I.6. Implicit functions.

I.6.1. Examples. Let us assume the equation $x^2 + y^2 = 1$ and the point $X_0 \equiv [x_0, y_0] \equiv [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$. (Draw the sketch.) It is clear that the equation defines some function $f : y = f(x)$ in the neighborhood of the point $x_0 = \frac{\sqrt{2}}{2}$, such that $f(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}$. Indeed, we get from the equation $y = +\sqrt{1-x^2}$ or $y = -\sqrt{1-x^2}$. Taking into account the condition $f(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}$ we get $y = f(x) = +\sqrt{1-x^2}$. This function has the following properties:

$f(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}$

The function is defined in some neighbourhood of x_0, i.e. in the interval $(\frac{\sqrt{2}}{2} - \delta; \frac{\sqrt{2}}{2} + \delta)$, (here $\delta = 1 - \frac{\sqrt{2}}{2}$).

If we substitute $f(x)$ into the relation $x^2 + y^2 = 1$ we get an identity: $x^2 + (f(x))^2 = 1 \Rightarrow 1 = 1$.

There is at most one such function. (There is no such function for instance if we choose the point $A \equiv [1, 0]$ or $A \equiv [-1, 0]$).

The graph of the function f locally coincides with the "graph" of the equation, i.e. there exists $\delta > 0$, such that

$$\left\{[x,y] \in (\frac{\sqrt{2}}{2} - \delta; \frac{\sqrt{2}}{2} + \delta) \times (\frac{\sqrt{2}}{2} - \delta; \frac{\sqrt{2}}{2} + \delta) \ : \ y = \sqrt{1-x^2}\right\} =$$

$$= \left\{[x,y] \in (\frac{\sqrt{2}}{2} - \delta; \frac{\sqrt{2}}{2} + \delta) \times (\frac{\sqrt{2}}{2} - \delta; \frac{\sqrt{2}}{2} + \delta) \ : \ x^2 + y^2 = 1\right\}.$$

This section deals with the conditions that ensure the existence of such a function, even if we are not able to express it explicitly from the originally given equation.

Let us assume an another example. There is given the equation

$$e^x + x - 10 = y + \tan(y) \qquad (I.6.1.)$$

It is easy to see that $f(x) = e^x + x - 10$ is a continuous increasing function for $x \in (-\infty; +\infty)$, $R(f) = (-\infty; +\infty)$ and function $g(y) = y + \tan y$ is also a continuous increasing function on each interval $y \in (-\frac{\pi}{2} + k\pi; \frac{\pi}{2} + k\pi), k \in \mathbf{N}$, $R(g) = (-\infty; +\infty)$. Due to these properties of the functions f, g it is clear that for every $x \in (-\infty; +\infty)$ there exists the unique $y \in (-\frac{\pi}{2}; \frac{\pi}{2})$ such that equation $(I.6.1)$ is satisfied. Hence, by means of $(I.6.1)$ a function ϕ_0 is defined with the domain of definition $D(\phi_0) = (-\infty; +\infty)$ and the range $R(\phi_0) = (-\frac{\pi}{2}; \frac{\pi}{2})$. Since the function value of ϕ_0 is defined as a solution of some equation and the analytic expression of the function value is not known, the function is called an *implicit* function.

If we substitute function ϕ_0 into relation $(I.6.1)$ we get an identity:

$$\forall x \in (-\infty; +\infty) : e^x + x - 10 = \phi_0(x) + \tan(\phi_0(x)).$$

We can keep repeating $k \in \mathbf{N}$: For every $x \in (-\infty; +\infty)$ there exists the unique $y \in (-\frac{\pi}{2} + k\pi; \frac{\pi}{2} + k\pi)$ such that equation $(I.6.1)$ is satisfied. Hence, by means of this relation we can define a function ϕ_k with the domain of definition $D(\phi_k) = (-\infty; +\infty)$ and the range $R(\phi_k) = (-\frac{\pi}{2} + k\pi; \frac{\pi}{2} + k\pi)$.

If some point $[x_0, y_0]$ satisfying relation $(I.6.1)$ is given, then this relation defines a unique function ϕ_k, such that $y_0 = \phi_k(x_0)$.

Relation $(I.6.1)$) can be written in the form $F(x, y) = 0$. The following theorem states sufficient conditions which ensure that the relation $F(x, y) = 0$ defines an implicit function.

