10. C is the circular cone with the radius of its basis $r = a > 0$ and the height $h > 0$. Surface σ is the boundary of C. The planar density of the mass distribution on σ is $\rho = const$. Evaluate the moment of inertia of σ with respect to the axis of cone C.

11. Evaluate the flux of vector field $\mathbf{f}$ through surface σ.

a) $\mathbf{f}(x,y,z) = y\,\mathbf{i} - x\,\mathbf{j} + z\mathbf{k}$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ z = 4 - x^2 - y^2/9,\ y \ge 0,\ z \ge 0\}$, σ is oriented so that the angle between its normal vector (at any point) and the vector $\mathbf{k} = (0,0,1)$ is acute (i.e. less than $\pi/2$).

b) $\mathbf{f}(x,y,z) = (0,0,y)$, σ is the triangle with the vertices $A = [0,0,0]$, $B = [5,0,1]$, $C = [1,4,1]$, oriented so that the angle between its normal vector (at any point) and the vector $\mathbf{k} = (0,0,1)$ is acute.

c) $\mathbf{f}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, σ is the cylindrical surface $x^2 + y^2 = 9$, $0 \le z \le 4$, oriented to the interior of the cylinder.

d) $\mathbf{f}(x,y,z) = (x^3, y^3, z^3)$, σ is the part (corresponding to $z \ge 0$) of the sphere with its center $S = [0,0,0]$ and radius $r = 2$, oriented so that its normal vector at the point $[0,0,2]$ is $\mathbf{n} = (0,0,1)$.

e) $\mathbf{f}(x,y,z) = x\,\mathbf{i} - x\,\mathbf{j} + y\,\mathbf{k}$, σ is the parallelogram $A = [0,0,0]$, $B = [0,3,3]$, $C = [-1,4,5]$, $D = [-1,1,2]$, oriented by the normal vector $\mathbf{n} = (1,-1,1)$.

f) $\mathbf{f}(x,y,z) = (x^2 - y^2,\ y^2 - z^2,\ z^2 - x^2)$, $\sigma = \sigma_1 \cup \sigma_2$, $\sigma_1 = \{[x,y,0] \in \mathbf{E}_3;\ x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\}$, $\sigma_2 = \{[x,0,z] \in \mathbf{E}_3;\ x^2 + z^2 \le 1,\ z \ge 0,\ z \ge 0\}$, the normal vector to σ_2 is $\mathbf{n} = (0,-1,0)$.

g) $\mathbf{f}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ y = 9 - \sqrt{x^2 + z^2},\ y \ge 3\}$, $\mathbf{n} \cdot \mathbf{j} \le 0$.

h) $\mathbf{f}(x,y,z) = (x^2, y^2, z^2)$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ y^2/16 + z^2/4 = 1,\ z \ge 0,\ 0 \le x \le 3\}$, the normal vector $\mathbf{n}$ at the point $P = [1,0,2]$ is $\mathbf{n} = (0,0,-1)$.

i) $\mathbf{f}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} - z\,\mathbf{k}$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2 + y^2 + z^2 = 4,\ x \ge 0\}$, the normal vector $\mathbf{n}$ at the point $P = [2,0,0]$ is $\mathbf{n} = \mathbf{i} = (1,0,0)$.

j) $\mathbf{f}(x,y,z) = (z,x,y)$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x + z = 2,\ x^2 + y^2 \le 4\}$, the normal vector is $\mathbf{n} = (1,0,1)/\sqrt{2}$.

12. Evaluate the flux of vector field $\mathbf{f}$ through the closed surface σ. If it is possible, apply the Gauss-Ostrogradsky theorem.

a) $\mathbf{f}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, σ is the sphere with the center at the point $S_0 = [x_0, y_0, z_0]$ and radius $r = a > 0$, oriented outward.

b) $\mathbf{f}(x,y,z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$, σ is the sphere with the center at the point $S_0 = [x_0, y_0, z_0]$ and radius $r = a > 0$, oriented outward.

c) $\mathbf{f}(x,y,z) = (x^2, y^2, z^2)$, σ is the sphere with the center at the point $S_0 = [x_0, y_0, z_0]$ and radius $r = a > 0$, oriented outward.

d) $\mathbf{f}(x,y,z) = (y, 2x, -z)$, σ is the boundary of the set $D = \{[x,y,z] \in \mathbf{E}_3;\ x^2 + y^2 \le a^2,\ -a \le z \le a\}$ $(a > 0)$, σ is oriented to the interior.

e) $\mathbf{f}(x,y,z) = (x^2, y^2, z^2)$, σ is the boundary of the set $D = \{[x,y,z] \in \mathbf{E}_3;\ -2 \le z \le 4 - x^2 - y^2,\ x^2 + y^2 \le 4\}$, σ is oriented to the interior.

f) $\mathbf{f}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, σ is the boundary of the set $D = \{[x,y,z] \in \mathbf{E}_3;\ y^2 + z^2 < x^2,\ 0 < x < 3\}$, σ is oriented to the exterior.

g) $\mathbf{f}(x,y,z) = (x^3, z, y)$, σ is the boundary of the set $D = \{[x,y,z] \in \mathbf{E}_3;\ x^2 + y^2 < z < 4\}$, σ is oriented to the exterior.

h) $\mathbf{f}(x,y,z) = 2xy\,\mathbf{i} - y^2\mathbf{j} + 2z\,\mathbf{k}$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2/4 + y^2/4 + z^2/9 = 1\}$, σ is oriented to the interior.

i) $\mathbf{f}(x,y,z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$, σ is the boundary of the set $D = \{[x,y,z] \in \mathbf{E}_3;\ x^2 + y^2 + z^2 \le a^2,\ y > 0\}$ $(a > 0)$, σ is oriented to the exterior.

j) $\mathbf{f} = (x - 1, y + 2, 2)$, σ is a closed surface in $\mathbf{E}_3$, oriented outwards, such that $m_3(Int\ \sigma) = 5$.

