Mathematics II A – Exam 3

1. Given real function of two variables: $f(x,y) = \ln(1+xy)$

a) Compute the gradient of the function f at point $T = [1;\ 2]$.
What is the geometrical interpretation of the vector $\mathrm{grad} f(T)$?

b) Find the directional derivative of the function f at point T in the direction given by the vector $\vec{s} = (3;1)$.

c) Write the formula for the differential $\mathrm{d}z$ of the function $z = f(x,y)$ at point $T = [1;\ 2]$.

d) Find the direction $\vec{u}$ in which the function f (at point T) doesn't changes its value.
What's the relation of this direction $\vec{u}$ to the $\mathrm{grad} f(T)$?

2. Given function $f(x,y) = x^2 + y^2 - y$ and domain (parabolic arc)
$L = \{[x;y] \in \mathbb{E}_2 : y = x^2; -1 \le x \le 1\}$.

a) *Justify the existence* of absolute (global) extrema of the given function f on the set L.
(Use the sufficient conditions theorem on the existence of global extrema.)

b) *Find* the absolute (global) extrema of the given function f on the set L.

3. a) Write the Green's theorem. *(complete theorem, including the assumptions, notation and the statement)*

b) Verify the assumptions of the Green's theorem for calculation of circulation of the vector field $\vec{f} = \left(\arctan x + y^2; \mathrm{e}^y - x^2\right)$ along the positively oriented curve $C = +\partial D$ enclosing the (annular) region $D = \{[x;y] \in \mathbb{E}_2 : 1 \le x^2 + y^2 \le 9; y \ge 0\}$.

c) Compute the line integral $\oint_C \vec{f} \cdot d\vec{s}$.

4. Body $M \subset \mathbb{E}_3$ is the region between the spheres of radius 1 and 2 centered at the origin.

a) Write the domain M in cartesian and also in spherical coordinates.
(including the bounds for all coordinates)

b) Use the spherical coordinates to compute $\displaystyle\iiint_M \frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\, dxdydz.$

5. Curve $C \subset \mathbb{E}_3$ is defined by its parametrization $X = P(t) = (t+1; \mathrm{e}^t; t^2)$ for $0 \le t \le 2$.
(C is oriented in agreement with $P(t)$.)

a) Calculate the work W performed moving a particle along the curve C in the presence of a force field $\vec{f}(x,y,z) = \left(z; y^2; x\right)$.

b) Compute $\mathrm{curl}\vec{f}$. Is the vector field $\vec{f}(x,y,z)$ potential (conservative)?

6. Surface (part of a plane inside a cylinder) is defined by

$$\sigma = \left\{[x;y,z] \in \mathbb{E}_3 : z = 6 - y; x^2 + y^2 \le 4\right\}$$

a) Compute the surface integral (of a scalar function) $\displaystyle\iint_\sigma xyz\, dp$.

b) Can the integral from a) express the mass of a thin plate having the shape of the given surface σ? Justify your answer.

Mathematics II B – Exam 3

1. Given real function of two variables: $f(x,y) = x^2 + \dfrac{y^4}{2} - 4xy$

a) Compute the partial derivatives of the first and second order of the given function f.

b) Find the local extrema of the function f, i.e. find their position, type and value.

c) Find the directional derivative of the function f at point $A = [3;2]$ in the direction given by the vector $\vec{u} = (3;4)$. Describe the behavior of the function in this direction (i.e. is the function decreasing or increasing, how fast?)

2. Function $y = f(x)$ is in the neighborhood of the point $P = [4;9]$ given implicitly by the relation $F(x,y) = \sqrt{x} + \sqrt{xy} + \sqrt{y} - 11 = 0$.

a) Compute the partial derivatives (of the first order) of the function $F(x,y)$ at point P.

b) Compute the value of the derivative $y' = f'(4)$
(i.e. find the slope of the tangent to the curve $F(x,y) = 0$ at point P).

c) Write the equation of the tangent to the graph of the function $y = f(x)$ at the point P.

3. a) Sketch the region $D = \{[x;y] \in \mathbb{E}_2 :\ 2 \le x^2 + y^2 \le 4; y \ge x\}$.

b) Transform the domain D into polar coordinates
(i.e. write the transformation formulas and corresponding bounds for variables).

c) Compute the double integral $\displaystyle\iint_D (xy + 10)\, dxdy$

4. Compute the volume of the solid body $B \subset \mathbb{E}_3$ bounded by the surfaces $z = 1$ and $z = \mathrm{e}^y$ over the rectangle $B_{xy} = \left\{[x;y] \in \mathbb{E}_2 : 0 \le x \le 1; 0 \le y \le \ln 2\right\}$

5. Given vector function: $\vec{f}(x,y) = \left(2xy + \dfrac{1}{y^2}; x^2 - \dfrac{2x}{y^3}\right)$ and oriented straight line segment $C \subset \mathbb{E}_2$ starting at point $M = [1;2]$ and ending at $N = [3;4]$.

a) Sketch the curve C and write down its parametrization $P(t)$.

b) Verify (the sufficient conditions) that the given vector field $\vec{f}(x,y)$ is potential (conservative) in $G = \{[x;y] \in \mathbb{E}_2 : y > 0\}$.

c) Compute the potential function $\varphi(x,y)$ of the vector field $\vec{f}(x,y)$.

d) Use the potential $\varphi(x,y)$ to compute the integral $\int_M^N \vec{f} \cdot d\vec{s}$.

6. a) Compute the surface integral (of a scalar function) $\displaystyle\iint_\sigma \frac{xy^2}{\sqrt{x^2+y^2+1}}\, dp$,

where $\quad \sigma = \left\{[x;y,z] \in \mathbb{E}_3 : 0 \le x \le 1; -x \le y \le x; z = xy\right\}$.

b) Write (one) possible physical (or geometrical) interpretation of the integral from a).