I.6.2. Theorem. *Let F be a function of two variables which are denoted x, y. We suppose that F and partial derivatives $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ are continuous in some neighbourhood $U(A)$ of the point $A = [a, b]$. We assume that $F(A) = 0$ and $\frac{\partial F}{\partial y}(A) \neq 0$. Then there are $\delta > 0, \varepsilon > 0$ such that the unique function f is defined in a way that satisfies the following properties:*

- a) $b = f(a)$
- b) $\forall x \in (a - \delta; a + \delta) : \ f(x) \in (b - \varepsilon; b + \varepsilon)$ and $F(x, f(x)) = 0$.
- c) f, f' are continuous in $(a - \delta; a + \delta)$
- d) $\forall x \in (a - \delta; a + \delta)$

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))} \qquad (I.6.2.)$$

Moreover, if all partial derivatives of F are continuous in a neighbourhood $U(A)$ up to the k-th order, then $f, f', ...f^{(k)}$ are continuous in $(a - \delta; a + \delta)$.

I.6.3. Remark. It is very simple to derive formula $(I.6.2.)$ from a) - c). Indeed, deriving the two sides of identity (see $(I.4.4)$

$$F(x, f(x)) = 0$$

we get

$$\frac{\partial F}{\partial x}(x, f(x)) + \frac{\partial F}{\partial y}(x, f(x)) \cdot f'(x) = 0. \qquad (I.6.3.)$$

(The left hand side is derived by means of the Chain rule for composite functions of several variables.) If we calculate $f'(x)$ from this relation we get $(I.6.2)$. Taking into account a) in Theorem $I.6.2$ we get

$$f'(a) = -\frac{\frac{\partial F}{\partial x}(A)}{\frac{\partial F}{\partial y}(A)}.$$

Deriving $(I.6.3)$ we get

$$\frac{\partial^2 F}{\partial x^2} + 2\frac{\partial^2 F}{\partial x \partial y} \cdot f'(x) + \frac{\partial^2 F}{\partial y^2} \cdot (f'(x))^2 + \frac{\partial F}{\partial y} \cdot f''(x) = 0. \qquad (I.6.4.)$$

From this relation we can express $f''(x)$.

We can calculate higher order derivatives of an implicit function in a similar way.

The next theorem states sufficient conditions which ensure that the relation $F(x,y,z)=0$ defines an implicit function.

I.6.4. Theorem. *Let F be a function of three variables which are denoted x,y,z. We suppose that F and partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ are continuous in some neighbourhood $U(A)$ of point $A=[a,b,c]$. We assume that $F(A)=0$ and $\frac{\partial F}{\partial z}(A)\neq 0$. Then there are $\delta>0,\varepsilon>0$ such that the unique function f is defined in a way that satisfies the following properties:*

a) $c=f(a,b)$

b) $\forall[x,y]\in(a-\delta;a+\delta)\times(b-\delta;b+\delta)\ :\ f(x,y)\in(c-\varepsilon;c+\varepsilon)$ *and* $F(x,y,f(x,y))=0$.

c) $f,\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ *are continuous in* $(a-\delta;a+\delta)\times(b-\delta;b+\delta)$

d) $\forall[x,y]\in(a-\delta;a+\delta)\times(b-\delta;b+\delta)$

$$\frac{\partial f}{\partial x}(x,y)=-\frac{\frac{\partial F}{\partial x}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))},\qquad \frac{\partial f}{\partial y}(x,y)=-\frac{\frac{\partial F}{\partial y}(x,y,f(x,y))}{\frac{\partial F}{\partial z}(x,y,f(x,y))}\qquad (I.6.5.)$$

Moreover, if all partial derivatives of F are continuous in a neighbourhood $U(A)$ up to the k-th order, then all partial derivatives of f are continuous in $(a-\delta;a+\delta)\times(b-\delta;b+\delta)$.

I.6.5. Example. Prove that the equation $F(x,y,z)\equiv z^3-xy+yz+y^3-2=0$ in some neighbourhood of the point $A\equiv[1,1,1]$ defines function f such that $F(x,y,f(x,y))=0$ in some neighbourhood of point $[1,1]$, and calculate the partial derivatives at this point.

