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**Contents**

# Introduction

The presented text approximately coincides with the contents of the Mathematics II course, taught at the Faculty of Mechanical Engineering in the second term. It deals with multi-variable calculus (i.e. with differential calculus of functions of more variables and with multiple, line and surface integrals). Functions of more variables appear more often in science than functions of one variable and their analysis leads to a large variety of applications. Since the problems we deal with have a more-dimensional character, their understanding requires not only a computational skill, but also a good space imagination. We therefore consider the text not as an independent textbook, but as a complementary material to the lectures and exercises where all the topics will be explained and commented in detail.

The text contains many well known theorems of applied mathematics, like the Green theorem, the Gauss–Ostrogradsky theorem, the Stokes theorem, etc. The conclusions of these theorems are certain integral formulas. The students often identify the theorems with these formulas. However, you should be aware that the important parts of all theorems are also their assumptions. It would be naive to think that the conclusive formulas hold in all cases. The opposite is true – they hold only in certain specific situations. The assumptions of the theorems represent the brief and simplest description of these situations and they are as important as the conclusions of the theorems.

Each chapter contains the section "Exercises" at the end. Many further exercises and solved examples can be found in the textbooks [1] and [3].

The authors wish to express their thanks to Mr. Robin Healey for carefully reading the text and correcting the language. If you still find some misprints or incorrect formulations in the text then they are only the authors who are responsible.

We believe that this text will be a useful study aid not only for students who attend the lectures and the exercises in English, but also for all other students who study in Czech.

# I. Functions of Several Real Variables

Functions of several real variables very often appear in mathematics and in science. You already know many formulas which can be understood as definitions of functions of several variables. For example, the formula $V = \pi r^2 h$, which determines the volume $V$ of a circular cylinder from its radius $r$ and height $h$ can be taken as the definition of function $V$, which depends on two real variables $r > 0, h > 0$.

## I.1. Euclidean space $\mathbf{E}_n$.

In this section, we will recall some notions that you know from the Mathematics I course. We will deal here with Euclidean space $\mathbf{E}_n$, subsets of $\mathbf{E}_n$ and its properties.

**I.1.1. $n$-dimensional arithmetic space.** If $n$ is a natural number (we use the notation: $n \in \mathbf{N}$) then the set of all ordered $n$–tuples of real numbers is denoted by $\mathbf{R}^n$. Let us define the sum of any two elements $[x_1, x_2, ..., x_n]$, $[y_1, y_2, ..., y_n]$ from $\mathbf{R}^n$ by the formula:

$$[x_1, x_2, ..., x_n] + [y_1, y_2, ..., y_n] = [x_1 + y_1, x_2 + y_2, ..., x_n + y_n]$$

and the product of any element $[x_1, x_2, ..., x_n]$ from $\mathbf{R}^n$ and any real number $\lambda$ by the formula

$$\lambda \cdot [x_1, x_2, ..., x_n] = [\lambda x_1, \lambda x_2, ..., \lambda x_n].$$

The set $\mathbf{R}^n$ with these two operations is called the *n-dimensional arithmetic space*. Its elements are called *arithmetic vectors*.

**I.1.2. Euclidean space $\mathbf{E}_n$ – definition.** Let us define the distance $\rho$ of any two elements $X = [x_1, x_2, ..., x_n]$, $Y = [y_1, y_2, ..., y_n]$ from $\mathbf{R}^n$ by the formula

$$\rho(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}.$$

The set $\mathbf{R}^n$ with this distance $\rho$ defined for all pairs of elements of $\mathbf{R}^n$ is called *n-dimensional Euclidean space*. This is denoted by $\mathbf{E}_n$.

**I.1.3. Zero element of $\mathbf{E}_n$.** The point $[0, 0, ..., 0]$ is called the *zero element of $\mathbf{E}_n$* or the *origin* of $\mathbf{E}_n$. The zero element is denoted $\mathcal{O}$.

**I.1.4. Remark.** Elements from $\mathbf{E}_n$ are often called *points*, because $\mathbf{E}_1$ can be imagined as a straight line, $\mathbf{E}_2$ as a plane, etc.

The distance between the zero element $\mathcal{O}$ and an arbitrary point $X$ of $\mathbf{E}_n$ is denoted $|X|$, i.e.

$$|X| = \rho(\mathcal{O}, X).$$

From this it follows that the distance between $X, Y \in \mathbf{E}_n$ can be expressed in the following way

$$\rho(X, Y) = |X - Y|,$$



where the difference $X - Y$ is understood in the sense of $n$-dimensional arithmetic space as $X + (-1) \cdot Y$.

In the following paragraphs, we will define some properties of subsets of $\mathbf{E}_n$, which play an important role in particular in definitions of continuity and limits of functions.

**I.1.5. Neighbourhoods in $\mathbf{E}_n$.** If $X \in \mathbf{E}_n$, then a *neighbourhood* of the point $X$ is any subset $\{Y \in \mathbf{E}_n : \rho(X, Y) < \varepsilon\}$ where $\varepsilon > 0$. The neighbourhood is denoted by $U_\varepsilon(X)$ or simply $U(X)$.

A *reduced neighbourhood* of the point $X \in \mathbf{E}_n$ is every set of the type $\{Y \in \mathbf{E}_n : 0 < \rho(X, Y) < \varepsilon\}$ where $\varepsilon > 0$, i.e. $U_\varepsilon(X) - X$. This neighbourhood will be denoted by $R_\varepsilon(X)$ or only $R(X)$.

**I.1.6. Interior point.** Let $M \subset \mathbf{E}_n$. A point $X \in M$ is called an *interior point of M* if there exists a neighbourhood $U(X)$ such that $U(X) \subset M$.

**I.1.7. Accumulation point.** Let $M \subset \mathbf{E}_n$. A point $X \in \mathbf{E}_n$ is called an *accumulation point of M* or a *point of accumulation of M* if in every reduced neighbourhood $R(X)$ there exists at least one point $Y$ which belongs to $M$.

**I.1.8. Remark.** If you read this definition carefully, you can see that if $X$ is an accumulation point of $M$ then in every neighbourhood $U(X)$ there exist an infinite number of points which belong to $M$.

From the definition it also follows that even if $X$ is an accumulation point of $M$ then it is possible: $X \notin M$.

**I.1.9. Isolated point.** Let $M \subset \mathbf{E}_n$. A point $X \in M$ is called an *isolated point of M* if in some reduced neighbourhood $R(X)$ there is no point which belongs to $M$.

**I.1.10. Remark.** The following assertion holds:

If $M$ is a subset in $\mathbf{E}_n$, $X$ is any point of $M$ and $X$ is not an isolated point of $M$, then $X$ is an accumulation point of $M$.

If $M$ is a subset in $\mathbf{E}_n$, $X$ is any point of $M$ and $X$ is not an accumulation point of $M$, then $X$ is an isolated point of $M$.

**I.1.11. Boundary point.** A point $X \in \mathbf{E}_n$ is called a *boundary point of a subset M* if in every $U(X)$ there exists at least one point which belongs to $M$ and at least one point which does not belong to $M$.

**I.1.12. Open set.** A subset $M$ of $\mathbf{E}_n$ is called an *open set in $\mathbf{E}_n$* (or shortly: an *open set*), if every point $X \in M$ is an interior point of $M$.

