Mathematics II A – Exam 2

1. Given real function of two variables: $f(x, y) = x^2 + y^2 - 2x - 4y$
and domain $D = \{[x; y] \in \mathbb{E}_2 : x \geq 0, y \geq x, y \leq 3\}$.

a) *Justify the existence* of absolute (global) extrema of the given function f on the set D.
(Use the sufficient conditions theorem on the existence of global extrema.)

b) *Find* the absolute (global) extrema of the given function f on the set D.

2. Given function $F(x, y) = y + \ln y + x^3$

a) Verify, that by $F(x, y) = 0$ is in the neighborhood of the point $T = [-1; 1]$ implicitly defined function $y = f(x)$ which has continuous first and second derivative.
(i.e. write down and verify all the assumptions of the corresponding theorem)

b) Write the equation of the tangent to the graph of the function $y = f(x)$ at the point T.

c) Compute $f''(-1)$ and decide if the function $f(x)$ is convex or concave (concave up or down) at $x = -1$.

d) Write down the Taylor's polynomial $T_2(x)$ of the function $f(x)$ centered at $x = -1$.

3. Given solid body $M \subset \mathbb{E}_3$ bounded by surfaces
$z = 0$, $z = 2 + \frac{1}{y}$, $y = 1$, $y = x$ and $y = 8 - x$.

a) Sketch the projection of the body M into the xy plane, i.e. sketch the region
$M_{xy} = \{[x; y] \in \mathbb{E}_2 : y \geq 1; y \leq x; y \leq 8 - x\}$.
Write this domain M_{xy} as an elementary domain of integration with respect to y axis.

b) Compute the volume of the given solid body M.

4. The solid body $B \subset \mathbb{E}_3$ (elliptic cylinder) is defined as
$B = \{[x; y; z] \in \mathbb{E}_3 : 16x^2 + 9y^2 \leq 144; -1 \leq z \leq 3\}$.

a) Transform the solid B into *generalized cylindrical coordinates*.
(i.e. write down the transformation formulas and determine the bounds for the transformed variables)

b) Use definition to compute the Jacobi matrix and Jacobian of the transformation from a).

c) Compute the integral $\iiint_B z^2 \sqrt{16x^2 + 9y^2}\, dxdydz$.

5. Given vector field $\vec{f}(x, y) = \left(x+y; x^2-y\right)$ and domain $\Omega = \{[x; y] \in \mathbb{E}_2 : x^2 \leq y \leq \sqrt{x}\}$.

a) Compute the integral $\oint_C \vec{f} \cdot ds$ where the closed curve $C = +\partial\Omega$ is positively oriented boundary of Ω.

b) Write down the Green's theorem (including all assumptions, notation and the statement).

c) Is the given vector field $\vec{f}(x, y)$ potential (conservative)? Why?

6. A three-dimensional thin plate has a shape of the surface
$Q = \left\{[x; y, z] \in \mathbb{E}_3 : x^2 + y^2 = 4; 0 \leq z \leq 4\right\}$
and (surface) density prescribed by $\rho(x, y, z) = \mathrm{e}^{-z}$.

a) Sketch the surface Q and suggest a suitable parametrization $P(u, v)$ of Q including the domain for parameters u and v.

b) Compute the mass m of the above described thin plate.

Mathematics II B – Exam 2

1. a) Find the local extrema of the function $f(x, y) = \mathrm{e}^x - x\mathrm{e}^y$.

(i.e. find the position of critical points, determine their type and compute the function value)

b) Compute the partial derivatives (of the first order) of the function $g(x, y) = (x + 3y)\mathrm{e}^{y-x^2}$.

2. Given real function of two variables $f(x, y) = \ln(4x^2 - y^2)$

a) Find the equation of tangent plane to the graph of the function $f(x, y)$ at point $[1; 1; ?]$.

b) Verify (by computation) that for given function f holds $\dfrac{\partial^2 f}{\partial x^2} = 4\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

c) Sketch the domain $D(f) \subset \mathbb{E}_2$, where the function f is defined.

3. Given region $\Omega = \left\{[x; y] \in \mathbb{E}_2 : \frac{1}{x} \le y \le \sqrt{x}; x \le 3\right\}$.

a) Sketch the domain Ω and determine the lower bound for variable x.

b) Compute the double integral $\displaystyle\iint_\Omega x^2 y \, dxdy$

c) Find two possible physical (or geometrical) interpretations of the integral from b).

4. a) Sketch the domain $M = \left\{[x; y, z] \in \mathbb{E}_3 : x^2 + y^2 \le 4; 1 \le z \le 5\right\}$ and transform it into cylindrical coordinates.

(i.e. write the transformation formulas, find the bounds for transformed variables)

b) Compute the triple integral $\displaystyle\iiint_M z\sqrt{x^2 + y^2} \, dxdydz$.

5. The curve $C \subset \mathbb{E}_2$ is defined by the graph of the function $y = x^3$ for $0 \le x \le 1$.

a) Sketch the curve C and write down its parametrization $P(t)$.
Find the tangent vector $\dot{P}(t)$ and its length $\|\dot{P}(t)\|$.

b) Compute the mass m of an object having the shape of the curve C along the which the density is distributed as $\rho(x, y) = \sqrt{1 + 9xy}$.

c) Given scalar potential $\varphi(x, y) = x^2 y + 2y^4$ of a vector field $\vec{f}(x, y)$.
Compute the vector field $\vec{f}(x, y)$ (from the given $\varphi(x, y)$ and definition of potential).

6. Oriented surface $Q \subset \mathbb{E}_3$ is defined by the parametric equation $X = P(u, v)$, where the parametrization is given by $P(u, v) = (u^2 - v; u; v^2)$ for $0 \le u \le 2$ and $-1 \le v \le 3$. Surface Q is oriented by a normal vector $\vec{n}_Q$ that has negative third (i.e. z) component.

a) Find out if the given surface Q is oriented in agreement or disagreement with the parametrization $P(u, v)$.

b) Compute the surface integral $\displaystyle\iint_Q (z; 0; y) \cdot d\vec{p}$.