Solution: We use the previous theorem. Function $F(x,y,z)\equiv z^3-xy+yz+y^3-2$ is polynomial, so it is defined and continuous in $\mathbf{E}_3$ and all (first order) partial derivatives are also defined and continuous in $\mathbf{E}_3$. Substituting A into the equation we get $F(1,1,1)=0$. For the partial derivatives we get the following expressions:

$$\frac{\partial F}{\partial x}(x,y,z)=-y,\qquad \frac{\partial F}{\partial y}(x,y,z)=-x+z+3y^2,\qquad \frac{\partial F}{\partial z}(x,y,z)=3z^2+y$$

Substituting the point $A\equiv[1,1,1]$ into these expressions we get:

$$\left.\frac{\partial F}{\partial x}(x,y,z)\right|_A=-1,\qquad \left.\frac{\partial F}{\partial y}(x,y,z)\right|_A=3,\qquad \left.\frac{\partial F}{\partial z}(x,y,z)\right|_A=4\neq 0$$

Thus, all conditions of the theorem are satisfied. The unique function $f(x,y)$ defined and continuous in some neighbourhood of $[1,1]$ exists, such that $f(1,1)=1$, $F(x,y,f(x,y))=0$ in some neighbourhood of $[1,1]$. Function f has continuous partial derivatives in some neighbourhood of $[1,1]$. Using the formulas from *d)* of the theorem for partial derivatives, substituting $[x,y]=[1,1]$, taking into account $f(1,1)=1$, we get:

$$\frac{\partial f}{\partial x}(1,1)=-\frac{\frac{\partial F}{\partial x}(1,1,f(1,1))}{\frac{\partial F}{\partial z}(1,1,f(1,1))}=-\frac{\frac{\partial F}{\partial x}(1,1,1)}{\frac{\partial F}{\partial z}(1,1,1)}=\left.\frac{-(-y)}{3z^2+y}\right|_{[1,1,1]}=\frac{1}{4},$$

$$\frac{\partial f}{\partial y}(1,1)=-\frac{\frac{\partial F}{\partial y}(1,1,f(1,1))}{\frac{\partial F}{\partial z}(1,1,f(1,1))}=-\frac{\frac{\partial F}{\partial y}(1,1,1)}{\frac{\partial F}{\partial z}(1,1,1)}=\left.\frac{-(-x+z+3y^2)}{3z^2+y}\right|_{[1,1,1]}=-\frac{3}{4}$$

I.7. Local extremes.

I.7.1. Remark. In order to distinguish between extremes of function f on a set and local extremes, an extreme of f on a set is often called a *global extreme of f on a set* or an *absolute extreme of f on a set*. A maximum on a set is therefore called a *global maximum of f on a set* or an *absolute maximum of f on a set*. Analogously, we can define a *global minimum of f on a set* or an *absolute minimum of f on a set*.

I.7.2. Local maxima and local minima. We suppose that f is a function of n variables $x_1,x_2,\dots,x_n$ defined in some subset D of $\mathbf{E}_n$ and A is an interior point of D.

If there exists a reduced neighbourhood $R(A)\subset D$ such that $\forall X: X\in R(A)\Rightarrow f(A)\geq f(X)$, then we say that function f has a *local maximum at point A*. Moreover, if there exists a reduced neighbourhood $R(A)\subset D$ such that $\forall X: X\in R(A)\Rightarrow f(A)>f(X)$, then we say that function f has a *strict local maximum at point A*.

A local minimum at a point and a strict local minimum is defined by analogy. If there exists a reduced neighbourhood $R(A)\subset D$ such that $\forall X: X\in R(A)\Rightarrow f(A)\leq f(X)$, then we say that function f has a *local minimum at point A*. Moreover, if there exists a reduced neighbourhood $R(A)\subset D$ such that $\forall X: X\in R(A)\Rightarrow f(A)<f(X)$, then we say that function f has a *strict local minimum at point A*.

Local maxima and local minima are called *local extremes*. It is assumed in these definitions that point A is an interior point of function f. These definitions can be extended in some sense to other cases.

I.7.3. Local maxima and local minima with respect to a set. We suppose that f is a function of n variables $x_1,x_2,\dots,x_n$ defined in some subset D of $\mathbf{E}_n$ and point $A\in D$. If there exists a reduced neighbourhood $R(A)$ such that $\forall X: X\in R(A)\cap D\Rightarrow f(A)\geq f(X)$, then we say that function f has a *local maximum with respect to set D at point A*.

Moreover, if there exists a reduced neighbourhood $R(A)$ such that $\forall X: X\in R(A)\cap D\Rightarrow f(A)>f(X)$, then we say that function f has a *strict local maximum with respect to set D at point A*.

By analogy, we can define a local minimum with respect to a set at a point, and a strict local minimum with respect to a set at a point.