**I.1.13. Examples.** See Fig. 1. The following sets are open in $\mathbf{E}_2$. Sketch the fourth one:

$$\emptyset, \qquad \mathbf{E}_2, \qquad \{X \in \mathbf{E}_2 : \rho(\mathcal{O}, X) < 5\},$$

$$\{[x,y] \in \mathbf{E}_2 : \rho([1,0],[x,y]) < 1\} \cup \{[x,y] \in \mathbf{E}_2 : x \in (-2;-1)\}$$

**I.1.14. Closure of a set.** Let $M \subset \mathbf{E}_n$. A *closure of subset M in* $\mathbf{E}_n$ (or shortly: a *closure of M*) is the name given to the union of $M$ with the set of all accumulation points of $M$. The closure of subset $M$ in $\mathbf{E}_n$ is denoted by $\overline{M}$.

**I.1.15. Closed set.** A subset $M$ of $\mathbf{E}_n$ is called a *closed set in* $\mathbf{E}_n$ (or shortly: a *closed set*), if $\overline{M} = M$.

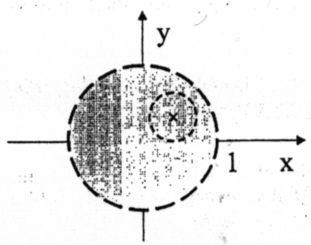


Fig. 1.

Each point of the set $\{X \in \mathbf{E}_2 : \rho(X,\mathcal{O}) < 1\}$ is an interior point, i.e. this set is open.

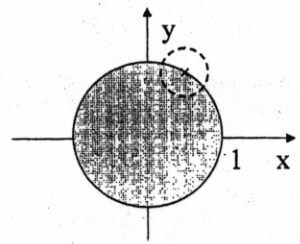


Fig. 2.

The set $\{X \in \mathbf{E}_2 : \rho(X,\mathcal{O}) \le 1\}$ contains each its boundary point, i.e. this set is closed.

**I.1.16. Examples.** The following sets are closed in $\mathbf{E}_2$:

$$\emptyset, \qquad \mathbf{E}_2, \qquad \{X \in \mathbf{E}_2 : \rho(\mathcal{O},X) \le 5\},$$

$$\{[x,y] \in \mathbf{E}_2 : \rho([1,2],[x,y]) \le 1\} \cup \{[x,y] \in \mathbf{E}_2 : x \in [-2;-1], y \in [0;1]\}$$

**I.1.17. Remark.** We can prove that the complement of an open set in $\mathbf{E}_n$ is a closed set. and vice versa.

**I.1.18. Boundary of a set.** Let $M \subset \mathbf{E}_n$. A *boundary of M* is the name given to a set of all boundary points of $M$. The boundary of $M$ is denoted by $\partial M$.

**I.1.19. Examples.**

The boundary of $\{X \in \mathbf{E}_2 : \rho(\mathcal{O},X) \le 5\}$ *is* $\{X \in \mathbf{E}_2 : \rho(\mathcal{O},X) = 5\}$

The boundary of $\{X \in \mathbf{E}_2 : \rho(\mathcal{O},X) < 5\}$ *is* $\{X \in \mathbf{E}_2 : \rho(\mathcal{O},X) = 5\}$

The boundary of $\{[x,y] \in \mathbf{E}_2 : x \in (-2;-1)\}$ is $\{[x,y] \in \mathbf{E}_2 : x = -2 \vee x = -1\}$.

**I.1.20. Remark.** We can prove that the boundary of an arbitrary set $M$ in $\mathbf{E}_n$ is a closed set. We can also prove that $\overline{M} = M \cup \partial M$, see Fig. 2.



**I.1.21. Line segment in $\mathbf{E}_n$.** Let $A, B \in \mathbf{E}_n$ and $A \ne B$. The set of points $X$ such that $X = A + t(B-A)$, $t \in [0;1]$ is called the *line segment in* $\mathbf{E}_n$ and it is denoted it by $\overline{AB}$.

**I.1.22. Remark.** The formula $X = A + t(B-A)$ should be understood in the sense of $n$-dimensional arithmetic space as $X = A + t \cdot (B + (-1) \cdot A)$.

**I.1.23. Polygonal line in $\mathbf{E}_n$.** Let $A_1, A_2, \dots A_r \in \mathbf{E}_n$, $r$ be a natural number $r \ge 2$ and $A_i \ne A_{i+1}, i = 1, 2, \dots, r-1$. The union of the line segments

$$\overline{A_1A_2} \cup \overline{A_2A_3} \cup \dots \cup \overline{A_{r-1}A_r}$$

is called a *polygonal line connecting points* $A_1, A_r$.

**I.1.24. Domain in $\mathbf{E}_n$.** Let $D$ be an arbitrary open set in $\mathbf{E}_n$. If for an arbitrary pair of points of $D$ there exists a polygonal line connecting these two points and entirely belonging to D, then D is called a *domain in* $\mathbf{E}_n$.

**I.1.25. Examples.**

The set $\{X \in \mathbf{E}_2 : \rho(\mathcal{O},X) < 5\}$ is a domain in $\mathbf{E}_2$.

The set $\{X \in \mathbf{E}_2 : \rho(\mathcal{O},X) \le 5\}$ is not a domain in $\mathbf{E}_2$.

The set $\{X \in \mathbf{E}_2 : \rho(\mathcal{O},X) < 5\} \cup \{X \in \mathbf{E}_2 : \rho([-10,0],X) < 5\}$ is not a domain in $\mathbf{E}_2$.

The set $\{X \in \mathbf{E}_2 : \rho(\mathcal{O},X) < 5\} \cup \{[5,0]\}$ is not a domain in $\mathbf{E}_2$.

**I.1.26. Bounded set in $\mathbf{E}_n$.** A subset $M$ of $E_n$ is called *bounded* if there exists $r > 0$ such that $\forall X \in M : \rho(\mathcal{O},X) \le r$.

**I.1.27. Examples.**

The set $\{[x,y] \in \mathbf{E}_2 : x \in (-2;-1)\}$ is not bounded.

The set $M = \{[x,y] \in \mathbf{E}_2 : x \in (-2;-1),\ y \in [0;1]\}$ is bounded, because it holds for instance $\forall X \in M : \rho(\mathcal{O},X) \le 3$.

**I.1.28. Limit of a sequence in $\mathbf{E}_n$.** The element $A \in \mathbf{E}_n$ is called the *limit of a sequence* $\{A^{(i)}\}$, $A^{(i)} \in \mathbf{E}_n$, for $i = 1, 2, \dots$ if

$$\forall U_\varepsilon(A)\ \exists n_0 \in \mathbf{N}\ \ \forall i \in \mathbf{N}: \ (i \ge n_0) \Rightarrow A^{(i)} \in U_\varepsilon(A).$$

(We read it: For every neighbourhood $U_\varepsilon(A)$ of the point $A$ there exists $n_0 \in \mathbf{N}$ so that for all $i \in \mathbf{N}$ it holds: If $i \ge n_0$, then $A^{(i)} \in U_\varepsilon(A)$. The fact that $A$ is the limit of the sequence $\{A^{(i)}\}$ is written down in this way: $\lim A^{(i)} = A$ or $A^{(i)} \to A$. We also say that the sequence $\{A^{(i)}\}$ is convergent or the sequence $\{A^{(i)}\}$ converges to point $A$.

**I.1.29. Remark.** The definition of the limit of a sequence in $\mathbf{E}_n$ uses the notion of a neighbourhood of a point in $\mathbf{E}_n$ but formally it is very similar to the definition of a limit of a sequence in $\mathbf{R}^*$, see [4], III.1.4.