13. Evaluate the circulation of vector field $\mathbf{f}$ around the closed curve C. If it is possible, apply the Stokes theorem.

a) $\mathbf{f}(x,y,z) = (y,z,x)$, C is the intersection of the cylindrical surface $x^2 + y^2 = 4$ and the plane $x + z = 0$. C is oriented so that its unit tangent vector at the point $[2,0,-2]$ is $\vec{\tau} = (0,1,0)$.

b) $\mathbf{f}(x,y,z) = (xy, yz, zx)$, $C = \{[x,y,z] \in \mathbf{E}_3;\ x + z = 1,\ y^2 + z^2 = 1\}$. C is oriented in accordance with the surface $\lambda = \{[x,y,z] \in \mathbf{E}_3;\ x + y = 1,\ y^2 + z^2 \le 1\}$ and the orientation of λ is given by the normal vector $\mathbf{n} = (1,0,1)/\sqrt{2}$.

c) $\mathbf{f}(x,y,z) = (z+1)\,\mathbf{i} + (x-y)\,\mathbf{j} + y\,\mathbf{k}$, C is a circle which is the intersection of the sphere $x^2 + y^2 + z^2 = 2$ with the plane $x + y + z = 0$. C is oriented clockwise as viewed from the point $[0,0,10]$.

d) $\mathbf{f}(x,y,z) = (-y/(x^2 + y^2),\ x/(x^2 + y^2),\ 2z)$, C is a circle $x^2 + y^2 = a^2$ $(a > 0)$, $z = h$ $(h > 0)$, oriented counterclockwise if viewed from the point $[0,0,2h]$.

VI. Potential and Solenoidal Vector Field

VI.1. Independence of the line integral of a vector function on the path. Potential vector field.

VI.1.1. Independence of the line integral of a vector function on the path. Let $\mathbf{f}$ be a k–dimensional vector field in domain $D \subset \mathbf{E}_k$ (for $k = 2$ or $k = 3$). Suppose that for any two curves C_1 and C_2 in D, such that $i.p.\,C_1 = i.p.\,C_2$ and $t.p.\,C_1 = t.p.\,C_2$, the line integrals $\int_{C_1} \mathbf{f} \cdot ds$ and $\int_{C_2} \mathbf{f} \cdot ds$ exist and are equal. Then we say that the *line integral of* $\mathbf{f}$ *does not depend on the path in* D.

VI.1.2. Theorem. *The line integral of vector function* $\mathbf{f}$ *does not depend on the path in domain* $D \subset \mathbf{E}_k$ *(for* $k = 2$ *or* $k = 3$*) if and only if the circulation of* $\mathbf{f}$ *around every closed curve* C *in* D *is equal to zero.*

Proof: a) Suppose at first that the line integral of $\mathbf{f}$ does not depend on the path in D, and C is a closed curve in D. Then C can be decomposed to the union of two curves K_1 and K_2 such that $t.p.\,K_1 = i.p.\,K_2$ and $t.p.\,K_2 = i.p.\,K_1$. Putting $C_1 = K_1$ and $C_2 = -K_2$, we get two curves in D with the same initial and terminal points. The independence of the line integral of $\mathbf{f}$ on the path implies that $\int_{C_1} \mathbf{f} \cdot d\mathbf{s} = \int_{C_2} \mathbf{f} \cdot d\mathbf{s}$. This implies

$$\oint_C \mathbf{f} \cdot d\mathbf{s} = \int_{K_1} \mathbf{f} \cdot d\mathbf{s} + \int_{K_2} \mathbf{f} \cdot d\mathbf{s} = \int_{C_1} \mathbf{f} \cdot d\mathbf{s} - \int_{C_2} \mathbf{f} \cdot d\mathbf{s} = 0.$$

b) Suppose now that the circulation of $\mathbf{f}$ along any closed curve in D is zero. Let C_1 and C_2 be two curves in D such that $i.p.\,C_1 = i.p.\,C_2$ and $t.p.\,C_1 = t.p.\,C_2$. Suppose for simplicity that curves C_1 and C_2 do not have any other common points, i.e. they do not intersect or touch at any other points. Then the union $C = C_1 \cup (-C_2)$ is a closed curve in D and so the circulation of $\mathbf{f}$ around C is zero. This implies:

$$\int_{C_1} \mathbf{f} \cdot d\mathbf{s} = \int_{C_1 \cup (-C_2)} \mathbf{f} \cdot d\mathbf{s} + \int_{C_2} \mathbf{f} \cdot d\mathbf{s} = \int_{C_2} \mathbf{f} \cdot d\mathbf{s}$$

A similar approach can be used in the case that the curves C_1 and C_2 have more common points than their initial and terminal points, and it can also be proved in this case that the line integrals of $\mathbf{f}$ on C_1 and C_2 are equal.

VI.1.3. Potential vector field. We say that the vector field $\mathbf{f}$ in domain $D \subset \mathbf{E}_k$ (for $k = 2$ or $k = 3$) is <u>*potential field*</u> in D if there exists a scalar field φ in D such that

$$\mathbf{f} = \operatorname{grad} \varphi$$

in D. Scalar function φ is called the <u>*potential*</u> of $\mathbf{f}$ in D.

VI.1.4. Remark. Recall that

$$\operatorname{grad} \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) \quad (\text{if } k = 2), \qquad \operatorname{grad} \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) \quad (\text{if } k = 3).$$

It is quite obvious that if $\mathbf{f}$ is a potential field in domain D and D' is a domain such that $D' \subset D$ then $\mathbf{f}$ is also a potential field in domain D'.

You will see later (in paragraph VI.2.4) that e.g. gravitational and electric fields are examples of potential fields.

A scalar function φ which is a potential of a potential vector field $\mathbf{f}$ has some properties which are in some sense analogous to the properties of an antiderivative. For example:

If $\mathbf{f}$ is a potential vector field in domain $D \subset \mathbf{E}_k$ then its potential φ is unique up to an additive constant.

This means that:

a) *$\varphi + c$ is also a potential of $\mathbf{f}$ in D for every real constant c.*

b) *Any other potential ζ of $\mathbf{f}$ in D differs from φ at most in an additive constant. In other words: If ζ is another potential of $\mathbf{f}$ in D then there exists a constant c such that $\zeta = \varphi + c$ in D.*

Assertion a) is very simple – the equation $\operatorname{grad} \varphi = \mathbf{f}$ in D immediately implies that $\operatorname{grad}(\varphi + c) = \mathbf{f}$ and so $\varphi + c$ is also a potential of $\mathbf{f}$ in D.

Assertion b) is also quite obvious: If φ and ζ are two potentials of $\mathbf{f}$ in D then $\mathbf{f} = \operatorname{grad} \varphi$ and $\mathbf{f} = \operatorname{grad} \zeta$ in D and so $\operatorname{grad} \zeta - \operatorname{grad} \varphi = \operatorname{grad}(\zeta - \varphi) = \vec{0}$. However the only function whose gradient is zero in domain D is a constant function. This implies the existence of a constant c such that $\zeta - \varphi = c$ and so $\zeta = \varphi + c$ in D.

