Exercises - Differential Calculus

1. Determine the domains of definition and the ranges of the given functions.

a) $f(x,y) = \mathrm{e}^{16-x^2-y^2}$ b) $f(x,y) = \dfrac{1}{x(y+3)}$ c) $f(x,y) = \ln(\mathrm{e}^2 + x^2 + y^2)$

d) $f(x,y) = \sqrt[4]{y-x}$ e) $f(x,y) = \sqrt{y-x^2}$ f) $f(x,y) = \cos(3x^2 - 2y + 5)$

g) $f(x,y,z) = \sqrt{x^2+y^2+z^2-1}$ h) $f(x,y,z) = yz\ \ln x$

i) $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$ j) $f(x,y,z) = \arctan(x+y+z)$

2. Compute the following limits, if they exist.

a) $\lim\limits_{[x,y]\to[0,0]} \dfrac{3x^2-y^2+5}{x^2+y^2+2}$ b) $\lim\limits_{[x,y]\to[0,0]} \dfrac{x^4}{x^4+y^2}$ c) $\lim\limits_{[x,y]\to[0,0]} \dfrac{\mathrm{e}^y \sin x}{x}$

d)* $\lim\limits_{\substack{[x,y]\to[0,0]\\ x\neq 0}} \dfrac{\mathrm{e}^y\ \sin x}{x}$ e)* $\lim\limits_{\substack{[x,y]\to[2,2]\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2}$ f)* $\lim\limits_{\substack{[x,y]\to[0,0]\\ x\neq y}} \dfrac{x+y}{x-y}$

3. On which sets $M \subset \mathbb{E}_2$ are the following functions continuous?

a) $f(x,y) = \dfrac{x+y}{x-y}$ b) $f(x,y) = \dfrac{x^2+y^4+1}{x^2+x-12}$ c) $f(x,y) = \dfrac{1}{x^2-2y}$

d) $f(x,y) = \ln \dfrac{y}{x}$ e) $f(x,y) = \cos(x^2+xy)$ f) $f(x,y) = \mathrm{e}^{1/(x+y)}$

4. On which sets $M \subset \mathbb{E}_3$ are the following functions continuous?

a) $f(x,y,z) = \dfrac{1}{x^2+z^2-4}$ b) $f(x,y,z) = \ln(xyz)$ c) $f(x,y,z) = \mathrm{e}^z \sin(x+y)$

d) $f(x,y,z) = \dfrac{x+y}{x-y}$ e) $f(x,y,z) = \ln \dfrac{1}{xyz}$ f) $f(x,y,z) = \dfrac{1}{|xy|+|z|}$

g) $f(x,y,z) = \dfrac{1}{\ln\sqrt{x^2+y^2+z^2}}$ h) $f(x,y,z) = \dfrac{y+4}{x^2y-xy+4x^2-4x}$

5. Compute the partial derivatives with respect to x and y.

a) $f(x,y) = x^2 - 7xy$ b) $f(x,y) = (x+2)^2 y$ c) $f(x,y) = x^2(3y-5)^7$

d) $f(x,y) = x\ \sin(xy)$ e) $f(x,y) = \ln(x^2 y)$ f) $f(x,y) = \dfrac{2x}{x - \sin\ y}$

g) $f(x,y) = \dfrac{x+y}{x-y}$ h) $f(x,y) = \ln(x^2-2y)$ i) $f(x,y) = \sqrt{x^2+y^2}$

j) $f(x,y) = \mathrm{e}^x\ \ln\ y$ k) $f(x,y) = \dfrac{1}{\tan(y/x)}$ l) $f(x,y) = y\,\mathrm{e}^{x^2 y}$

6. Compute the partial derivatives with respect to x, y, and z.

a) $f(x,y,z) = \dfrac{x^5y^2}{z^3}$ b) $f(x,y,z) = x - \sqrt{y^2+z^2}$

c) $f(x,y,z) = \arctan(x+y+z)$ d) $f(x,y,z) = xy + yz + zx$

e) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ f) $f(x,y,z) = \dfrac{1}{\sqrt{(x^2+y^2+z^2)}}$

g) $f(x,y,z) = x^2 \sin^2 y \cos z^2$ h) $f(x,y,z) = \dfrac{x^2}{\sqrt{y^2+z^2}}$

7. Compute the second-order partial derivatives of the following functions.

a) $f(x,y) = x^2y + \cos y + y \sin x$ b) $f(x,y) = x\,\mathrm{e}^y + y + x^5y^4 - 13$

c) $f(x,y) = \mathrm{e}^{x+3y} + x \ln y + y \ln x + 3$ d) $f(x,y) = y + x^2y + 4y^3x - \ln(y^2+x)$

e) $f(x,y) = y^2 + y(\sin x - x^4)$ f) $f(x,y) = x^2 + 5xy + \sin(xy) + x\,\mathrm{e}^{y^2/2}$

8. Compute the gradient of the function f at the point A and the directional derivative $\partial f/\partial \mathbf{s}(A)$.