On the other hand, the definition of the limit can be overwritten in the following way:

$$\forall \varepsilon > 0 \ \exists n_0 \in \mathbf{N} \ \forall i \in \mathbf{N}: \ (i \geq n_0) \Rightarrow \rho(A^{(i)}, A) < \varepsilon$$

This means that the limit of a sequence in $\mathbf{E}_n$ can also be defined in another way: The element $A \in \mathbf{E}_n$ is called the limit of a sequence $\{A^{(i)}\}$, $A^{(i)} \in \mathbf{E}_n$, for $i = 1, 2, \dots$ if the sequence of real numbers $\{\rho(A^{(i)}, A)\}$ converges to $0 \in \mathbf{R}$.

**I.1.30. Theorem.** *Every sequence in $\mathbf{E}_n$ has at most one limit.*

The proof is analogous to the proof of Theorem III.1.8, see [4]. The neighbourhoods in $\mathbf{R}^*$ must be replaced by the neighbourhoods in $\mathbf{E}_n$, but the scheme of the proof is the same.

**I.1.31. Remark.** Note that $A^{(i)} = [a_1^{(i)}, a_2^{(i)}, \dots, a_n^{(i)}]$. The question is what is the relation between the convergency of the sequence $\{A^{(i)}\}$ and the convergency of sequences $\left\{a_1^{(i)}\right\}$, $\left\{a_2^{(i)}\right\}$, ... $\left\{a_n^{(i)}\right\}$. The next theorem shows that this relation is very natural.

**I.1.32. Theorem.** *The sequence $\{A^{(i)}\}$, $A^{(i)} = [a_1^{(i)}, a_2^{(i)}, \dots, a_n^{(i)}] \in \mathbf{E}_n$, for $i = 1, 2, \dots$ converges to the point $A = [a_1, a_2, \dots a_n] \in \mathbf{E}_n$ if and only if every sequence $\left\{a_r^{(i)}\right\}$, converges to the number $a_r$, for $r = 1, 2, \dots, n$.*

**I.1.33. Remark.** The proof of this theorem is based on inequality:

$$|a_r^{(i)} - a_r| \leq \rho(A^{(i)}, A) \leq \sqrt{n} \cdot \max_{s \in \{1,2,\dots,n\}} |a_s^{(i)} - a_s| \quad \text{for } i = 1, 2, \dots$$

**I.1.34. Example.** Find the limit of the sequence $\{X^{(k)}\}$ in $\mathbf{E}_3$ where

$$X^{(k)} = \left[\frac{\sin(k)}{k}, \frac{k^2 - 7k}{6 - 5k - 2k^2}, \frac{4}{k}\right].$$

Solution: First, we find the limits of each coordinate:

$$\lim \frac{\sin(k)}{k} = 0, \quad \lim \frac{k^2 - 7k}{6 - 5k - 2k^2} = -\frac{1}{2}, \quad \lim \frac{4}{k} = 0$$

Further, using Theorem *I*.1.32 we get $\lim X^{(k)} = [0, -\frac{1}{2}, 0]$.

## I.2. Real functions of several real variables.

**I.2.1. Real functions of $n$ real variables – the definition.** If $M \subset \mathbf{E}_n$, $n \in \mathbf{N}$, then each mapping of $M$ to $\mathbf{E}_1$ is called a *real function of n real variables* (shortly: a *function*).



**I.2.2. Domain of definition, range, graph.** A function is a special case of a mapping and the notions "domain of definition of a mapping" and a "range of mapping" are known from secondary school. Hence, the notions "domain of definition of a function" (shortly: domain of a function) and "range of a function" (shortly: range of a function) can be regarded as known. In accordance with the notation which is used in connection with general mappings, $D(f)$ will be the domain of definition and $R(f)$ will be range of function $f$.

A *graph* of function $f$ of variables $x_1, x_2, \dots, x_n$ is the set

$$G(f) = \{[X, f(X)] \in \mathbf{R}^{n+1} : X = [x_1, x_2, \dots, x_n] \in D(f)\}$$

**I.2.3. Remark.** Let $f$ be a function of $n$ variables and let $X = [x_1, x_2, \dots, x_n]$ belong to its domain. The value of function $f$ is denoted by $f(X)$ or by $f(x_1, x_2, \dots, x_n)$.

**I.2.4. Example.**

Let $f$ be a function of two variables $x, y$, which is defined for all $[x, y] \in D(f) = [3; +\infty) \times \mathbf{R}$ by its function value: $f(x, y) = \sqrt{x - 3}$.

Let $g$ be a function of a single variable $x$, which is defined for all $x \in D(g) = [3; +\infty)$ by its function value: $g(x) = \sqrt{x - 3}$.

Although the values of functions are defined in both cases by the same formula, $f$ and $g$ are different functions with different domains of definition $D(f) \subset \mathbf{E}_2$, $D(g) \subset \mathbf{E}_1$.

**I.2.5. Operations with functions.** Let $f, g$ be functions of variables $x_1, x_2, \dots, x_n$, $D(f), D(g) \subset \mathbf{E}_n$. A *sum of functions* $f$ and $g$ is a function $h$ such that $h(X) = f(X) + g(X)$ for $X = [x_1, x_2, \dots, x_n] \in D(f) \cap D(g)$. Thus $D(h) = D(f) \cap D(g)$. We use the notation $h = f + g$.

We define a *difference of functions* and a *product of functions* $f$ and $g$ by analogy. A *quotient of functions* $f$ and $g$ can also be defined similarly – however, its domain is the set $[D(f) \cap D(g)] - \{X \in D(g) : g(X) = 0\}$.

**I.2.6. Restriction of a function.** Suppose that $f$ is a function and $M \subset D(f)$. A function which is defined only on $M$ and which assigns to each $X \in M$ the same value as function $f$ is called the *restriction* of function $f$ to set $M$, and it is denoted by $f|_M$. The set of all function values of function $f$ on set $M$ can be denoted by $R(f|_M)$ or by $f(M)$.

**I.2.7. Composite function.** We assume that function $f$ of $n$ variables $y_1, y_2, \dots, y_n$ is defined for each $Y = [y_1, y_2, \dots y_n] \in D \subset \mathbf{E}_n$ and functions $\phi_1, \phi_2, \dots, \phi_n$ of $m$ variables $x_1, x_2, \dots, x_m$ are defined for each $X = [x_1, x_2, \dots x_m] \in \Omega \subset \mathbf{E}_m$. Let

$$[\phi_1(X), \phi_2(X), \dots, \phi_n(X)] \in D \quad \text{for } X \in \Omega.$$

Then the function

$$F(X) = f(\phi_1(X), \phi_2(X), \dots, \phi_n(X))$$

defined for each $X \in \Omega$ is called a *composite* function.

**I.2.8. Remark.** We denote as $\phi$ the mapping defined by

$$\phi(X) = [\phi_1(X), \phi_2(X), ..., \phi_n(X)] \quad \text{for } X \in \Omega.$$

The mapping $\phi$ is called a *vector valued function of m variables*, with $D(\phi) \subset \mathbf{E}_m$ and $R(\phi) \subset \mathbf{E}_n$.

The fact that $F$ is defined as a composite function by the relation

$$F(X) = f(\phi_1(X), \phi_2(X), ..., \phi_n(X)) \quad \text{for } X \in \Omega$$

is denoted

$$F = f \circ \phi.$$

**I.2.9. A bounded function.** Function $f$ is called *bounded above* (or *upper bounded*) if there exists a number $K \in \mathbf{R}$ such that $\forall X \in D(f) : f(X) \leq K$. We can by analogy define the function *bounded below* (or *lower bounded*). Function $f$ is called *bounded* if it is bounded above and bounded below.