I.7.4. Theorem. Necessary condition of local extremes of differentiable functions. *If function f is differentiable at point $A \in \mathbf{E}_n$ and f has a local extreme at point A then*

$$(\operatorname{grad} f)(A) = \mathcal{O}.$$

I.7.5. Critical points. An interior point A of the domain of a function f where

$$(\operatorname{grad} f)(A) = \mathcal{O}$$

or where at least one partial derivative at point A does not exist is a so called <u>*critical point of f.*</u>

An interior point A of a set G which is contained in the domain of a function f where

$$(\operatorname{grad} f)(A) = \mathcal{O}$$

or where at least one partial derivative at point A does not exist is called a <u>*critical point of f on set G.*</u>

I.7.6. Remark. Theorem *I.7.4* implies that the only points where a function f can ever have a global extreme on a set G are critical the points of function f on set G or the boundary points of set G.

I.7.7. Theorem. Sufficient condition of local extremes of differentiable functions of two variables. *Let f be a function of two variables, and let f be differentiable at point A and $(\operatorname{grad} f)(A) = \mathcal{O}$. We assume that there exist all partial derivatives of the second order in a neighbourhood $U(A)$ which are continuous at point A. Denoting*

$$\Delta_2(A) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(A), & \frac{\partial^2 f}{\partial x \partial y}(A) \\ \frac{\partial^2 f}{\partial x \partial y}(A), & \frac{\partial^2 f}{\partial y^2}(A) \end{vmatrix}, \quad \Delta_1(A) = \frac{\partial^2 f}{\partial x^2}(A),$$

we have:

a) If $\Delta_2(A) > 0$ and $\Delta_1(A) > 0$ then function f has a strict local minimum at point A.

b) If $\Delta_2(A) > 0$ and $\Delta_1(A) < 0$ then function f has a strict local maximum at point A.

c) If $\Delta_2(A) < 0$ then function f has no local extreme at point A.

I.7.8. Theorem. Sufficient condition of local extremes of differentiable functions of n variables. *Let f be a function of n variables, and let f be differentiable at point A and $(\operatorname{grad} f)(A) = \mathcal{O}$. We assume that there exist all partial*

derivatives of the second order in a neighbourhood $U(A)$ which are continuous at point A. We use the following notation:

$$\Delta_k(A) = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2}(A), & \frac{\partial^2 f}{\partial x_1 \partial x_2}(A), & \dots, & \frac{\partial^2 f}{\partial x_1 \partial x_k}(A) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(A), & \frac{\partial^2 f}{\partial x_2^2}(A), & \dots, & \frac{\partial^2 f}{\partial x_2 \partial x_k}(A) \\ & & & \\ \frac{\partial^2 f}{\partial x_k \partial x_1}(A), & \frac{\partial^2 f}{\partial x_k \partial x_2}(A), & \dots, & \frac{\partial^2 f}{\partial x_k^2}(A) \end{vmatrix}, \quad \text{for } k = 1, 2, \dots, n$$

Assuming $\Delta_k(A) \neq 0$ for $k = 1, 2, \dots, n$ we have:

a) If $\Delta_k(A) > 0$ for $k = 1, 2, \dots, n$ then function f has a strict local minimum at point A.

b) If $(-1)^k \Delta_k(A) > 0$ for $k = 1, 2, \dots, n$ then function f has a strict local maximum at point A.

c) In other cases function f has no local extreme at point A.

I.7.9. Example. Find all local extremes of function $f : f(x, y, z) = x^2 + 3z^2 + 3y - xz - xy$.

Solution: Function f is defined in $\mathbf{E}_3$. We find all critical points of f. We calculate the partial derivatives:

$$\frac{\partial f}{\partial x}(x, y, z) = 2x - y - z, \quad \frac{\partial f}{\partial y}(x, y, z) = 3 - x, \quad \frac{\partial f}{\partial z}(x, y, z) = 6z - x$$

The partial derivatives are defined and continuous in $\mathbf{E}_3$. Using the necessary condition of a local extreme, we solve the system $(\operatorname{grad} f)(X) = \mathcal{O}$, i.e.

$$\begin{array}{rcrcrcl} 2x & - & y & - & z & = & 0 \\ 3 & - & x & & & = & 0 \\ 6z & - & x & & & = & 0 \end{array}$$

From the second equation we get $x = 3$, substituting this value into the third equation we get $z = \frac{1}{2}$ and, finally, the first equation implies $y = \frac{11}{2}$. Thus, the unique critical point of f is the point $A = [3, \frac{11}{2}, \frac{1}{2}]$.

