Mathematics II A – Exam 1

1. Given real function of two variables: $f(x,y) = \mathrm{e}^{-xy}\cos y$

a) Compute the gradient of the function f at point $T = [\pi;\ 0]$.
What is the geometrical interpretation of the vector $\mathrm{grad} f(T)$?

b) Find the directional derivative of the function f at point T in the direction given by the vector $\vec{s} = (2;3)$. Describe the behavior of the function in this direction.
(i.e. is the function decreasing or increasing, how fast?)

c) Write the formula for the differential $\mathrm{d}z$ of the function $z = f(x,y)$ at point $T = [\pi;\ 0]$.

d) Does the function f has a local extremum at the point T? Justify your answer.

2. The function $z = f(x,y)$ is given in the proximity of the point $P = [1;1;1]$ by the implicit formula $F(x,y,z) = xyz + x^2 + y^2 + z^2 - 4 = 0$

a) Compute the (first order) partial derivatives of the function $f(x,y)$ at the point $[1;1]$.

b) Write the equation of the plane tangent to the surface $F(x,y,z) = 0$ at the point P.

c) Use the equation of the tangent plane to find the approximate value of $f(1.2;0.9)$.

d) Find the (position of) critical points of the function $f(x,y)$ (where $\mathrm{grad} f = \vec{0}$).

3. a) Write the Fubini theorem for the double integral.
(complete theorem, including the assumptions, notation and the statement)

b) Compute the double integral $\displaystyle\iint_\Omega \sin x \; dxdy$ where
$\Omega = \{[x;y] \in \mathbb{E}_2 : 0 \le x \le \pi; x \le y \le 2x\}$.

c) Write (one) possible physical (or geometrical) interpretation of the integral from b).

4. Solid body $M \subset \mathbb{E}_3$ is bounded by the (conical) surface $z = \sqrt{x^2+y^2}$ and sphere $x^2+y^2+z^2 = 1$. It has constant density $\rho(x,y,z) = 1$.

a) Use the triple integral (in spherical coordinates) to compute the mass m of the body M.

b) Compute the static moment of the body M with respect to xy plane, i.e. compute m_{xy}.

c) Compute the z-coordinate of the center of mass of the body M, i.e. find z_{CM}.

5. Given vector field: $\vec{f}(x,y) = \left(2x\mathrm{e}^{-y}; 2y - x^2\mathrm{e}^{-y}\right)$.

a) Using the sufficient condition(s) verify that the given vector field $\vec{f}(x,y)$ is potential (conservative) in $\mathbb{E}_2$.

b) Compute its potential $\varphi(x,y)$. Verify (from definition) that the obtained scalar field $\varphi(x,y)$ is a potential of the given vector field $\vec{f}(x,y)$.

c) Compute the line integral $\int \vec{f} \cdot d\vec{s}$ from point $P = [1;0]$ to point $Q = [2;1]$.

6. Surface $\sigma = \left\{[x;y,z] \in \mathbb{E}_3 : z = \sqrt{x^2+y^2}; 1 \le z \le 3\right\}$ is oriented by a normal vector $\vec{n}_\sigma$ that has negative last (third, z) component.

a) Sketch the surface σ including its orientation. Suggest a suitable parametrization $P(u,v)$ of σ and determine, if the given surface is oriented in agreement with this parametrization.

b) Compute the surface integral (of a vector function) $\displaystyle\iint_\sigma (-x;-y;z^3)\cdot d\vec{p}$.

Mathematics II B – Exam 1

1. Given real function of two variables: $f(x,y) = x^3 + y^3 - 3x^2 - 3y + 10$

a) Compute the partial derivatives of the first and second order of the given function f.

b) Find the local extrema of the function f, i.e. find their position, type and value.

c) Find the directional derivative of the function f at point $[0;0]$ in the direction given by the vector $\vec{s} = (2;3)$. Describe the behavior of the function in this direction (i.e. is the function decreasing or increasing, how fast?)

2. Function $y = f(x)$ is in the neighborhood of the point $P = [\pi;0]$ given implicitly by the relation $F(x,y) = \sin(x+y) - y^2 \cos x = 0$.

a) Compute the partial derivatives (of the first order) of the function $F(x,y)$.

b) Compute the value of the derivative $y' = f'(\pi)$
(i.e. find the slope of the tangent to the curve $F(x,y) = 0$ at point P).

c) Write the equation of the tangent to the graph of the function $y = f(x)$ at the point P.

3. a) Sketch the region $D = \{[x;y] \in \mathbb{E}_2 : \ 1 \le x^2 + y^2 \le 4; x \ge 0; y \ge x\}$.

b) Transform the domain D into polar coordinates
(i.e. write the transformation formulas and corresponding bounds for variables).

c) Compute the double integral $\displaystyle\iint_D \frac{1}{(x^2+y^2)^{3/2}}\, dxdy$

4. a) Compute the triple integral $\displaystyle\iiint_\Omega 6xy \; dxdydz$ where

$$\Omega = \{[x;y,z] \in \mathbb{E}_3 : 0 \le x \le 1; 0 \le y \le \sqrt{x}; 0 \le z \le 1 + x + y\}$$

b) Write (one) possible physical (or geometrical) interpretation of the integral from a).

5. Given vector function: $\vec{f}(x,y) = \left(2y^{3/2}; 3x\sqrt{y}\right)$ and oriented curve $C \subset \mathbb{E}_2$ defined by the graph of the function $y = x^2$ starting at point $P = [1;?]$ and ending at $Q = [2;?]$.

a) Sketch the curve C and write down its parametrization $P(t)$.

b) Use the parametrization $P(t)$ to compute the line integral $\int_C \vec{f} \cdot d\vec{s}$.

c) The given vector field $\vec{f}(x,y)$ is potential (do not verify). Compute its potential $\varphi(x,y)$.

d) Use the potential $\varphi(x,y)$ to compute the same integral as in b), i.e. the integral $\int_P^Q \vec{f} \cdot d\vec{s}$.

6. a) Compute the surface integral (of a scalar function) $\displaystyle\iint_\sigma y \; dp$ where

$$\sigma = \left\{[x;y,z] \in \mathbb{E}_3 : 0 \le x \le 1; 0 \le y \le 2; z = x + y^2\right\}$$

b) Write (one) possible physical (or geometrical) interpretation of the integral from a).