Assume further that $M \subset D(f)$. Function $f$ is called *bounded above on set M* if there exists a number $K \in \mathbf{R}$ such that $\forall X \in M : f(X) \leq K$. We can similarly define the function *bounded below on set M* and the notion of a function *bounded on set M*.

**I.2.10. Extremes of a function.** We say that function $f$ has its *maximum* at the point $A \in D(f)$ if $\forall X \in D(f) \; : \; f(A) \geq f(X)$. We write:

$$\max f = f(A).$$

Analogously, we can define the *minimum* of a function $f$. We denote it $\min f$.

Suppose that $M \subset D(f)$. We say, that function $f$ has its *maximum on set M at point $A \in M$* if $\forall X \in M \; : \; f(A) \geq f(X)$. We write: $\max_M f = f(A)$. Other often used notations of the maximum of function $f$ on a set $M$ are

$$\max_{M} f, \; \max_{X \in M} f(X).$$

Analogously, we can also define the *minimum of function f on set M*. We denote it:

$$\min_M f, \; \min_{M} f, \; \min_{X \in M} f(X).$$

The maxima and minima of function $f$ are called the *extremes* (or *extrema* ) of function $f$.

The maxima and minima of function $f$ on a set are called the *extremes on a set* of function $f$.

**I.2.11. Remark.** Obviously, the extremes of function $f$ are special cases of the extremes of $f$ on a set.



## I.3. Limits and continuity

**I.3.1. Limit of a function.** Assume that $C \in \mathbf{E}_n$, $y \in \mathbf{R}^*$ and the domain of definition of a function $f$ contains some reduced neighbourhood $R(C)$ of point $C$. If for each sequence $\{X^{(i)}\}$ in $R(C)$ the implication

$$\left\{X^{(i)}\right\} \to C \Rightarrow f(X^{(i)}) \to y$$

is true, then we say that function $f$ has the *limit at point C* equal to $y$. We write

$$\lim_{X \to C} = y.$$

Assume that $y \in \mathbf{R}^*$ and the domain of definition of a function $f$ contains the following set:

$$R_r(\infty) = \{X \in \mathbf{E}_n, \; |X| > r\} \quad \text{for some } r > 0$$

If for each sequence $\{X^{(i)}\}$ in $R_r(\infty)$ the implication

$$\left\{|X^{(i)}|\right\} \to +\infty \Rightarrow f(X^{(i)}) \to y$$

is true, then we say that function $f$ has the *limit at infinity* equal to $y$. We write

$$\lim_{|X| \to \infty} = y.$$

Further, we generalize the definition of the limit of a function at a point in order to be able to define a limit not only at point $X$ for which there exists a reduced neighbourhood $R(X)$ such that $R(X) \subset D(f)$.

**I.3.2. Limit of a function with respect to a set.** Assume that the domain of definition of function $f$ contains some $M \subset \mathbf{E}_n$, $C \in \mathbf{E}_n$ is an accumulation point of $M$, and $y \in \mathbf{R}^*$. If for each sequence $\{X^{(i)}\}$ in $M$ the implication

$$\left\{X^{(i)}\right\} \to C \Rightarrow f(X^{(i)}) \to y$$

is true, then we say that function $f$ has the *limit at point C with respect to set M* equal to $y$. We write

$$\lim_{\substack{X \in M \\ X \to C}} f(X) = y.$$

Assume that the domain of definition of function $f$ contains some $M \subset \mathbf{E}_n$, such that each set

$$R_r(\infty) = \{X \in \mathbf{E}_n, \; |X| > r\} \quad r > 0$$

contains at least one point of $M$, and assume that $y \in \mathbf{R}^*$. If for each sequence $\{X^{(i)}\}$ in $M$ the implication

$$\left\{|X^{(i)}|\right\} \to +\infty \Rightarrow f(X^{(i)}) \to y$$

is true, then we say that function $f$ has the *limit at infinity with respect to set M* equal to $y$. We write

$$\lim_{\substack{X \in M \\ |X| \to +\infty}} = y.$$

**I.3.3. Remark.**

From the definition it follows that if there exists $\lim_{X\to C} f(X)$ then there exists $\lim_{\substack{X\in M\\ X\to C}} f(X)$ for each $M \subset D(f)$ with accumulation point $C$ and

$$\lim_{\substack{X\in M\\ X\to C}} f(X) = \lim_{X\to C} f(X).$$

The next theorem is an easy consequence of Theorem III.1.8, see [4].

**I.3.4. Theorem.**

*Function $f$ can have at any point $C \in \mathbf{E}_n$ at most one limit.*
*Function $f$ can have at infinity at most one limit.*
*Function $f$ can have at any point $C \in \mathbf{E}_n$ at most one limit with respect to a set $M$.*
*Function $f$ can have at infinity at most one limit with respect to a set $M$.*

**I.3.5. Example.** Let $f$ be a function of two variables defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x\cdot y = 0 \\ 0 & \text{if } x\cdot y \neq 0 \end{cases} \quad \text{for } [x,y] \in \mathbf{E}_2$$

Prove that the function has no limit at point $[0,0]$.

Solution: Assume the sequence $\left\{\left[\frac{1}{n},\frac{1}{n}\right]\right\}$. This sequence converges to point $[0,0]$, see I.1.32. The function value $f\left(\frac{1}{n},\frac{1}{n}\right) = 0$ for all $n = 1,2,...,$ hence $f\left(\frac{1}{n},\frac{1}{n}\right) \to 0$.

Assume the sequence $\left\{\left[\frac{1}{n},0\right]\right\}$. This sequence also converges to point $[0,0]$. The function value $f\left(\frac{1}{n},0\right) = 1$ for all $n = 1,2,...,$ hence $f\left(\frac{1}{n},0\right) \to 1$.
Thus, the function has no limit at point $[0,0]$.

**I.3.6. Example.** Prove that function $f$ from the previous example has a limit at point $[0,0]$ with respect to the subset $M = \{[x,y] \in \mathbf{E}_2 : x > 0 \wedge y > 0\}$.

Solution: We can see that $[0,0]$ is an accumulation point of $M$. It is clear from the definition of $f$ that for all points $X \in M$ we get $f(X) = 0$. Hence, for *every* sequence $\{X^{(i)}\}$, $X^{(i)} \in M$ it holds $f(X^{(i)}) \to 0$. Thus, the function $f$ has limit 0 at point $[0,0]$ with respect to set $M$.

The following theorem is fully analogous with the theorem about the limit of the functions of one real variable. It concerns the limit of the sum, difference, product and quotient of two functions of several real variables, and it can easily be proved by means of Theorem III.2.13, see [4].

We use the symbol "#," which has the meaning of any of the symbols "+", "−", "·", "/" here.



**I.3.7. Theorem.** *Let $C \in \mathbf{E}_n$, $a,b \in \mathbf{R}^*$. Let $\lim_{X\to C} f(X) = a$, $\lim_{X\to C} g(X) = b$.*

*Then $\lim_{X\to C} [f(X)\#g(X)] = a\#b$ (if the expression has a sense).*

*Let $a,b \in \mathbf{R}^*$. Let $\lim_{|X|\to+\infty} f(X) = a$, $\lim_{|X|\to+\infty} g(X) = b$. Then*

$$\lim_{|X|\to+\infty} [f(X)\#g(X)] = a\#b \text{ (if the expression has a sense).}$$

**I.3.8. Continuity of a function at a point.** We say that function $f$ is *continuous at the point* $C \in D(f) \subset \mathbf{E}_n$ if

$$\lim_{X\to C} f(X) = f(C).$$

**I.3.9. Continuity of a function.** We say that function $f$ is continuous if $f$ is continuous at each point $C \in D(f)$.