Another analogy between the potential φ of a potential vector field and the antiderivative to a function of one variable is the similarity of formula (IV.1) (see the next theorem) and the Newton–Leibnitz formula (II.6).

VI.1.5. Theorem. *If $\mathbf{f}$ is a continuous and potential vector field in domain D, φ is a potential of $\mathbf{f}$ in D and C is a curve in D then*

$$\int_C \mathbf{f} \cdot d\mathbf{s} = \varphi(t.p.\,C) - \varphi(i.p.\,C). \tag{VI.1}$$

Proof: Since φ is a potential of $\mathbf{f}$ in D, $\mathbf{f}$ is equal to $\operatorname{grad}\varphi$ in D. Suppose that C is a simple smooth curve in D and P is its parametrization defined in the interval $\langle a, b \rangle$ such that curve C is oriented in accordance with P. Let us denote by $x(t)$, $y(t)$ and $z(t)$ the coordinate functions of parametrization P. Then

$$\begin{aligned}
\int_C \mathbf{f} \cdot d\mathbf{s} &= \int \operatorname{grad}\varphi \cdot d\mathbf{s} = \int_a^b \operatorname{grad}\varphi(P(t)) \cdot \dot{P}(t)\, dt = \int_a^b \Big(\frac{\partial \varphi}{\partial x}(x(t), y(t), z(t)), \\
&\quad \frac{\partial \varphi}{\partial y}(x(t), y(t), z(t)),\ \frac{\partial \varphi}{\partial z}(x(t), y(t), z(t))\Big) \cdot (\dot{x}(t),\ \dot{y}(t),\ \dot{z}(t))\, dt = \\
&= \int_a^b \Big[\frac{\partial \varphi}{\partial x}(x(t), y(t), z(t))\, \dot{x}(t) + \frac{\partial \varphi}{\partial y}(x(t), y(t), z(t))\, \dot{y}(t) + \\
&\quad + \frac{\partial \varphi}{\partial z}(x(t), y(t), z(t))\, \dot{z}(t)\Big]\, dt = \int_a^b \frac{d}{dt}\, \varphi(x(t), y(t), z(t))\, dt = \\
&= \varphi(x(b), y(b), z(b)) - \varphi(x(a), y(a), z(a)) = \varphi(t.p.\,C) - \varphi(i.p.\,C).
\end{aligned}$$

The same equality can also be proved in the case when C is a simple piecewise–smooth curve.

The next theorem is perhaps the most important theorem in this section.

VI.1.6. Theorem. *Suppose that $\mathbf{f}$ is a continuous vector field in domain $D \subset \mathbf{E}_k$ (for $k = 2$ or $k = 3$). Then the next two conditions are equivalent:*

a) *$\mathbf{f}$ is a potential vector field in D.*

b) *The line integral of $\mathbf{f}$ does not depend on the path in D.*

Proof: The implication a) $\Longrightarrow$ b) is the consequence of formula (VI.1).

Let us now prove the opposite implication, i.e. b) $\Longrightarrow$ a). Suppose that condition b) is fulfilled. Let us denote by U, V and W the components of $\mathbf{f}$. Choose a point

$A \in D$. The point A was chosen arbitrarily, but we take it as a fixed point from now. Let $X = [x, y, z]$ be any other point of D. Let us define

$$\varphi(x, y, z) \;=\; \int_C \mathbf{f} \cdot ds \tag{VI.2}$$

where C is a curve in D such that $i.p.\,C = A$ and $t.p.\,C = X$. (It follows from the independence of the line integral of $\mathbf{f}$ on the path in D that the value of $\varphi(x, y, z)$ does not depend on the concrete choice of the curve C connecting the points A and $X = [x, y, z]$.) We claim that $\operatorname{grad}\varphi = \mathbf{f}$ in D. To prove this, it is sufficient to show that

$$\frac{\partial\varphi}{\partial x}(X) = U(X), \quad \frac{\partial\varphi}{\partial y}(X) = V(X) \quad \text{and} \quad \frac{\partial\varphi}{\partial z}(X) = W(X). \tag{VI.3}$$

Let us prove for example the first of these equalities. Using the definition of the partial derivative of φ with respect to x at point $X = [x, y, z]$, we obtain

$$\frac{\partial\varphi}{\partial x}(X) \;=\; \frac{\partial\varphi}{\partial x}(x, y, z) \;=\; \lim_{h\to 0} \frac{\varphi(x+h, y, z) - \varphi(x, y, z)}{h}.$$

$\varphi(x+h, y, z)$ can be expressed as the line integral of $\mathbf{f}$ on the simple piecewise – smooth curve which is the union of C and the line segment XY leading from point X to the point $Y = [x+h, y, z]$. The unit tangent vector on XY is $\vec{\tau} = (1, 0, 0)$. Thus, we get

$$\frac{\partial\varphi}{\partial x}(X) \;=\; \lim_{h\to 0} \frac{1}{h}\Big[\int_{C\cup XY} \mathbf{f} \cdot ds - \int_C \mathbf{f} \cdot ds\Big] \;=\; \lim_{h\to 0} \frac{1}{h}\int_{XY} \mathbf{f} \cdot ds \;=$$

$$=\; \lim_{h\to 0} \frac{1}{h}\int_{XY} (U, V, W) \cdot (1, 0, 0)\; ds \;=\; \lim_{h\to 0} \frac{1}{h}\int_{XY} U\, ds \;=\; U(X).$$

The equalities in (VI.3) show that $\mathbf{f} = \operatorname{grad}\varphi$ in D and so the vector field $\mathbf{f}$ is potential in D.

VI.1.7. Remark. If $\mathbf{f}$ is a potential vector field in domain $D \subset \mathbf{E}_k$ (for $k = 2$ or $k = 3$) then the line integral of $\mathbf{f}$ is independent on the path in D (by Theorem VI.1.6) and this means that the circulation of $\mathbf{f}$ on every closed curve in D is zero (by Theorem VI.1.2). If $\mathbf{f}$ has a physical meaning of a force then we can say that the work done by force $\mathbf{f}$ over every closed curve C is zero. Due to this fact, potential vector fields are also often called *conservative fields*.

