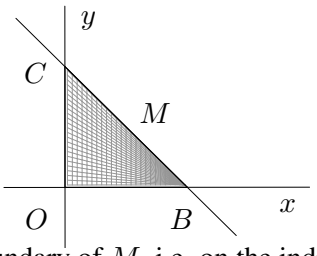
***Example* 207.** $f(x, y) = x^2 + y^2 + xy - x - y,\ M = \{[x; y] \in \mathbb{E}_2 : x \geq 0, y \geq 0,\ x + y \leq 1\}$

Solution : We verify the existence of absolute extrema: The given set M is closed and bounded. The given function f is continuous on $\mathbb{E}_2$, hence also on M. This is sufficient for the existence of absolute extrema.

1) determine stationary points in the interior of the set M and their function values:

$$\left.\begin{array}{l} f_x \equiv 2x + y - 1 = 0 \\ f_y \equiv 2y + x - 1 = 0 \end{array}\right\} \ x = y = \frac{1}{3} \Longrightarrow$$

$$A_1 = \left[\frac{1}{3}, \frac{1}{3}\right] \in M, \quad f(A_1) = -\frac{1}{3}\ ;$$



2) determine suspicious points on the boundary of M, i.e. on the individual sides of the triangle.

$$OB : y = 0 \Longrightarrow \quad h_1(x) = f(x, 0) = x^2 - x,\ h_1'(x) = 2x - 1 = 0 \quad \Longrightarrow x = \frac{1}{2},$$

$$A_2 = \left[\frac{1}{2}; 0\right] \in M,\ f(A_2) = -\frac{1}{4},$$

$$OC : x = 0 \Longrightarrow \quad h_2(y) = f(0, x) = y^2 - y,\ h_2'(y) = 2y - 1 = 0 \quad \Longrightarrow y = \frac{1}{2},$$

$$A_3 = \left[0; \frac{1}{2}\right] \in M,\ f(A_3) = -\frac{1}{4},$$

$$BC : y = 1 - x \Longrightarrow h_3(x) = f(x, 1 - x) = x^2 + (1 - x)^2 + x(1 - x) - x - 1 + x = -x^2 - x,$$

$$h_3'(x) = -2x - 1 = 0 \quad \Longrightarrow \quad x = -\frac{1}{2}, \qquad \left[-\frac{1}{2}; \frac{3}{2}\right] \notin M\ ;$$

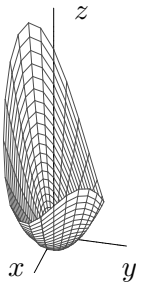
3) determine the values of the function $f(x, y)$ at the vertices O, B, C

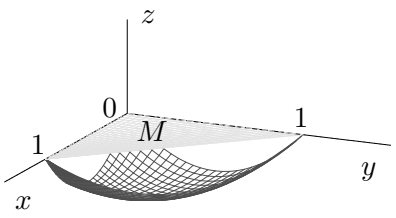
$$f(O) = f(0, 0) = 0, \quad f(B) = f(1, 0) = 0, \quad f(C) = f(0, 1) = 0.$$

From all the computed function values, choose the smallest and the largest.

The global maximum occurs at the points $O, B, C,\ f_{\max}(O) = f_{\max}(B) = f_{\max}(C) = 0$

and similarly the global minimum at the point $A_1,\ f_{\min}(A_1) = -\frac{1}{3}$.





■

***Example* 208.** $z = x^2 + y^2 - 6x + 6y,\ M = \{[x;y] \in \mathbb{E}_2 : x^2 + y^2 \le 4\}$

Solution : Geometrically, this means finding extrema of the z-coordinate on the paraboloid bounded by the intersection of the paraboloid $z = x^2 + y^2 - 6x + 6y$ with the cylindrical surface $x^2 + y^2 = 4$.

1) Stationary points in the interior of the set M:

$$\left.\begin{array}{l} z_x = 2x - 6 = 0 \Longrightarrow x_1 = 3 \\ z_y = 2y + 6 \qquad \Longrightarrow y_1 = -3 \end{array}\right. \quad \Longrightarrow \quad A = [3;-3], \quad A \notin M.$$

2) Points on the boundary of the set M are written in parametric form:

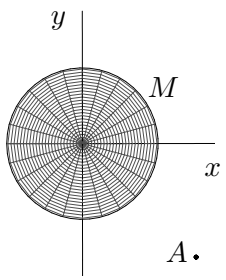
$$\left.\begin{array}{l} x = 2\cos t \\ y = 2\sin t \end{array}\right\}, \quad t \in \langle 0; 2\pi\rangle$$

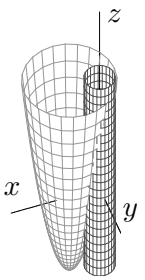
Then successively:

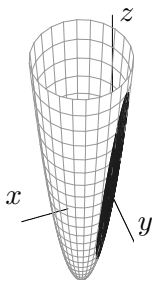
$$\tilde{z} = z(2\cos t, 2\sin t) = 4 - 12\cos t + 12\sin t,$$
$$\frac{d\,\tilde{z}}{d\,t} = 12\sin t + 12\cos t = 0,$$
$$\Longrightarrow \qquad \sin t = -\cos t, \qquad t_1 = \frac{3}{4}\pi, \quad t_2 = \frac{7}{4}\pi,$$

$t_1 = \dfrac{3}{4}\pi \Longrightarrow B = [-\sqrt{2};\sqrt{2}]$, $z(B) = 4 + 12\sqrt{2}$ is the global maximum and

$t_2 = \dfrac{7}{4}\pi \Longrightarrow C = [\sqrt{2};-\sqrt{2}]$, $z(C) = 4 - 12\sqrt{2}$ is the global minimum.







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• A function $f = f(x,y)$ is given.

a) State the necessary condition for a local extremum of a differentiable function of n variables at a point A.

b) State the sufficient conditions for a local minimum (respectively maximum) of the function $f(x,y)$ at the point A.

c) Investigate the local extrema of the given function f, i.e. determine their location, type, and function value.

209. $f(x,y) = x^2 + 12y^2 - 6xy + 4x$ [Strict local min. at the point $[-8;-2]$, $f(-8,-2) = -16$]

210. $f(x,y) = x^3 + y^2 - 12x + 6y$ [Strict local min. at the point$[2;-3]$, $f(2,-3) = -25$, *at* the point$[-2;-3]$ there is no extremum.]

211. $f(x,y) = 2y - y^2 - x\,\mathrm{e}^x$ [Strict local max. at the point $[-1;1]$, $f = 1 + 1/\mathrm{e}$]

212. $f(x,y) = x^3 + y^2 - 2x^2 - 2xy + 6$ [Strict local min. at the point$[2;2]$, $f(2,2) = 2$, *at* the point$[0;0]$ there is no extremum.]

213. $f(x,y) = x^3 + 3y^2 - 6xy - 9x + 8$ [Strict local min. at the point$[3;3]$, $f(3,3) = -19$, *at* the point$[-1;-1]$ there is no extremum.]

214. $f(x,y) = x^2 + 2x + y^4 - 4y + 7$ [Strict local min. at the point $[-1;1]$, $f(-1,1) = 3$]

215. $f(x,y) = xy + \frac{50}{x} + \frac{20}{y}$ [Strict local min. at the point $[5;2]$, $(f(5,2)=30)$]

216. $f(x,y) = x^2 + xy + y^2 - 6\ln x$ $\left[\begin{array}{r} f_{\min}(2,-1) = 3 - 6\ln 2, \\ \text{the point } [-2;1] \notin D(f) \end{array}\right]$

• Determine the local extrema of the function:

217. $z = x^2 + y - x\sqrt{y} - 6x + 12$ $[z_{\min}(4,4) = 0]$

218. $z = x^2 + y^2 + 6x - 4y$ $[z_{\min}(-3,2) = -13]$

219. $z = \ln(1 - x^2 - y^2)$ $[z_{\max}(0,0) = 0]$

220. $z = x^3 + 8y^3 - 6xy + 5$ $[z_{\min}\left(1, \frac{1}{2}\right) = 4,$ at the point $[0;0]$ there is no extremum]

221. $z = \mathrm{e}^{x/2}(x + y^2)$ $\left[z_{\min}(-2,0) = -\frac{2}{e}\right]$

222. $z = x^3 + y^3 + \frac{9}{2}x^2 - 3y - 12x$ $\left[\begin{array}{r} z_{\min}(1,1) = -\frac{17}{2},\ z_{\max}(-4,-1) = 58 \\ \text{at the points} [1;-1], [-4;1] \text{ there are no extrema} \end{array}\right]$

223.* $x^2 + y^2 + z^2 - 4x + 6y - 2z + 13 = 0$ $[z_{\min}(2,-3) = 0,\ z_{\max}(2,-3) = 2]$

224.* $x^2 + xy - z^2 + z + y + 5 = 0$ $\left[\begin{array}{r} \text{at the stationary points } [-1;2;3],\ [-1;2;-2] \\ \text{there are no extrema} \end{array}\right]$

225.* $f(x,y,z) = x^2 + y^2 + z^2 - 6x + 2y + 6z$ $[f(2,-1,-3)) = -14$, local min.]

