Mathematics II A – Exam 3

1. Given function of two variables: $f(x,y) = \left(x^2 + y^2\right)\mathrm{e}^{-x}$

a) Compute the partial derivatives of the first and second order of the given function $f(x,y)$.

b) Find the critical points of the function f. *(Determine their position, type and function value.)*

2. Given function $f(x,y) = 6 - x^2y - \ln(xy)$ and point $A = [4; 1/4]$.

a) Write the equation of the tangent plane and normal line to the graph of the function $z = f(x,y)$, at the point $P = [4; 1/4; ?]$.

b) Find the direction $\vec{s}$ of the steepest descent of the function f at the point A. How fast is the function decreasing in this direction (i.e. find the slope of the tangent)?

c) Write the formula for the differential $\mathrm{d}f(A)$ of the given function f at point A.

d) Write down and sketch the domain $D \subset \mathbb{E}_2$, where the function $f(x,y)$ is differentiable. *Use the theorem on the sufficient conditions of differentiability.*

3. Domain $\Omega \subset \mathbb{E}_2$ is defined as elementary domain of integration with respect to y axis by $\Omega = \{[x;y] \in \mathbb{E}_2 : 0 \le y \le 4; \sqrt{y} \le x \le 2\}$.

a) Rewrite the given domain Ω as elementary domain of integration with respect to x axis.

b) Compute the mass of a thin flat plate having the shape of Ω where the density ρ is defined by the function $\rho(x,y) = \sqrt{x^3 + 1}$. *(Hint: use the result from a).)*

4. Solid body M is given by $M = \left\{[x; y, z] \in \mathbb{E}_3 : x^2 + y^2 \le 4; 0 \le z \le y\right\}$.

a) Sketch the body M and its projection M_{xy} into the xy plane. *(Hint: note that $y \ge 0$.)* Transform the body M into cylindrical coordinates. *(i.e. write the transformation formulas, find the bounds for transformed variables)*

b) Compute the triple integral $\iiint_M z \, dxdydz$.

c) Write (one) possible physical (or geometrical) interpretation of the integral from a).

5. The curve $C \subset \mathbb{E}_3$ is given by its parametrization $P(t) = (e^t; e^t; t)$, where $t \in \langle -1; 1\rangle$ (C is oriented in agreement with its parametrization P).

a) Compute the work done by the force field $\vec{f}(x,y,z) = \left(\frac{3z}{y}; 4x; -y\right)$ acting along the curve C.

b) Write down the necessary condition for the vector field $\vec{f}(x,y,z)$ to be potential (conservative). Verify if this condition is satisfied for the given vector field $\vec{f}$ from a).

6. a) Compute the surface integral (of a scalar function) $\iint_\sigma (x^2z + y^2z)\, dp$, where the surface σ is a hemisphere given by $\sigma = \left\{[x; y, z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 = 4; z \ge 0\right\}$ *(Hint: use the parametrization based on spherical coordinates)*

b) Write two possible physical interpretations of the integral from a).

Mathematics II B – Exam 3

1. Given function of two variables: $f(x,y) = -x^3 + 4xy - 2y^2 + 1$

a) Compute the partial derivatives of the first and second order of the given function $f(x,y)$.

b) Find the critical points of the function f. *(Determine their position, type and function value.)*

2. Given function $f(x,y) = \sqrt{1 + xy^2}$

a) Compute the gradient of the function $f(x,y)$ at point $A = [2; -2]$.

b) Write down the equation of the tangent plane to the graph of the given function $f(x,y)$ at point $P = [2; -2; ?]$.

c) Use the equation of the tangent plane to find the approximate value of $f(2.5; -2.2)$.

3. a) Sketch the region $\Omega \subset \mathbb{E}_2$ that is bounded by the lines $x = 0$, $x = 3 - y$ and $x = y/2$.

b) Compute the double integral $\iint_\Omega 2xy\, dxdy$.

c) Write two possible physical (or geometrical) interpretations of the integral from b).

4. Body B is defined by $B = \{[x; y, z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 \leq 1; x \geq 0; y \geq 0; z \geq 0\}$.

a) Transform the body B into spherical coordinates.

(i.e. write the transformation formulas, find the bounds for transformed variables)

b) Compute the volume of B, i.e. compute $V = \iiint_B 1\, dxdydz$.

5. Given vector field $\vec{f}(x,y) = \left(2xy + y^3; x^2 + 3xy^2 + 2y\right)$.

a) Compute the line integral of $\vec{f}(x,y)$ over the oriented line segment C_1 from the point $M = [1; 1]$ to the point $N = [1; 4]$.

b) Find the circulation of the given field $\vec{f}(x,y)$ over a negatively oriented circle $C_2 = \{[x; y] \in \mathbb{E}_2 : x^2 + y^2 = 5\}$.

c) Is any of the following functions φ_i the potential of the above given vector field $\vec{f}$? Justify your answers!

i) $\varphi_1(x,y) = y + x^5 y + xy^3$

ii) $\varphi_2(x,y) = x^2 y + xy^3 + y^2 + 3$

iii) $\varphi_3(x,y) = xy^3 + y^2 + x^2 y$

6. Surface $Q \subset \mathbb{E}_3$ is defined as a graph of the function $z = g(x,y) = x\mathrm{e}^y$ for $x \in \langle 0; 1\rangle$ and $y \in \langle 0; 1\rangle$. Surface Q is oriented by a normal vector $\vec{n}_Q$ that has positive first (i.e. x) component.

a) Chose a suitable parametrization of the given surface Q. Find out if the surface Q is oriented in agreement or disagreement with the chosen parametrization.

b) Compute the surface integral (of a vector function) $\iint_Q (xy; 4x^2; yz) \cdot d\vec{p}$.