**I.3.10. Remark.** If you read the definition *I*.3.8 and definition *I*.3.1 carefully, you will see that function $f$ can be continuous at point $C$ only if it is defined in some neighbourhood of $C$ (i.e. if $D(f)$ contains some neighbourhood $U(C)$. But this condition is satisfied for each point of $D(f)$ only if $D(f)$ is an open set. We will now study a more general situation.

**I.3.11. Continuity of a function at a point with respect to a set.** Let $M \subset D(f) \subset \mathbf{E}_n$ and $C$ be an accumulation point of $M$. We say that function $f$ is *continuous at point $C$ with respect to set $M$* if

$$\lim_{\substack{X\in M\\ X\to C}} f(X) = f(C).$$

Let $C$ be an isolated point of $M$. Then we also say that function $f$ is continuous at point $C$ with respect to set $M$.

**I.3.12. Continuity of a function on a set.** Let $M \subset D(f) \subset \mathbf{E}_n$. We say that function $f$ is *continuous on set $M$* if $f$ is continuous at each point $C \in M$ with respect to set $M$.

**I.3.13. Remark.** You can see from the definition that if a function $f$ is continuous on a set $M$ and $M_1 \subset M$ then function $f$ is continuous on set $M_1$.

**I.3.14. Theorem (on continuity of the sum, difference, product, quotient, and absolute value).** *If functions $f$ and $g$ are continuous at point $C$, then also the functions $f+g$, $f-g$, $f\cdot g$, and $|f|$ are continuous at $C$. If, in addition $g(C) \neq 0$ then the function $f/g$ is also continuous at point $C$.*

(This part of the theorem is also valid when we replace "continuity at point $C$" by "continuity at point $C$ with respect to set $M$".)

*If functions $f$ and $g$ are continuous on set $M$, then also the functions $f+g$, $f-g$, $f\cdot g$, and $|f|$ are continuous on set $M$. If, in addition $g(X)\neq 0$ for all $X\in M$ then the function $f/g$ is also continuous on set $M$.*

**I.3.15. Example.** Let $f$ be a function of two variables defined by

$$g(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2} & \text{if } [x,y]\neq[0,0]\\ 0 & \text{if } [x,y]=[0,0]\end{cases}\quad \text{for } [x,y]\in \mathbf{E}_2$$

a) Prove that function $g$ is continuous on $\mathbf{E}_2-[0,0]$.

b) Prove also that function $g$ is continuous on axis $x$, i.e. on the set $\{[x,y]\in\mathbf{E}_2 : x\in\mathbf{R},\ y=0\}$.

Solution:

a) From the previous theorem it follows that function $g$ is continuous at any point $[x,y]\neq[0,0]$. At point $[0,0]$ the value of function $g$ is defined, but $g$, we claim, has no limit at point $[0,0]$. We show the proof by contradiction. We suppose that there exists $\lim\limits_{X\to[0,0]} g(X)$. We denote $M=\{[x,y]\in\mathbf{E}_2 : y=mx\}$. Then for every $M$ we have

$$\lim_{\substack{X\in M\\ X\to[0,0]}} g(X)=\lim_{X\to[0,0]} g(X).$$

We calculate the limit at point $[0,0]$ with respect to each defined set $M$:

$$\lim_{\substack{X\in M\\ X\to[0,0]}} g(X)=\lim_{\substack{y=mx\\ [x,y]\to[0,0]}}\frac{2xy}{x^2+y^2}=\lim_{[x,mx]\to[0,0]}\frac{2xy}{x^2+y^2}=$$

$$=\lim_{x\to0}\frac{2xmx}{x^2+m^2x^2}=\frac{2m}{1+m^2}.$$

Because the value depends on $m$, we have a contradiction, see Remark *I.3.3*. Hence, the limit of $g$ at $[0,0]$ does not exist, and function $g$ is not continuous at $[0,0]$.

b) The set $\{[x,y]\in\mathbf{E}_2 : x\in\mathbf{R}\wedge y=0\}$ is a set $M$ from a) with $m=0$. From the same calculations (where now $m=0$) we get

$$\lim_{\substack{X\in M\\ X\to[0,0]}} g(X)=0.$$

We have g(0,0)=0, so $g$ is continuous at $[0,0]$ with respect to $\{[x,y] : x\in\mathbf{R}\wedge y=0\}$. (In other points is $g$ continuous.) Hence, $g$ is continuous on axis $x$.

**I.3.16. Theorem (on continuity of a composite function).** *We assume that function $f$ is continuous at point $B=[b_1,b_2,...b_n]$, functions $\phi_1,\phi_2,...,\phi_n$ are continuous at point $A=[a_1,a_2,...a_m]$ and $B=[\phi_1(A),\phi_2(A),...,\phi_n(A)]$. Let us denote $\phi=[\phi_1,\phi_2,...,\phi_n]$, i.e. $\phi$ is a vector valued function defined by coordinate functions $\phi_1,\phi_2,...,\phi_n$. Then composite function $F=f\circ\phi$ is continuous at point $A$.*

*We assume now that function $f$ is continuous on set $D$, functions $\phi_1,\phi_2,...,\phi_n$ are continuous on set $\Omega$ and $\phi(X)=[\phi_1(X),\phi_2(X),...,\phi_n(X)]\in D$ for $X\in\Omega$. Then the composite function $F=f\circ\phi$ is continuous on set $\Omega$.*



**I.3.17. Theorem (Darboux' property).** *If function $f$ is continuous on a domain $M$ and $A,B$ are any two points from $M$, then to any given number $Y$ between $f(A)$ and $f(B)$ and to any polygonal line $L\subset M$ connecting $A$, $B$ there exists a point $X\in L$ such that $f(X)=Y$.*

**I.3.18. Theorem.** *Function $f$, which is continuous on a bounded closed set $M$, has its maximum and minimum on this set $M$. (Thus, $\max\limits_{X\in M} f(X)$ and $\min\limits_{X\in M} f(X)$ exist.)*

## I.4. Partial derivatives, differentials.

When we hold all but one of the independent variables constant and derive with respect to that one variable, we get a partial derivative. For example, the partial derivative of a function $f(x,y)$ with respect to $x$ at point $[x_0,y_0]$ is the value of the derivative of the function of one real variable $f(x,y_0)$ at point $x_0$.