226.* $f(x,y,z) = x^2 + 2y^2 + z^3 - 12yz - 6x$ $\left[\begin{array}{r} f(3,36,12) = -873, \text{local min.}, \\ \text{at the point } [3;0;0] \text{ there is no extremum} \end{array}\right]$

227.* $z = \ln(xy) + x^2 + y^2 - 4x - 4y + 7$

$$\left[\begin{array}{r} \text{local max. at the point } A_1 = \left[1 - \frac{\sqrt{2}}{2}; 1 - \frac{\sqrt{2}}{2}\right], \\ \text{local min. at the point } A_2 = \left[1 + \frac{\sqrt{2}}{2}; 1 + \frac{\sqrt{2}}{2}\right], \\ \text{other stationary points } A_3 = \left[1 + \frac{\sqrt{2}}{2}; 1 - \frac{\sqrt{2}}{2}\right],\ A_4 = \left[1 - \frac{\sqrt{2}}{2}; 1 + \frac{\sqrt{2}}{2}\right] \end{array}\right]$$

228. a) Investigate the local extrema of the function $f(x,y) = 2x^2 + y^2 - xy + 3x + y + 1$.
b) Justify the existence and find the absolute extrema of this function on the segment AB, where $A = [0;2], B = [1;1]$.

$\left[\begin{array}{r} \text{a) local } f_{\min}(-1,-1) = -1, \\ \text{b) } \mathrm{f}_{\min}(1/2, 3/2) = 6,\ \mathrm{f}_{\max}(0,2) = \mathrm{f}_{\max}(1,1) = 7 \end{array}\right]$

• A function f and a set M are given.
a) Justify the existence of absolute extrema of the function f on the given set M.
b) Investigate the absolute extrema, i.e. determine their location and compute the value of the maximum and minimum of the function f on the set M.

229. $f(x,y) = x + \ln x - y^2,\ \ M = \{\,[x;y] \in \mathbb{E}_2\,;\ y = x + 1,\ 1/4 \le x \le 1\}$
$\left[\begin{array}{r} f_{\min}(1,2) = -3,\ f_{\max}(1/2, 3/2) = -7/4 - \ln 2 \doteq -2,4, \\ f(1/4, 5/4) = -21/16 - \ln 4 \doteq -2,7 \text{ is not an extremum} \end{array}\right]$

230. $f(x,y) = x^2 + xy - 3x - y,\ \ M = \{\,[x;y] \in \mathbb{E}_2\,;\ x + y \le 3,\ x \ge 0,\ y \ge 0\,\}.$
$[f_{\min}(0,3) = -3,\ f_{\max}(0,0) = f_{\max}(3,0) = 0]$

231. $f(x,y) = x^2 - 2x + y^2,\ \ M = \{\,[x;y] \in \mathbb{E}_2\,;\ x^2 + y^2 \le 9,\ y \ge 0\,\}$

[$f_{\min}(1,0) = -1, \quad f_{\max}(-3,0) = 15$
To investigate boundary points, you may use
polar coordinates, but the problem can also be solved without them.]

• Determine the global extrema of the function $z = f(x,y)$ on the set M :

232. $f(x,y) = x^2 + y^2 - xy,\ M = \{[x;y] \in \mathbb{E}_2 : |x| + |y| \le 1\}$

[global min. $f(0,0) = 0$
global max. $f(1,0) = f(0,1) =$
$= f(-1,0) = f(0,-1) = 1$]

233. $f(x,y) = x^2 - y^2,\ M = \{[x;y] \in \mathbb{E}_2 : x^2 + y^2 \le 4\}$ (use polar coordinates)

[global min. $f(0,\pm 2) = -4$
global max. $f(\pm 2,0) = 4$]

234. $f(x,y) = xy(x-a)(y-b),\ M = \{[x;y] \in \mathbb{E}_2 : 0 \le x \le a,\ 0 \le y \le b\}$

[global max. $f(\frac{a}{2},\frac{b}{2}) = \frac{a^2b^2}{16}$
global min. $f(0,0) = f(a,0) = f(0,b) = f(a,b) = 0$]

235. $f(x,y) = xy(2-x-y),\ M = \{[x;y] \in \mathbb{E}_2 : x \ge 0,\ y \ge 0,\ x+y \le 1\}$

[global max. $f(\frac{1}{2},\frac{1}{2}) = \frac{1}{4}$
global min. $f(x,0) = f(0,y) = 0$]

236. $f(x,y) = x^2 + y^2 - xy + x + y - 1,\ M = \{[x;y] \in \mathbb{E}_2 : x \le 0,\ y \ge 0, x - y + 3 \ge 0\}$

[global min. $f(-\frac{1}{2},0) = -\frac{5}{4}$
global max. $f(0,3) = 11$]