Mathematics II A – Exam 1

1. Given real function of two variables: $f(x, y) = y^2 + xy - x^2$
and domain $\Omega = \{[x; y] \in \mathbb{E}_2 : 0 \leq x \leq 2; 0 \leq y \leq 2\}$.

a) Find the *local* extrema of the given function f, i.e. find their position, type and value.

b) Find the *absolute (global)* extrema of the given function f on the set Ω.

2. Given function $F(x, y) = \ln(xy) - 2x$

a) Verify, that by $F(x, y) = 0$ is in the neighborhood of the point $T = [1; e^2]$ implicitly defined function $y = f(x)$ which has continuous first and second derivative.
(i.e. write down and verify all the assumptions of the corresponding theorem)

b) Write the equation of the tangent to the graph of the function $y = f(x)$ at the point T.

c) Compute $f''(1)$ and decide if the function $f(x)$ is convex or concave (concave up or down) at $x = 1$.

d) Write down the Taylor's polynomial $T_2(x)$ of the function $f(x)$ centered at $x = 1$.

3. a) Write down the Green's theorem. *(complete, including the assumptions, notation and the statement)*

b) Sketch the domain $D = \{[x; y] \in \mathbb{E}_2 : 1 \leq x \leq 2; 1 \leq y \leq x^2\}$ and the curve $C = +\partial D$ which is a positively oriented boundary of the domain D.

c) Compute the circulation $\oint_C \vec{f} \cdot d\vec{s}$, where $\vec{f} = \left(\frac{x}{y}, 2 + 3x\right)$.

4. The solid body $B \subset \mathbb{E}_3$ (part of a ball) is defined as
$$B = \{[x; y; z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 \leq 4; z \geq 1\}.$$

a) Transform the solid B into *cylindrical coordinates*.
(i.e. write down the transformation formulas and determine the bounds for the transformed variables)

b) Use definition to compute the Jacobi matrix and Jacobian of the transformation from a).

c) Compute the volume of the body B.

5. a) Use the sufficient conditions to verify that the vector field
$\vec{f}(x, y) = \left(y e^x + \sin y; e^x + x \cos y\right)$ is potential (in $\mathbb{E}_2$).

b) Compute the potential $\varphi(x, y)$ of the above given vector field $\vec{f}(x, y)$.

c) Given scalar field $\psi(x, y, z) = xy^2 + y e^{3z}$.
Compute $\vec{g}(x, y, z) = \operatorname{grad} \psi(x, y, z)$ and $\operatorname{curl} \vec{g}(x, y, z)$.

d) Is the vector field $\vec{g}(x, y, z)$ potential (conservative)? Why?

6. Let Q be the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the xy-plane ($z = 0$).

a) Compute the surface area (using surface integral of scalar function) of the (elliptical) part σ of the boundary of the body Q lying in the plane $x + z = 6$.

b) Use the Gauss (divergence) theorem to compute the flux of the vector field
$\vec{f}(x, y, z) = (x^2 + \sin z; xy + \cos z; e^y)$ through the surface ∂Q (boundary of Q) oriented by the outward normal, i.e. compute the $\oiint_{\partial Q} \vec{f} \cdot d\vec{p}$.

Mathematics II B – Exam 1

1. a) Find the local extrema of the function $f(x,y) = \ln x + 2\ln y - x - 4y$.

(i.e. find the position of critical points, determine their type and compute the function value)

b) Write down and sketch the domain $D(f) \subset \mathbb{E}_2$, where the function f is defined.

2. Function $y = f(x)$ is in the neighborhood of the point $P = [8; 2]$ defined implicitly by the relation $F(x,y) = x + \sqrt{xy} - 2y - 8 = 0$.

a) Compute the partial derivatives (of the first order) of the function $F(x,y)$.

b) Compute the value of the derivative $y' = f'(8)$
(i.e. find the slope of the tangent to the curve $F(x,y) = 0$ at point P).

c) Write the equation of the tangent to the graph of the function $y = f(x)$ at the point P.

3. a) Sketch the region $\Omega = \left\{[x;y] \in \mathbb{E}_2 : \ -1 \le x \le 1; 0 \le y \le x^2\right\}$.

b) Compute the double integral $\displaystyle\iint_\Omega \left(x^2 + \sqrt{y}\right) dx dy$

c) Find (one) possible physical (or geometrical) interpretation of the integral from b).

4. Given solid body $M = \left\{[x;y,z] \in \mathbb{E}_3 : x^2 + y^2 \le 25; 0 \le z \le \mathrm{e}^{-x^2-y^2}\right\}$

a) Transform the given body M into cylindrical coordinates.

(i.e. write down the transformation formulas and determine the bounds for the transformed variables)

b) Compute the volume of the body M.

5. The curve $C = +\partial D$ is a positively oriented boundary of the unit disc $D = \left\{[x;y] \in \mathbb{E}_2 : x^2 + y^2 \le 1\right\}$.

a) Find out, if the vector field $\vec{f}(x,y) = (xy^2; x)$ is potential (conservative).

b) Compute the circulation $\displaystyle\oint_C \vec{f} \cdot d\vec{s}$.

Hint: You can, but don't have to, use the Green's theorem and polar coordinates.

6. Given surface $\sigma = \left\{[x;y,z] \in \mathbb{E}_3 : z = \dfrac{1}{2}(6 - 2x - y); x \ge 0; y \ge 0; z \ge 0\right\}$

a) Sketch the given surface σ.

b) Compute the surface integral (of a scalar function) $\displaystyle\iint_\sigma (y^2 + 2yz)\, dp$.

c) Write (one) possible physical interpretation of the integral from b).