The path independence of the line integral in potential vector field $\mathbf{f}$ has an important physical meaning: It says that the amount of work done by force $\mathbf{f}$ in moving from point $A \in D$ to point $B \in D$ is the same for all paths in D, leading from A to B. This is known to hold for example in a gravitational or an electric field, where the amount of work it takes to move a mass particle or a charge from point A to point B depends only on the position of A and B and not on the path taken between A and B.

Since the value of the integral $\int_C \mathbf{f} \cdot ds$ of potential vector field $\mathbf{f}$ depends only on the position of the initial point and the terminal point of curve C, this integral is often written as

$$\int_C \mathbf{f} \cdot ds \;=\; \int_A^B \mathbf{f} \cdot ds$$

where $A = i.p.\,C$ and $B = t.p.\,C$.

We have shown the proofs of the last two theorems because they are very instructive and their ideas can also be used in other situations. Formula (VI.1) provides a very simple way of evaluating the line integral of a potential vector field when the potential φ of $\mathbf{f}$ is known. On the other hand, the idea of the proof of Theorem VI.1.6 shows a method of finding a potential φ of a vector field $\mathbf{f}$ provided that it is known that $\mathbf{f}$ is a potential vector field. This approach will be applied to concrete example (see paragraph VI.2.1). We will also show another method of finding a potential φ of $\mathbf{f}$ – see examples VI.2.1., VI.2.2 and VI.2.4.

We have seen that the potential vector field $\mathbf{f}$ in domain D has interesting and useful properties – especially that its line integral does not depend on the path in D (see Theorem VI.1.6) and moreover, its line integral can be evaluated by means of formula (VI.1) (see Theorem VI.1.5). It is therefore very important to be able to recognize whether a given vector field in domain D is or is not a potential vector field in D. The next paragraphs will deal with this question. We will distinguish between the two–dimensional case (see paragraphs VI.I.8–VI.1.13) and the three–dimensional case (see paragraphs VI.1.14–VI.1.18).

VI.1.8. Theorem. (Potential field in $\mathbf{E}_2$ – the necessary condition.) *Suppose that $\mathbf{f} = (U, V)$ is a potential vector field in domain $D \subset \mathbf{E}_2$ Suppose that the components U and V of $\mathbf{f}$ are continuously differentiable functions in D. Then*

$$\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0 \quad \text{in } D. \tag{VI.4}$$

Proof: If φ is a potential of $\mathbf{f}$ in D then $\mathbf{f} = (\partial\varphi/\partial x, \partial\varphi/\partial y)$. Hence we have

$$U \;=\; \frac{\partial\varphi}{\partial x}, \qquad V \;=\; \frac{\partial\varphi}{\partial y}$$

in D. This form of U and V, together with the information about the continuity of the partial derivatives of U and V in D (which means the continuity of the second partial derivatives of φ in D), easily implies condition (VI.4).

VI.1.9. Remark. Condition (VI.4) is a necessary but not a sufficient condition. This means that the two–dimensional vector field $\mathbf{f}$ in domain $D \subset \mathbf{E}_2$ whose components are continuously differentiable functions in D can be a potential vector field in D only if condition (VI.4) is fulfilled, but on the other hand the validity of condition (VI.4) itself does not guarantee that vector field $\mathbf{f}$ really is a potential field in D. We can demonstrate this through the next example.

VI.1.10. Example. The vector function $\mathbf{f} \;=\; (U, V) \;=\; \Big(-\dfrac{y}{x^2+y^2},\; \dfrac{x}{x^2+y^2}\Big)$ satisfies condition (VI.4) in the domain

$$D = \big\{[x, y] \in \mathbf{E}_2;\; x^2 + y^2 > 0\big\}.$$

(Verify it for yourself!)

Let us now evaluate the circulation of $\mathbf{f}$ around the circle $C_r: x^2+y^2=r^2$ (where $r>0$), whose orientation is positive (see paragraph IV.5.3). It can easily be checked that the mapping $P: x=\phi(t)=r\cos t,\ y=\psi(t)=r\sin t$ (for $t\in\langle 0,2\pi)$) is a parametrization of C_r which also generates the positive orientation of C_r. Thus, we have

$$\oint_{C_r}\mathbf{f}\cdot d\mathbf{s} = \oint_{C_r} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy =$$

$$= \int_0^{2\pi}\Big[-\frac{r\sin t}{r^2}(-r\sin t)+\frac{r\cos t}{r^2}\,r\cos t\Big]\,dt = \int_0^{2\pi}dt = 2\pi.$$

Thus, although the components of $\mathbf{f}$ satisfy condition (VI.4), $\mathbf{f}$ is not a potential vector field in D because we have shown that the circulation of $\mathbf{f}$ around at least one closed curve in D is different from zero. (See Theorem VI.1.2.)

Our next goal is to formulate sufficient conditions which will guarantee that a given two-dimensional vector field will be potential in domain $D\subset \mathbf{E}_2$. We will therefore need the notion of a so called simply connected domain in $\mathbf{E}_2$:

VI.1.11. A simply connected domain in $\mathbf{E}_2$. Domain $D\subset\mathbf{E}_2$ is said to be *simply connected* if each closed curve C in D can be contracted to a point in D without ever leaving D.

A simply connected domain in $\mathbf{E}_2$ can also be defined as such domain $D\subset\mathbf{E}_2$ that the interior of every closed curve C in D is a subset of D.

Roughly speaking, domains which have bounded "holes" (and look like a Swiss cheese) are not simply connected, while domains which do not have such holes are simply connected. The examples of domains which are simply connected are: the whole plane $\mathbf{E}_2$, the half-plane, $\mathbf{E}_2$ minus a half-line, interiors of closed curves, etc. Examples of domains that are not simply connected include: $\mathbf{E}_2$ minus one point (e.g. domain D from example VI.1.10), $\mathbf{E}_2$ minus a bounded subset and an open disk minus one point.

VI.1.12. Theorem. (Potential field in $\mathbf{E}_2$ – sufficient conditions.) *Let*

a) *D be a simply connected domain in $\mathbf{E}_2$ and*

b) *$\mathbf{f}=(U,V)$ be a vector field in D, whose components U, V are continuously differentiable functions in D and they satisfy the condition*

$$\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}=0 \quad \text{in } D. \tag{VI.4}$$

Then $\mathbf{f}$ is a potential vector field in D.