a) $f(x,y) = x^2 + 2xy - 3y^2$, $A = [1,1]$, $\mathbf{s} = (3,4)$

b) $f(x,y,z) = x^2 + 2y^2 - 3z^3 - 17$, $A = [1,1,1]$, $\mathbf{s} = (1,1,1)$

c) $f(x,y,z) = \cos(xy) + \mathrm{e}^{yz} + \ln(zx)$, $A = [1,0,0.5]$, $\mathbf{s} = (1,2,2)$

9. Write the equation of the tangent plane and the normal line at the point $[x_0, y_0, f(x_0,y_0)]$ to the surface which is the graph of the function $z = f(x,y)$.

a) $f(x,y) = 2x + y^2$, $x_0 = 1$, $y_0 = 2$ b) $f(x,y) = \sin(x-y)$, $x_0 = 0$, $y_0 = \pi$

c) $f(x,y) = \sqrt{x} - y$, $x_0 = 4$, $y_0 = 1$ d) $f(x,y) = \ln(x+y)$, $x_0 = 1$, $y_0 = 0$

10. Write the equation of the tangent plane and the normal line at the point A to the surface given by the equation $f(x,y,z) = 0$.

a) $f(x,y,z) = x^2 + y^2 + z^2 - 3 = 0$, $A = [1,1,1]$

b) $f(x,y,z) = \cos(\pi x) - x^2y + \mathrm{e}^{xz} + yz - 4 = 0$, $A = [0,1,2]$

11. Investigate the local extrema of the given functions.

a) $f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y$ b) $f(x,y) = x^2 + xy + 3x + 2y + 5$

c) $f(x,y) = 5xy - 7x^2 + 3x - 6y + 2$ d) $f(x,y) = x^2 - 4xy + y^2 + 6y + 2$

e) $f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$ f) $f(x,y) = x^2 - y^2 - 2x + 4y + 6$

g) $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$ h) $f(x,y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$

i) $f(x,y) = 4xy - x^4 - y^4 - 11$ j) $f(x,y) = x^4 + y^4 + 4xy + 7$

12. Investigate the existence, location, and values of the absolute extrema of the given functions on the specified sets.

a) $f(x,y) = 2x^2 - 4x + y^2 - 4y + 2, \quad M = \{[x,y];\ x \geq 0,\ y \leq 2,\ y \geq 2x\}$

b) $f(x,y) = x^2 - xy + y^2 + 7, \quad M = \{[x,y];\ x \geq 0,\ y \leq 4,\ y \geq x\}$

c) $f(x,y) = x^2 + xy + y^2 - 6x + 2, \quad M = \{[x,y];\ 0 \leq x \leq 5,\ -3 \leq y \leq 3\}$

d) $f(x,y) = x^2 + xy + y^2 - 6x, \quad M = \{[x,y];\ 0 \leq x \leq 5,\ -3 \leq y \leq 0\}$

e) $f(x,y) = 48xy - 32x^3 - 24y^2, \quad M = \{[x,y];\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$

f) $f(x,y) = x^2 - y^2, \quad M = \{[x,y];\ x \geq -1,\ y \geq -1,\ x + 2y \leq 2\}$

13. Show that the equation $F(x,y) = 0$ implicitly defines a function $y = f(x)$ in a neighborhood of the point $[x_0, y_0] \in \mathbb{E}_2$. Compute the derivative of this function at x_0 and write the equation of the tangent line to its graph at the point $[x_0, y_0]$.

a) $F(x,y) = x^3 + y^3 - 6xy + 4, \quad [x_0, y_0] = [1,1]$

b) $F(x,y) = x^2 - 3xy + 5xy^2 - 6y^2 - 32, \quad [x_0, y_0] = [2,3]$

c) $F(x,y) = x\,\mathrm{e}^{y/x} - \mathrm{e}^5, \quad [x_0, y_0] = [1,5]$

d) $F(x,y) = \arctan(2x + y), \quad [x_0, y_0] = [0,0]$

14. Show that the equation $F(x,y,z) = 0$ implicitly defines a function $z = f(x,y)$ in a neighborhood of the point $[x_0, y_0, z_0] \in \mathbb{E}_3$. Compute its partial derivatives at the point $[x_0, y_0]$ and write the equation of the tangent plane to its graph at the point $[x_0, y_0, z_0]$.

a) $F(x,y,z) = z^3 - xy + yz + y^3 - 2 = 0, \quad [x_0, y_0, z_0] = [1,1,1]$

b) $F(x,y,z) = x^2 - 2y^2 + z^2 - 4x + 2z - 5 = 0, \quad [x_0, y_0, z_0] = \left[-1,\ \sqrt{\tfrac{3}{2}},\ 1\right]$

c) $F(x,y,z) = xz^2 - x^2y + y^2z + 2x - y = 0, \quad [x_0, y_0, z_0] = [0,1,1]$

d) $F(x,y,z) = \sin(x+y) + \sin(y+z) + \sin(x+z) = 0, \quad [x_0, y_0, z_0] = [\pi, \pi, \pi]$