Mathematics II A – Exam 2

1. Given real function of two variables: $f(x,y) = \mathrm{e}^y\left(y^2 - x^2\right)$.

a) Compute the partial derivatives of the first and second order of the given function $f(x,y)$.

b) Find the local extrema of the function f. *(Determine their position, type and function value.)*

2. The function $z = f(x,y)$ is given in the proximity of the point $P = [0;0;0]$ by the implicit formula $F(x,y,z) = x\mathrm{e}^y + y\mathrm{e}^z + z\mathrm{e}^x = 0$.

a) Compute the (first order) partial derivatives of the function $f(x,y)$ at the point $[0;0]$.

b) Write the equation of the plane tangent to the graph of the function $f(x,y)$ at the point P.

c) Use the equation of the tangent plane to find the approximate value of $f(0.2;-0.1)$.

d) Find the value of the directional derivative of function $f(x,y)$, at point $[0;0]$ in the direction defined by the vector $\vec{v} = (2;-1)$.

3. a) Write the Fubini theorem for the double integral.

(Complete theorem, including the assumptions, notation and the statement.)

b) Sketch the region $M = \left\{[x;y] \in \mathbb{E}_2 : x \geq 0; y \geq x; y \leq 1\right\}$.
Write this domain M as an elementary domain of integration with respect to y axis.

c) Compute the double integral $\iint_M \sin\left(y^2\right)\,dxdy$. *(Hint: Use the result from b).)*

4. Compute the volume of solid body $\Omega \subset \mathbb{E}_3$ bounded by the paraboloids
$z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$.

(Hint: Find the intersection of the two paraboloids. Use transformation to cylindrical or polar coordinates.)

5. Given vector field $\vec{f}(x,y) = \left(x^2 + \sin y; (x+1)\cos y\right)$.

a) Verify that the given vector field $\vec{f}(x,y)$ is potential (conservative) in $\mathbb{E}_2$.

b) Compute the scalar potential $\varphi(x,y)$ of the given vector field $\vec{f}$.

c) Find the work done by the given force $\vec{f}(x,y)$ acting from the point $M = [0;\pi/3]$ to $N = [1;\pi/4]$.

d) Compute the *divergence* and *curl* of the vector field $\vec{q} = \left(z^2 \sin y; xz^2 \cos y; 2xz \sin y\right)$.

6. Given surface $Q \in \mathbb{E}_3$ defined by its parametrization $P(u,v) = (u+v; u-v; 1+2u+v)$, where $u \in \langle 0;2\rangle$ and $v \in \langle 0;1\rangle$. *(Surface Q is oriented in agreement with $P(u,v)$.)*

a) Compute the surface integral (of scalar function) $\iint_Q (x+y+z)\,dp$.

b) Compute the surface integral (of vector function) $\iint_Q \left(z\mathrm{e}^{xy}; -3z\mathrm{e}^{xy}; xy\right) \cdot d\vec{p}$.

Mathematics II B – Exam 2

1. a) Find the equation of the tangent plane to the graph of the function $f(x,y) = \dfrac{x}{\sqrt{y}}$ at the point $T = [4; 4; ?]$.

b) Given function (of three variables) $g(x,y,z) = z\mathrm{e}^{2x+3y}$ and point $P = [3; -2; 4]$. Compute the gradient of the function g at point P and find the value of the directional derivative of function g, at point P in the direction defined by the vector $\vec{v} = (2; -1; -3)$.

2. a) Find the local extrema of the function $f(x,y) = x^2 + xy + \dfrac{1}{2}y^2 - 2x + y$.
(Determine their position, type and function value.)

b) Verify (by computation) that for $h(x,y) = \sqrt[3]{x^2 + y^4 + 8}$ holds $x\,\dfrac{\partial h}{\partial y} = 2y^3\dfrac{\partial h}{\partial x}$.

3. Given region (elliptical segment) $D = \big\{[x; y] \in \mathbb{E}_2 :\ 4x^2 + y^2 \leq 4; x \geq 0; y \geq 0\big\}$.

a) Compute mass of a flat plate having the shape of D and density defined by $\rho(x,y) = xy^3$.

b) Determine the surface area of the domain D, i.e. compute $A = \displaystyle\iint_D 1\,dxdy$.

4. Solid body $W \subset \mathbb{E}_3$ is defined as a bounded region between the the planes $z = x + y$ and $z = 3x + 5y$, lying over the the rectangle $W_{xy} = \big\{[x; y] \in \mathbb{E}_2 : x \in \langle 0; 3\rangle; y \in \langle 0; 2\rangle\big\}$.

a) Compute the triple integral $\displaystyle\iiint_W z\,dxdydz$.

b) What could be the physical interpretation of the integral from a)?

5. The curve $C \subset \mathbb{E}_3$ is defined by the parametrization $X = P(t)$:

$P(t) = (\cos t; \sin t; t)$, where $t \in \langle 0; \pi\rangle$. *(Curve C is oriented in agreement with $P(t)$.)*

a) Compute the line integral (of scalar function) $\displaystyle\int_C (xy + z)\,ds$.

b) Compute the line integral (of vector function) $\displaystyle\int_C (x; y; z)\cdot d\vec{s}$.

6. Surface $S \subset \mathbb{E}_3$ is defined as a part of the graph of the function $z = 8 - x - 2y$ lying in the first octant.

a) Compute the surface area of the surface S.

b) Compute the surface integral $\displaystyle\iint_S \mathrm{e}^{-z}\,dp$.