Proof: We will prove that $\mathbf{f}$ is a potential vector field in D if we show that the circulation of $\mathbf{f}$ around every closed curve in D equals zero. (See Theorem VI.1.2.) Thus, let C be a closed curve in D. Since D is simply connected, $Int\ C\subset D$. Applying Green's theorem (see paragraph IV.5.5), we obtain

$$\oint_C \mathbf{f}\cdot d\mathbf{s} = \pm\iint_{Int\ C}\Big(\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}\Big)\,dx\,dy = 0.$$

(The "+" sign is valid if C is positively oriented and "–" holds if C is negatively oriented. However, since the integral on the right hand side equals zero, the signs are not important.)

VI.1.13. Example. We have seen in example VI.1.10 that the vector field

$$\mathbf{f}=(U,V)=\Big(-\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2}\Big)$$

is not potential in the domain $D=\{[x,y]\in\mathbf{E}_2;\ x^2+y^2>0\}$. However, it is potential in any sub-domain $D'\subset D$ which is simply connected! For instance, $\mathbf{f}$ is potential in the upper half-plane $D'=\{[x,y]\in\mathbf{E}_2;\ y>0\}$. This follows immediately from Theorem VI.1.12. (The validity of condition (VI.4) was already mentioned in example VI.1.10.)

Thus, we know that vector field $\mathbf{f}$ is potential in domain D', but we do not know the potential φ of $\mathbf{f}$ in D'. We will deal with methods of finding the potential in Section VI.2, and we will also return to this example. (See example VI.2.2.)

The following paragraphs deal with three-dimensional potential vector fields. However, you can find many analogies with the contents of paragraphs VI.1.8–VI.1.13, which deal with two-dimensional potential vector fields.

VI.1.14. Theorem. (Potential field in $\mathbf{E}_3$ – the necessary condition.) *Suppose that $\mathbf{f}$ is a potential vector field in domain $D\subset\mathbf{E}_3$. Suppose that the components of $\mathbf{f}$ are continuously differentiable functions in D. Then*

$$\operatorname{curl}\mathbf{f} = \vec{0} \quad \text{in } D. \tag{VI.5}$$

Proof: Condition (VI.5) is an immediate consequence of formula (V.7) and the fact that $\mathbf{f}$ can be expressed in the form $\mathbf{f}=\operatorname{grad}\varphi$ for some scalar function φ (the potential of $\mathbf{f}$ in D).

VI.1.15. Remark. Suppose that vector field $\mathbf{f}$ has the components U, V and W. Writing $\operatorname{curl}\mathbf{f}$ in components (see paragraph V.5.2), we can observe that condition (VI.5) says the same as the three equations

$$\frac{\partial W}{\partial y}-\frac{\partial V}{\partial z}=0,\qquad \frac{\partial U}{\partial z}-\frac{\partial W}{\partial x}=0,\qquad \frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}=0. \tag{VI.6}$$

It can be observed that Theorem VI.1.8 (dealing with the two-dimensional case) is a consequence of the more general Theorem VI.1.14. Indeed, if we have a two-dimensional potential vector field (U,V) in domain $D\subset\mathbf{E}_2$ then $\mathbf{f}=(U,V,0)$ is a three dimensional potential vector field in the three-dimensional domain $D\times\mathbf{R}$. Applying Theorem VI.1.14 now to this vector field, writing condition (VI.5) in the form of equations (VI.6) and using the fact that U and V do not depend on z, we can see that the first two equations in (VI.6) are automatically satisfied and the third equation in (VI.6) is identical with (VI.4).

VI.1.16. Remark. Analogously to condition (VI.4), condition (VI.5) is the necessary condition, but it is not a sufficient condition! This may be seen from the three-dimensional version of example VI.1.10: The vector function

$$\mathbf{f} = (U, V, W) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

satisfies condition (VI.5) in the domain $D = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2 > 0\}$, but vector field $\mathbf{f}$ is still not potential in D, because the circulation of $\mathbf{f}$ around the circle C_r: $x^2+y^2 = r^2$, $z = 0$ (where $r > 0$) is either 2π or -2π, in dependence on the chosen orientation of C_r. (This can be computed similarly as in example VI.1.10.)

However, condition (VI.5) becomes a sufficient condition if it is completed by the assumption about the form of domain D.

VI.1.17. A simply connected domain in $\mathbf{E}_3$. Domain $D \subset \mathbf{E}_3$ is said to be *simply connected* if each closed curve C in D can be contracted to a point in D without ever leaving D.

VI.1.18. Theorem. (Potential field in $\mathbf{E}_3$ – sufficient conditions.) *Let*

a) D be a simply connected domain in $\mathbf{E}_3$ and

b) $\mathbf{f}$ be a vector field in D, whose components are continuously differentiable functions in D and they satisfy the condition

$$\text{curl}\ \mathbf{f} = \vec{0} \quad \text{in } D. \tag{VI.5}$$

Then $\mathbf{f}$ is a potential vector field in D.

VI.1.19. Remark. A vector field $\mathbf{f}$ can also be potential in a domain D which is not simply connected. However, this cannot be verified by means of Theorem VI.1.12 (in the two-dimensional case) or Theorem VI.2.18 (in the three-dimensional case).

Theorems VI.1.8, VI.1.12, VI.1.14 and VI.1.18 will be applied to concrete examples in the next section.

VI.2. How to find a potential.

In this section, we will deal with two methods of finding a potential φ of a potential vector field. We will explain these methods with concrete examples.

VI.2.1. Example. $\mathbf{f} = (y^2 + y \cos x + 6x,\ 2xy + \sin x + 5)$. a) Is $\mathbf{f}$ a potential field in $\mathbf{E}_2$? b) If yes, find its potential. c) Compute the integral $\int_C \mathbf{f} \cdot d\mathbf{s}$ on curve C which is the part of the parabola $y = x^2 + 2$ from point $[0,2]$ to point $[2,6]$.

a) The components of vector field $\mathbf{f}$ are continuously differentiable functions and you can easily check that they satisfy equation (VI.4) in $\mathbf{E}_2$. The whole plane $\mathbf{E}_2$ is a simply connected domain. Thus, by Theorem VI.1.12, $\mathbf{f}$ is a potential field in $\mathbf{E}_2$. Let us denote by φ its potential.