**I.4.1. Partial derivative of a function.** Let $f$ be a function of $n$ real variables $x_1,x_2,...,x_n$ and $A=[a_1,a_2,...,a_n]\in\mathbf{E}_n$. If there exists a finite limit

$$\lim_{h\to0}\frac{f(a_1,a_2,...,a_{k-1},a_k+h,a_{k+1},...,a_n)-f(a_1,a_2,...,a_{k-1},a_k,a_{k+1},...,a_n)}{h}$$

then its value is called the *partial derivative of f with respect to $x_k$ at point A*. It is denoted by

$$\frac{\partial f}{\partial x_k}(A)\quad or\quad \left.\frac{\partial f}{\partial x_k}\right|_A.$$

Let us assume the set of all points $A$ for which $\frac{\partial f}{\partial x_k}(A)$ exists. The function defined by its function value $\frac{\partial f}{\partial x_k}(A)$ in this set is called the *partial derivative of f with respect to $x_k$*. This function is denoted by

$$\frac{\partial f}{\partial x_k}.$$

From the definition it follows that

$$D\left(\frac{\partial f}{\partial x_k}\right)\subset D(f).$$

**I.4.2. Remark.** Let $g$ be a function of one real variable $x_k$ defined in the following way:

$$g(x_k)=f(a_1,a_2,...,a_{k-1},x_k,a_{k+1},...,a_n)$$

From the definition it follows that the partial derivative of $f$ with respect to $x_k$ at the point $A = [a_1, a_2, ..., a_n]$ is defined as the derivative of function $g$ at point $a_k$. Hence, to calculate the partial derivative with respect to $x_k$ we assume other variables to be constant and calculate the derivative of a function of one variable $x_k$. Thus, all theorems about calculation of derivatives also hold for partial derivatives.

Let now $f : f(x,y)$ then $\frac{\partial f}{\partial x}(a,b) = \left.\frac{d(f(x,b))}{dx}\right|_a = \tan\alpha$, see Fig. 3:

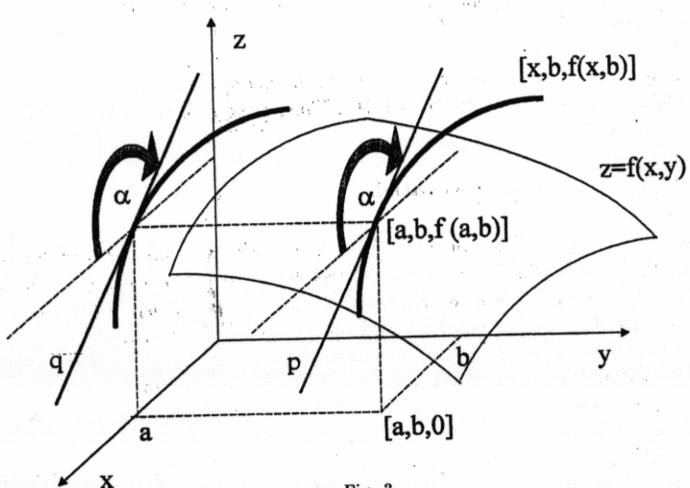


Fig. 3.

**I.4.3. Example.** We calculate the partial derivatives of function $f$ of three variables $x, y, z$ given by the formula $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $[x,y,z] \in \mathbf{E}_3$. Deriving the expression with respect to $x$ we regard $y$ and $z$ as constants and we get (using the formula about the derivatives of composite functions of one variable):

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{1}{2}\frac{1}{\sqrt{x^2+y^2+z^2}}2x = \frac{x}{\sqrt{x^2+y^2+z^2}}.$$

Analogously, we get

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial f}{\partial z}(x,y,z) = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

$D(f) = \mathbf{E}_3, \quad D(\frac{\partial f}{\partial x}) = D(\frac{\partial f}{\partial y}) = D(\frac{\partial f}{\partial z}) = \mathbf{E}_3 - \{[0,0,0]\}.$

**I.4.4. Remark.** Let us suppose that the function $f(x,y)$ has partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at point $[x_0, y_0]$. What can we say about the behaviour of the function in



the neighbourhood of point $[x_0, y_0]$? For example, is this function continuous at this point?

**I.4.5. Example.**

$$f(x,y) = \begin{cases} 1 & \text{if } x \cdot y = 0 \\ 0 & \text{if } x \cdot y \neq 0 \end{cases}$$

The partial derivatives at point $[0,0]$ exist, but the function is not continuous at $[0,0]$, see *I*.3.5. (Because $f(x,0) = f(0,y) = 1$ for $x, y \in \mathbf{R}$, we get $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.)

In the next paragraph we will solve the question of the functions that can be well approximated by a linear function in the neighbourhood of some point.

**I.4.6. Differentials.** Let function $f$ be defined in a neighbourhood $U(A)$ of point $A = [a_1, a_2, ..., a_n] \in \mathbf{E}_n$ and for every $X \in U(A)$ let the following relation be satisfied:

$$f(X) - f(A) = [\alpha_1(x_1 - a_1) + \alpha_2(x_2 - a_2) + ... + \alpha_n(x_n - a_n)] + \varepsilon(X) \qquad (I.4.1.)$$

where $\alpha_1, \alpha_2, ..., \alpha_n$ are some real numbers, $\varepsilon(X)$ is a function continuous at point $A$, $\varepsilon(A) = 0$, and

$$\lim_{X \to A} \frac{\varepsilon(X)}{\rho(X,A)} = 0. \qquad (I.4.2.)$$

Then the function is called ***differentiable*** at point $A$ and the linear expression

$$[\alpha_1(x_1 - a_1) + \alpha_2(x_2 - a_2) + ... + \alpha_n(x_n - a_n)]$$

is called the *total differential of function f at point A*, and is denoted by $df(A)$.

**I.4.7. Remark.** If the function is differentiable at point $A$, it follows from the definition that it must be defined in some neighbourhood of this point.

Relation (*I*.4.1) means that the function value $f(X)$ can be approximated by the linear function

$$f(A) + df(A),$$

$$\text{i.e.} \quad f(A) + [\alpha_1(x_1 - a_1) + \alpha_2(x_2 - a_2) + ... + \alpha_n(x_n - a_n)]. \qquad (I.4.3.)$$

The "error" function of this approximation equals $\varepsilon(X)$. From the relation

$$\lim_{X \to A} \frac{\varepsilon(X)}{\rho(X,A)} = 0$$

it follows that this "error" is essentially less than the distance between $X$ and $A$.

**I.4.8. Geometrical meaning.** If $n = 2$ then the graph of $f(X)$ in the neighbourhood $U(A)$ is a surface in $\mathbf{E}_3$ which contains the point $[A, f(A)]$. The graph of function (*I*.4.3) is a plane which contains the point $[A, f(A)]$. Relations (*I*.4.1); (*I*.4.2) mean that the plane is the tangent plane to the graph of function $f(X)$ at the point $[A, f(A)]$.

We will formulate two theorems which state the relation between partial derivatives and the differentiability of a function. Differentiability is an important property of a function, being the condition in a number of theorems.

**I.4.9. Theorem. Necessary condition of differentiability at a point.** *If function $f(X)$ is differentiable at point $A$ then $f$ is continuous at $A$, and there exist partial derivatives at this point*

$$\frac{\partial f}{\partial x_1}(A), \frac{\partial f}{\partial x_2}(A), ..., \frac{\partial f}{\partial x_n}(A)$$

*and the constants from the definition of a differential $\alpha_1, \alpha_2, ..., \alpha_n$ are equal to these derivatives, i.e.*

$$df(A) = \frac{\partial f}{\partial x_1}(A)(x_1 - a_1) + \frac{\partial f}{\partial x_2}(A)(x_2 - a_2) + ... + \frac{\partial f}{\partial x_n}(A)(x_n - a_n).$$

**I.4.10. Theorem. Sufficient condition of differentiability at a point.** *If the function $f$ of $n$ variables $x_1, x_2, ..., x_n$ has its partial derivatives*

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}$$

*in a neighbourhood $U(A)$ and the derivatives all are continuous at point $A$, then function $f$ is differentiable at point $A$.*

The next assertion is an easy consequence of the previous theorem.