1st method of finding a potential. It follows from the definition of the potential (see paragraph VI.1.3) that $\mathbf{f} = \text{grad}\ \varphi$ in $\mathbf{E}_2$. This means that

$$\frac{\partial\varphi}{\partial x} = y^2 + y \cos x + 6x, \qquad \frac{\partial\varphi}{\partial y} = 2xy + \sin x + 5. \tag{VI.7}$$

Integrating the first equality in (VI.7) with respect to x, we obtain

$$\varphi(x,y) = xy^2 + y \sin x + 3x^2 + C_1(y). \tag{VI.8}$$

(C_1 is the constant of integration which arose by the integration with respect to x. So it is a constant with respect to x. However, it can generally depend on y.) Integrating now the second equality in (VI.7) with respect to y, we get

$$\varphi(x,y) = xy^2 + y \sin x + 5y + C_2(x). \tag{VI.9}$$

(C_2 is the constant of integration which appeared after the integration with respect to y. So it cannot depend on y. However, it can depend on x.) Comparing (VI.8) and (VI.9), we get

$$\begin{aligned} xy^2 + y \sin x + 3x^2 + C_1(y) &= xy^2 + y \sin x + 5y + C_2(x), \\ 3x^2 + C_1(y) &= 5y + C_2(x). \end{aligned}$$

This is satisfied if we put e.g. $C_1(y) = 5y$ and $C_2(x) = 3x^2$. Substituting this either to (VI.8) or to (VI.9) and using the uniqueness of the potential up to an additive constant (see paragraph VI.1.4), we get

$$\varphi(x,y) = xy^2 + y \sin x + 3x^2 + 5y + \mathit{const}. \tag{VI.10}$$

2nd method of finding a potential. This method follows the proof of Theorem VI.1.6, and the potential is constructed by means of formula (VI.2): Choose $O = [0,0]$ as a fixed point and $X = [x_0, y_0]$ as a "variable" point and put $\varphi(x_0,y_0) = \int_C \mathbf{f} \cdot d\mathbf{s}$ where C is an arbitrary curve with the initial point O and the terminal point X. Let us choose curve C so that the computation of the line integral is as simple as possible. For instance: Put $C = OX' \cup X'X$ where OX' is the line segment leading from point O to the point $X' = [x_0, 0]$ and $X'X$ is the line segment leading from point X' to point X. Then we have:

$$\varphi(x_0,y_0) = \left(\int_{OX'} + \int_{X'X}\right) (y^2 + y \cos x + 6x)\, dx + (2xy + \sin x + 5)\, dy.$$

Since $y = 0$ and x varies from 0 to x_0 on the line segment OX', we have $dy = 0$ on OX' and

$$\int_{OX'} (y^2 + y \cos x + 6x)\, dx + (2xy + \sin x + 5)\, dy = \int_0^{x_0} 6x\, dx = 3x_0^2.$$

Further, $x = x_0$ and y varies from 0 to y_0 on $X'X$. Hence $dx = 0$ on $X'X$ and

$$\int_{X'X} (y^2 + y \cos x + 6x)\, dx + (2xy + \sin x + 5)\, dy =$$

$$= \int_0^{y_0} (2x_0 y + \sin x_0 + 5)\, dy = x_0 y_0^2 + y_0 \sin x_0 + 5y_0.$$

Thus, the value of the potential φ at point X is

$$\varphi(x_0, y_0) = \left(\int_{OX'} + \int_{X'X} \right) (y^2 + y \cos x + 6x)\, dx + (2xy + \sin x + 5)\, dy =$$

$$= 3x_0^2 + x_0 y_0^2 + y_0 \sin x_0 + 5y_0.$$

Writing $[x, y]$ instead of $[x_0, y_0]$ and taking into account that the potential is determined uniquely up to an additive constant, we obtain formula (VI.10).

c) Using now Theorem VI.1.5, we can evaluate the given line integral:

$$\int_C \mathbf{f} \cdot d\mathbf{s} = \int_{[0,2]}^{[2,6]} \mathbf{f} \cdot d\mathbf{s} = \varphi(2, 6) - \varphi(0, 2) = 104 + 6 \sin 2.$$

VI.2.2. Example. We already know from example VI.1.13 that the vector field

$$\mathbf{f} = (U, V) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

is potential in the domain $D' = \{[x, y] \in \mathbf{E}_2;\ y > 0\}$. Using the first or the second method for computation of the potential, we can find that the potential of vector field $\mathbf{f}$ in D' is $\varphi(x, y) = -\arctan(x/y) + const.$

VI.2.3. Example. $\mathbf{f} = (y^2 + x^2,\ x - y)$. a) Is $\mathbf{f}$ a potential field in $\mathbf{E}_2$? b) If yes, find its potential.

a) If we denote by U and V the components of $\mathbf{f}$ then we can easily compute that

$$\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 1 - 2y \quad \text{in } \mathbf{E}_2.$$

Thus, condition (VI.4) is not satisfied and so applying Theorem VI.1.8, we can see that vector field $\mathbf{f}$ is not a potential field in $\mathbf{E}_2$.

VI.2.4. Example. It is known from physics that a particle with the mass M at the point $X_0 = [x_0, y_0, z_0]$ generates the gravitational field

$$\mathbf{g} = -\kappa M \frac{(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}}$$

in $D = \mathbf{E}_3 - \{X_0\}$. Since $\mathbf{g}$ satisfies condition (VI.5) in $\mathbf{E}_3 - \{X_0\}$ (Verify this for yourself!) and the domain $\mathbf{E}_3 - \{X_0\}$ is simply connected (Why?), $\mathbf{g}$ is a potential vector field in $\mathbf{E}_3 - \{X_0\}$.

Potential φ of $\mathbf{g}$ satisfies

$$\frac{\partial \varphi(x, y, z)}{\partial x} = -\kappa M \frac{x - x_0}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}}.$$

Integrating this equation with respect to x, we obtain

$$\varphi(x, y, z) = \frac{\kappa M}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}} + C_1(y, z)$$

where $C_1(y, z)$ is the constant of integration. Putting the partial derivatives of φ with respect to y and z equal to the second and the third components of $\mathbf{g}$ and integrating with respect to y and z, we can obtain the same formulas for φ, only with $C_2(x, z)$ or $C_3(x, y)$ instead of $C_1(y, z)$. Comparing all three expressions of φ, we can see that we can put $C_1(y, z) = C_2(x, z) = C_3(x, y) = 0$ and so we get the potential of the gravitational field $\mathbf{g}$ in $\mathbf{E}_3 - \{X_0\}$:

$$\varphi(x, y, z) = \frac{\kappa M}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}} + const.$$

You can verify that the electric field generated by a charge Q at point X_0 is also a potential field in $\mathbf{E}_3 - \{X_0\}$ and its potential φ has a similar form.

VI.3. Solenoidal vector field.

VI.3.1 Solenoidal vector field. A vector field $\mathbf{f}$ in domain D is called <u>*solenoidal*</u> if its flux through any closed surface σ in D is zero. (Note that the flux of $\mathbf{f}$ through surface σ was defined in paragraph V.4.2 as $\iint_\sigma \mathbf{f} \cdot d\mathbf{p}$.)

VI.3.2. Theorem. (Solenoidal field in $\mathbf{E}_3$ – the necessary condition.) *Suppose that $\mathbf{f}$ is a solenoidal vector field in domain $D \subset \mathbf{E}_3$. Suppose that the components of $\mathbf{f}$ are continuously differentiable functions in D. Then*

$$\operatorname{div} \mathbf{f} = 0 \quad \text{in } D. \qquad \text{(VI.11)}$$

Proof: By contradiction. Suppose that there exists point $X_0 \in D$ such that $\operatorname{div} \mathbf{f}(X_0) \neq 0$. We can suppose that $\operatorname{div} f(X_0) > 0$ without loss of generality. It follows from the continuity of partial derivatives of the components of $\mathbf{f}$ that there exists a neighbourhood $U(X_0) \subset D$ such that $\operatorname{div} \mathbf{f} > 0$ in all points of $U(X_0)$. Let σ be a sphere with the center X_0 and with such a small radius that $\sigma \subset U(X_0)$. Using the Gauss–Ostrogradsky theorem (see paragraph V.6.3), we obtain

$$\iint_\sigma \mathbf{f} \cdot d\mathbf{p} = \pm \iiint_{Int\ \sigma} \operatorname{div} \mathbf{f}\ dx\, dy\, dz$$

where the "+" sign holds if σ is oriented to its exterior and the "−" sign holds in the opposite case. The integral on the right hand side is positive because $Int\ \sigma \subset U(X_0)$ and $\operatorname{div} \mathbf{f} > 0$ in $U(X_0)$. Thus, the flux of $\mathbf{f}$ through the closed surface σ is different from zero and so vector field $\mathbf{f}$ is not solenoidal in D. This is the desired contradiction.

VI.3.3. Remark. Analogously to conditions (VI.4) and (VI.5), condition (VI.11) is the necessary condition, but it is not a sufficient condition! This may be shown through the following example: The vector function

$$\mathbf{f} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{[x^2 + y^2 + z^2]^{3/2}}$$

satisfies condition (VI.11) in the domain $D = \mathbf{E}_3 - O$ where $O = [0, 0, 0]$. (You can check this for yourself.) However, $\mathbf{f}$ is not a solenoidal field in D. We can prove it so that we show that its flux through some closed surface σ in D is different from zero. Thus, let σ be for instance a sphere with the center O and radius R, oriented to its exterior. The flux of $\mathbf{f}$ through σ cannot be evaluated by means of the Gauss–Ostrogradsky theorem (see paragraph V.6.3) because the components of $\mathbf{f}$ do not satisfy the assumption of this theorem. (They are not continuous at point O which belongs to $Int\ \sigma$.) However, the surface integral $\iint_\sigma \mathbf{f}\cdot d\mathbf{p}$ can be computed by means of parametrization P discussed in paragraph V.2.10:

$$x = \phi(u,v) = R\cos u\cos v,$$
$$y = \psi(u,v) = R\sin u\cos v,$$
$$z = \vartheta(u,v) = R\sin v$$

for $u \in \langle 0, 2\pi\rangle$, $v \in \langle -\pi/2, \pi/2\rangle$. The vector $P_u \times P_v$ is

$$P_u(u,v)\times P_v(u,v) = \left(R^2\cos u\cos^2 v,\ R^2\sin u\cos^2 v,\ R^2\sin v\cos v\right).$$

Using now formula (V.6) and applying Fubini's theorem III.3.2, we get

$$\iint_\sigma \mathbf{f}\cdot d\mathbf{p} = \int_0^{2\pi}\left(\int_{-\pi/2}^{\pi/2}\left(\cos^2 u\cos^3 v + \sin^2 u\cos^3 v + \sin^2 v\cos v\right)dv\right)du =$$
$$= \int_0^{2\pi}\left(\int_{-\pi/2}^{\pi/2}\cos v\,dv\right)du = 4\pi.$$

Condition (VI.11) becomes a sufficient condition if it is completed by an assumption about the form of domain D:

VI.3.4. Theorem. (Solenoidal field in $\mathbf{E}_3$ – sufficient conditions.) *Let*

a) D be a domain in $\mathbf{E}_3$ such that if σ is any closed surface in D then $Int\ \sigma \subset D$,

b) $\mathbf{f}$ be a vector field in D, whose components are continuously differentiable functions in D and they satisfy the condition

$$\operatorname{div}\mathbf{f} = 0 \quad in\ D. \qquad \text{(VI.11)}$$

Then $\mathbf{f}$ is a solenoidal vector field in D.

P r o o f : Let σ be a closed surface in D. The flux of $\mathbf{f}$ through σ can be evaluated by means of the Gauss–Ostrogradsky theorem and if we also use condition (VI.11), we obtain

$$\iint_\sigma \mathbf{f}\cdot d\mathbf{p} = \pm\iiint_{Int\ \sigma}\operatorname{div}\mathbf{f}\ dx\,dy\,dz = 0.$$

(The sign in front of the triple integral depends on the orientation of σ. However, it is not important because the integral is equal to zero.)

VI.3.5. Example. Vector field $\mathbf{f}$ from paragraph VI.3.3 is solenoidal in the domain $G = \{[x,y,z]\in\mathbf{E}_3;\ z>0\}$. It can be verified that it satisfies condition (VI.11) in G and moreover, domain G has property a) formulated in Theorem VI.3.4.

VI.4. Exercises.