**I.4.11. Theorem. Sufficient condition of differentiability on an open set.** *If function $f$ has partial derivatives*

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}$$

*in an open set $M$ which all are continuous on this set $M$, then function $f$ is differentiable at every point of $M$.*

**I.4.12. Example.**

We find the equation of the tangent plane to the graph of function $f$ given by the formula $f(x,y) = x^2 + y^2 + 2x - y - 7$ at the point $T = [A, f(A)]$, $A = [3,4]$.

Solution: We easily get $f(A) = 3^2 + 4^2 + 6 - 4 - 7 = 20$. There exist partial derivatives $\frac{\partial f}{\partial x}(x,y) = 2x + 2$, $\frac{\partial f}{\partial y}(x,y) = 2y - 1$. We can see that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are defined and continuous on $\mathbf{E}_2$, and therefore on some neighbourhood of $A = [3,4]$. Hence, $f$ is differentiable at $A$ and

$$df(A) = \left.\frac{\partial f}{\partial x}\right|_A (x - a_1) + \left.\frac{\partial f}{\partial y}\right|_A (y - a_2) = 8(x-3) + 7(y-4).$$



The tangent plane is the graph of the function $f(A) + df(A)$. The equation of the graph of this function is:

$$z = f(A) + df(A) \quad \text{i.e.} \quad z = f(A) + \left.\frac{\partial f}{\partial x}\right|_A (x - a_1) + \left.\frac{\partial f}{\partial y}\right|_A (y - a_2).$$

Hence we get:

$$z = 20 + 8(x-3) + 7(y-4)$$

We can calculate the partial derivatives of composite functions of several variables by using the so called Chain rule. You already know the analogy of this rule for functions of one variable. Compare the two formulations.

**I.4.13. Derivatives of composite functions - Chain rule.** If functions $\phi_1(X)$, $\phi_2(X)$, ..., $\phi_n(X)$ are differentiable at the point $A = [a_1, a_2, ..., a_m]$ and function $f$ is differentiable at the point $B = [\phi_1(A), \phi_2(A), ..., \phi_n(A)]$ then the composite function $F$ defined by the formula

$$F(X) = f(\phi_1(X), \phi_2(X), ..., \phi_n(X))$$

in some neighbourhood of point $A$ is differentiable at this point and for $k = 1, 2, ..., m$

$$\frac{\partial F}{\partial x_k}(A) = \frac{\partial f}{\partial y_1}(B)\frac{\partial \phi_1}{\partial x_k}(A) + \frac{\partial f}{\partial y_2}(B)\frac{\partial \phi_2}{\partial x_k}(A) + ... + \frac{\partial f}{\partial y_n}(B)\frac{\partial \phi_n}{\partial x_k}(A). \qquad (I.4.4.)$$

**I.4.14. Examples.**

Let $f$ be a function of two variables $u, v$ and let $\phi_1, \phi_2$ be functions of two variables $x, y$. A composite function $F$ is defined by its function value:

$$F(x,y) = f(\phi_1(x,y), \phi_2(x,y))$$

Find the expressions of $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$. (Functions $f, \phi_1, \phi_2$ are assumed differentiable and $[\phi_1(x,y), \phi_2(x,y)] \in D(f)$ if $[x,y] \in D(\phi_1) \cap D(\phi_2)$.)

Solution: To simplify the expressions, let us denote as $\phi$ the vector valued function defined by its function value:

$$\phi(x,y) \equiv [\phi_1(x,y), \phi_2(x,y)] \quad \text{for} \quad [x,y] \in D(\phi) \equiv D(\phi_1 \cap \phi_2)$$

Using the Chain rule we get:

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial f}{\partial u}(\phi(x,y))\frac{\partial \phi_1}{\partial x}(x.y) + \frac{\partial f}{\partial v}(\phi(x,y))\frac{\partial \phi_2}{\partial x}(x,y)$$

$$\frac{\partial F}{\partial y}(x,y) = \frac{\partial f}{\partial u}(\phi(x,y))\frac{\partial \phi_1}{\partial y}(x,y) + \frac{\partial f}{\partial v}(\phi(x,y))\frac{\partial \phi_2}{\partial y}(x,y)$$

Shortly:

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial \phi_1}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial \phi_2}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial \phi_1}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial \phi_2}{\partial y}$$

Let $g$ be a function of two variables $u, v$ and let $\phi, \psi$ be functions of one variable $x$, and $\phi(x) \equiv x$. A composite function $G$ is defined by its function value:

$$G(x) = f(x, \psi(x))$$

Find the expression of $\frac{dG}{dx}$. (Functions $g, \psi$ are assumed differentiable and $[x, \psi(x)] \in D(g)$ if $x \in D(\psi)$.)

Solution: Using the Chain rule we get:

$$\frac{dG}{dx}(x) = \frac{\partial f}{\partial u}(x, \psi(x)) \cdot 1 + \frac{\partial f}{\partial v}(x, \psi(x))\frac{d\psi}{dx}(x)$$

i.e.

$$G'(x) = \frac{\partial f}{\partial u}(x, \psi(x)) + \frac{\partial f}{\partial v}(x, \psi(x))\psi'(x)$$

or shortly:

$$G' = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\psi'$$

**I.4.15. Higher order derivatives.** Let the function $f$ of $n$ variables $x_1, x_2, ..., x_n$ have a partial derivative $\frac{\partial f}{\partial x_k}$ in subset $M_1$, $k \in \{1, 2, ..., n\}$. This partial derivative is also a function of $n$ variables. If there exists a partial derivative of this function, i.e.

$$\frac{\partial\left(\frac{\partial f}{\partial x_k}\right)}{\partial x_l}, \quad l \in \{1, 2, ..., n\}$$

in some set $M_2 \subset M_1$ then it is called a partial derivative of the second order, and it is denoted by

$$\frac{\partial^2 f}{\partial x_l \partial x_k}, \qquad \frac{\partial^2 f}{\partial x_k^2} \ \text{(if } k = l\text{)}.$$

**I.4.16. Example.** Find the all partial derivatives of the first and second order of the function $f : f(x, y) = e^{xy^2}$.

Solution:

$$\frac{\partial f}{\partial x}(x, y) = e^{xy^2} y^2, \qquad \frac{\partial f}{\partial y}(x, y) = e^{xy^2} 2xy,$$



$$\frac{\partial^2 f}{\partial x^2}(x, y) = e^{xy^2} y^4, \qquad \frac{\partial^2 f}{\partial y^2}(x, y) = e^{xy^2} 2xy\, 2xy + e^{xy^2} 2x = 2xe^{xy^2}(2xy^2 + 1),$$

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = e^{xy^2} 2xy\, y^2 + e^{xy^2} 2y = 2ye^{xy^2}(xy^2 + 1),$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = e^{xy^2} y^2\, 2xy + e^{xy^2} 2y = 2ye^{xy^2}(xy^2 + 1)$$

Function $f$ and all partial derivatives of the first and second order are defined and continuous in $\mathbf{E}_2$.

**I.4.17. Remark.** By analogy, we can define partial derivatives of the third order etc.

In the previous example we derived $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, but in general $\frac{\partial^2 f}{\partial x_l \partial x_k} \neq \frac{\partial^2 f}{\partial x_k \partial x_l}$. The next theorem states the sufficient conditions which ensure that partial derivatives differing by the order of differentiation define the same functions.

**I4.18. Theorem.** *Let function $f$ have partial derivatives $\frac{\partial f}{\partial x_k}$, $\frac{\partial f}{\partial x_l}$, $k, l = 1, 2, ..., n$, $k \neq l$ in a neighbourhood $U(A)$. Let $\frac{\partial^2 f}{\partial x_l \partial x_k}$ be continuous at point $A$. Then there exists $\frac{\partial^2 f}{\partial x_k \partial x_l}(A)$ and*

$$\frac{\partial^2 f}{\partial x_k \partial x_l}(A) = \frac{\partial^2 f}{\partial x_l \partial x_k}(A).$$

## I.5. Gradients, directional derivatives.

**I.5.1. Gradients.** If function $f$ of $n$ variables denoted by $x_1, x_2, ... x_n$ has all its partial derivatives at point $A$, then the vector

$$\left[\frac{\partial f}{\partial x_1}(A), \frac{\partial f}{\partial x_2}(A), ..., \frac{\partial f}{\partial x_n}(A)\right] \in \mathbf{E}_n$$

is called the underbar*gradient of function $f$ at point $A$*, and it is denoted by

$$(\operatorname{grad} f)(A), \qquad (\nabla f)(A), \qquad (\operatorname{grad} f)|_A, \qquad \nabla f|_A.$$

If the gradient of a function $f$ exists at points of some set $M$, the vector function given by the relation $\Phi(X) = (\operatorname{grad} f)(X)$, $X \in M$ is called the gradient of a function $f$ and it is denoted

$$\operatorname{grad} f \quad \text{or} \quad \nabla f.$$

**I.5.2. Directional derivatives.** In this paragraph we will generalize the notion of a partial derivative.

Let $f$ be a function of $n$ real variables $x_1, x_2, ..., x_n$, $A = [a_1, a_2, ..., a_n]$ be some given point $A \in \mathbf{E}_n$ and $\vec{s} = (s_1, s_2, ..., s_n) \neq \mathcal{O}$ a given vector. We denote by $\vec{S}$

$$\vec{S} = (S_1, S_2, ...S_n) = \frac{\vec{s}}{|\vec{s}|} = \left(\frac{s_1}{|\vec{s}|}, \frac{s_2}{|\vec{s}|}, ..., \frac{s_n}{|\vec{s}|}\right)$$

If there exists the limit

$$\lim_{t \to 0} \frac{f(A + \vec{S}\,t) - f(A)}{t} = \lim_{t \to 0} \frac{f(a_1 + S_1 t, a_2 + S_2 t, ..., a_n + S_n t) - f(a_1, a_2, .., a_n)}{t}$$

it is called the <u>*directional derivative of f in direction $\vec{s}$ at point A*</u>, and it is denoted by $\frac{\partial f}{\partial \vec{s}}(A)$.

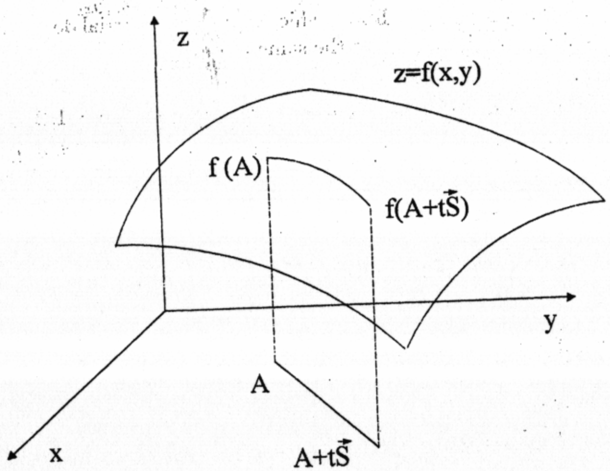


Fig. 4.

**I.5.3. Remark.** It is clear that this definition is a generalization of the notion of a partial derivative. Indeed, if we choose in this definition for instance $\vec{s} = (1, 0, 0, ..., 0)$ we will get the definition of $\frac{\partial f}{\partial x_1}(A)$.

**I.5.4. Remark.** It is also clear that this definition is identical with the definition of the derivative of the following function of one real variable $t$ at point $0$ :

$$f(a_1 + S_1 t, a_2 + S_2 t, ..., a_n + S_n t)$$



This is a composite function of function $f$ and a vector function $\Phi$, the value of which is defined by the formula

$$\Phi(t) = [a_1 + S_1 t, a_2 + S_2 t, ..., a_n + S_n t].$$

Assuming differentiability of $f$ and using formula $(I.4.4)$ we get

$$\frac{\partial f}{\partial \vec{s}}(A) = \frac{d\Phi}{dt}(0) =$$

$$= \frac{\partial f}{\partial x_1}(A)\frac{d(a_1 + S_1 t)}{dt}(0) + \frac{\partial f}{\partial x_2}(A)\frac{d(a_2 + S_2 t)}{dt}(0) + ... + \frac{\partial f}{\partial x_n}(A)\frac{d(a_n + S_n t)}{dt}(0) =$$

$$= \frac{\partial f}{\partial x_1}(A)\,S_1 + \frac{\partial f}{\partial x_2}(A)\,S_2 + ... + \frac{\partial f}{\partial x_n}(A)\,S_n \;=\; \vec{S} \cdot (\operatorname{grad} f)(A).$$

Hence, assuming the differentiability of a function $f$ at point $A$ we derive a formula for the directional derivative at point $A$ :

$$\frac{\partial f}{\partial \vec{s}}(A) = \frac{\vec{s} \cdot (\operatorname{grad} f)(A)}{|\vec{s}|}$$

**I.5.5. Remark.** Because $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\alpha$, where $\alpha$ is the angle between $\vec{a}$, $\vec{b}$ we get:

$$\frac{\partial f}{\partial \vec{s}}(A) = \frac{\vec{s} \cdot (\operatorname{grad} f)(A)}{|\vec{s}|} = |(\operatorname{grad} f)(A)|\,\cos\alpha$$

Thus, if the angle $\alpha$ between $(\operatorname{grad} f)(A)$ and $\vec{s}$ equals zero, the directional derivative is maximal. i.e. the gradient at a point is the direction in which the increment of the function (in a sufficiently small neighbourhood) is maximal.

**I.5.6. Example.** Find the directional derivative $\frac{\partial f}{\partial \vec{s}}(A)$ if $A \equiv [1, 2]$, $\vec{s} = (1, 1)$ and $f : f(x, y) = x^2 + xy$.

Solution: We define the unit vector $\vec{S}$ :

$$|\vec{s}| = \sqrt{1^2 + 1^2} = \sqrt{2}, \qquad \vec{S} = \frac{\vec{s}}{|\vec{s}|} = \frac{(1, 1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Further, we get expressions for the partial derivatives of $f$

$$\frac{\partial f}{\partial x}(x, y) = 2x + y, \qquad \frac{\partial f}{\partial y}(x, y) = x.$$

These partial derivatives are defined and continuous in $\mathbf{E}_2$. Thus, function $f$ is differentiable at each point of $\mathbf{E}_2$, in particular at the point $A \equiv [1, 2]$. Hence,

$$\frac{\partial f}{\partial \vec{s}}(A) = (\operatorname{grad} f)|_A \cdot \vec{S} = \left.\frac{\partial f}{\partial x}\right|_A S_1 + \left.\frac{\partial f}{\partial y}\right|_A S_2 = 4\,\frac{1}{\sqrt{2}} + 1\,\frac{1}{\sqrt{2}} = \frac{5}{\sqrt{2}}.$$