1. Find maximaum domains in $\mathbf{E}_2$ in which the given vector field $\mathbf{f}$ is defined. Verify whether $\mathbf{f}$ is a potential field in these domains. If yes, find the potential φ of $\mathbf{f}$ and evaluate the integral $\int_A^B \mathbf{f}\cdot d\mathbf{s}$.

a) $\mathbf{f}(x,y) = y^2, 2xy),\quad A=[1,3],\ B=[3,2]$

b) $\mathbf{f}(x,y) = \dfrac{-\mathbf{i}+\mathbf{j}}{(x-y)^2},\quad A=[1,2],\ B=[4,1]$

c) $\mathbf{f}(x,y) = (x^2, y^2),\quad A=[0,0],\ B=[3,5]$

d) $\mathbf{f}(x,y) = \left(y^2 - \dfrac{x}{\sqrt{y-x^2}} - 1,\ 2xy + \dfrac{1}{2\sqrt{y-x^2}}\right),\quad A=[0,1],\ B=[1,2]$

e) $\mathbf{f}(x,y) = \dfrac{y^2}{\sqrt{x}}\mathbf{i} + 4y\sqrt{x}\,\mathbf{j},\quad A=[1,2],\ B=[4,-2]$

f) $\mathbf{f}(x,y) = \left(1 - y^2 + \dfrac{1}{2\sqrt{y^2+x}},\ \dfrac{y}{\sqrt{y^2+x}} - 2xy\right),\quad A=[-3,2],\ B=[3,1]$

g) $\mathbf{f}(x,y) = (xy, x+y),\quad A=[0,0],\ B=[1,1]$

h) $\mathbf{f}(x,y) = \dfrac{-y\,\mathbf{i}+x\,\mathbf{j}}{(x-y)^2},\quad A=[2,1],\ B=[6,2]$

i) $\mathbf{f}(x,y) = (x^3y^2 + x,\ y^2 + yx^4),\quad A=[3,-1],\ B=[1,5]$

j) $\mathbf{f}(x,y) = (1 + y^2\sin 2x,\ -2y\cos^2 x),\quad A=[\pi,1],\ B=[\pi/2,2]$

k) $\mathbf{f}(x,y) = \left(\ln y - \dfrac{e^y}{x^2},\ \dfrac{e^y}{x} + \dfrac{x}{y}\right),\quad A=[1,1],\ B=[1,2]$

l) $\mathbf{f}(x,y) = \left(\dfrac{x-2y}{(y-x)^2} + x,\ \dfrac{y}{(y-x)^2} - y^2\right),\quad A=[0,1],\ B=[1,4]$

m) $\mathbf{f}(x,y) = \dfrac{4x\,\mathbf{i}+y\,\mathbf{j}}{4x^2+y^2-4},\quad A=[0,0],\ B=[0,2]$

n) $\mathbf{f}(x,y) = (y\sin x,\ y - \cos x),\quad A=[0,1],\ B=[5,2]$

o) $\mathbf{f}(x,y) = (\cos(2y) + y + x,\ y - 2x\sin(2y) + x),\quad A=[0,0],\ B=[-2,2]$

p) $\mathbf{f}(x,y) = (y^2, 2xy),\quad A=[2,1],\ B=[0,0]$

2. Function φ is the potential of vector field $\mathbf{f}$ in domain $D\subset\mathbf{E}_3$. Find D (a maximum possible), $\mathbf{f}$ and evaluate the work done by vector field f on curve C leading from point A to point B.

a) $\varphi(x,y,z) = xy + xz + yz,\quad A=[-1,2,-1],\ B=[3,4,1]$

b) $\varphi(x,y,z) = \ln|x^2+y^2+z^2-1|,\quad A = -1,1,2],\ B=[-3,4,-1]$

3. Find maximum domains in $\mathbf{E}_3$ in which the given vector field $\mathbf{f}$ is defined. Verify whether $\mathbf{f}$ is a potential field in these domains. If yes, find the potential φ.

a) $\mathbf{f}(x,y,z) = \left(\frac{y^2}{z}, \frac{2xy}{z}, -\frac{xy^2}{z^2}\right)$

b) $\mathbf{f}(x,y,z) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 2z\right)$

c) $\mathbf{f}(x,y,z) = \frac{-x\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}/2}{\sqrt{z - x^2 - y^2}}$

d) $\mathbf{f}(x,y,z) = \left(\frac{2x-y}{x^2+y^2}, \frac{x+2y}{x^2+y^2}, \ln z\right)$

e) $\mathbf{f}(x,y,z) = \left(\frac{1}{\sqrt{x}} \sin z - y^2, -2xy, 2\sqrt{x} \cos z\right)$

f) $\mathbf{f}(x,y,z) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$

g) $\mathbf{f}(x,y,z) = \left(\frac{z}{x-y}, \frac{z}{y-x}, \ln(x-y) + \frac{1}{\sqrt{z}}\right)$

h) $\mathbf{f}(x,y,z) = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$

i) $\mathbf{f}(x,y,z) = \left(e^x(y+z^2), e^x, ze^x\right)$

j) $\mathbf{f}(x,y,z) = \left(x-y, y^2, x+z\right)$

k) $\mathbf{f}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 2z\right)$

l) $\mathbf{f}(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, \frac{1}{z}\right)$

4. Find maximum domains in $\mathbf{E}_3$ in which the given vector field $\mathbf{f}$ is defined. Verify whether $\mathbf{f}$ is a solenoidal field in these domains.

a) $\mathbf{f}(x,y,z) = (y^2, z^2, x^2)$

b) $\mathbf{f}(x,y,z) = (z-y,\ x-z,\ y-z)$

c) $\mathbf{f}(x,y,z) = (x,\ y,\ -2z)$

d) $\mathbf{f}(x,y,z) = \left(\frac{x}{y}, \sqrt{z-x^2}, -\frac{z}{y}\right)$

e) $\mathbf{f}(x,y,z) = (y,\ z,\ x^2)$

f) $\mathbf{f}(x,y,z) = (xy,\ 1-y^2,\ yz)$

g) $\mathbf{f}(x,y,z) = \frac{(yz,\ xz,\ -xy)}{y^2+z^2-1}$

h) $\mathbf{f}(x,y,z) = (5x^2,\ y-z,\ \ln z)$

5. $\mathbf{f}(x,y,z) = \left(\frac{x-y+z}{(x^2+y^2+z^2)^{3/2}}, \frac{x+y-z}{(x^2+y^2+z^2)^{3/2}}, \frac{-x+y+z}{(x^2+y^2+z^2)^{3/2}}\right)$

a) Show that $\operatorname{div} \mathbf{f} = 0$ in $\mathbf{E}_3 - \{[0,0,0]\}$.

b) Evaluate the flux of $\mathbf{f}$ through the sphere with the center at the origin and radius $r = 1$, oriented outward.

c) Decide whether $\mathbf{f}$ is a solenoidal vector field in $\mathbf{E}_3 - \{[0,0,0]\}$. (Why?)

d) Find a domain in $\mathbf{E}_3$ where vector field $\mathbf{f}$ is solenoidal.

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