Now we will use the sufficient condition of the existence of an extreme. We calculate all partial derivatives of the second order:

$$\frac{\partial^2 f}{\partial x^2}(x, y, z) = 2, \quad \frac{\partial^2 f}{\partial y \partial x}(x, y, z) = \frac{\partial^2 f}{\partial z \partial x}(x, y, z) = -1,$$

$$\frac{\partial^2 f}{\partial y^2}(x, y, z) = \frac{\partial^2 f}{\partial z \partial y}(x, y, z) = 0, \quad \frac{\partial^2 f}{\partial z^2}(x, y, z) = 6$$

Hence,

$$\Delta_3 = \begin{vmatrix} 2, & -1, & -1 \\ -1, & 0, & 0 \\ -1, & 0, & 6 \end{vmatrix} = -6, \qquad \Delta_2 = \begin{vmatrix} 2, & -1 \\ -1, & 0 \end{vmatrix} = -1, \qquad \Delta_1 = |2| = 2.$$

The condition in *a)* of the previous theorem is not satisfied, and the condition in *b)* is also not satisfied. Using *c)* we can conclude that the function f has no local extreme in $\mathbf{E}_2$. From this it also follows that the function f has no (global) extreme on $\mathbf{E}_2$.

In the next example we pay attention to a procedure for finding global extremes of a function on a set.

I.7.10. Example. Find the global extremes of function $f : f(x,y) = \dfrac{x^3}{3} + xy^2 - 4xy$ on the set $G = \left\{[x,y] \in \mathbf{E}_2 : y \geq \dfrac{x^2}{3} \wedge y \leq 3x\right\}$. (Draw the sketch of G.)

Solution: function f is a function defined and continuous in $\mathbf{E}_2$, and set G is a bounded closed subset of $\mathbf{E}_2$, so the (global) extremes of f on G exist, see Theorem *I.3.18*.

A function can have global extremes at critical points or at boundary points only, see Remark *I.7.6*.

A) Firstly, we find all critical points – interior points of G where $(\operatorname{grad} f)(X) = \mathcal{O}$ or where the function is not differentiable. We calculate the partial derivatives:

$$\frac{\partial f}{\partial x}(x,y) = x^2 + y^2 - 4y = x^2 + y(y-4), \qquad \frac{\partial f}{\partial y}(x,y) = 2xy - 4x = 2x(y-2)$$

Partial derivatives are defined and continuous in $\mathbf{E}_2$, so f is differentiable. Using the necessary condition of a local extreme we solve the system $(\operatorname{grad} f)(X) = \mathcal{O}$, i.e.

$$\begin{aligned} x^2 + y(y-4) &= 0 \\ 2x(y-2) &= 0 \end{aligned}$$

From the second equation we get $x = 0 \vee y = 2$.

α) Let $x = 0$; then from the first equation we get $y = 0 \vee y = 4$.

β) Let $y = 2$; then from the first equation we get $x = -2 \vee x = 2$.

Thus, we get the points: $[0,0], [0,4], [2,-2], [2,2]$. However, only point [2,2] is an interior point of G, $([0,4], [2,-2] \notin G, \ [0,0] \in \partial G)$. We denote $A_0 \equiv [2,2]$, $f(A_0) = -\dfrac{16}{3} = -5.\overline{3}$.

B) Now we will investigate the boundary of G. The boundary of G can be divided into two parts:

$$\Gamma_1 = \left\{[x,y] \in \mathbf{E}_2 : y = \frac{x^2}{3}, x \in [0;9]\right\}, \qquad \Gamma_2 = \{[x,y] \in \mathbf{E}_2 : y = 3x, x \in [0;9]\}$$

Part Γ_1: The function value of f on Γ_1 depends only on one variable:

$$F_1(x) \equiv f(x, \frac{x^2}{3}) = \frac{x^3}{3} + x\frac{x^4}{9} - 4x\frac{x^2}{3} = \frac{x^5}{9} - x^3$$

Part Γ_2: The function value of f on Γ_2 also depends only on one variable:

$$F_2(x) \equiv f(x, 3x) = \frac{28}{3}x^3 - 12x^2$$

Ba) We will investigate these functions on the open interval $x \in (0;9)$, (and we will evaluate the function values at $x = 0$ and $x = 9$ in part Bb)). We get the critical points of F_1 from the relation:

$$F_1'(x) = \frac{5}{9}x^4 - 3x^2 = 0$$

Thus, $x = 0, \sqrt{\frac{27}{5}}, -\sqrt{\frac{27}{5}}$. However, only the point $x = \sqrt{\frac{27}{5}}$ is an interior point of the interval $[0;9]$. After evaluation of the y-coordinate ($y = \frac{x^2}{3} = \frac{9}{5}$) we denote $A_1 \equiv [\sqrt{\frac{27}{5}}, \frac{9}{5}]$. We can compute the function value of f at $A_1 : f(A_1) = F_1(\sqrt{\frac{27}{5}}) \doteq -5.019389$.

Part Γ_2:

$$F_2'(x) = 28x^2 - 24x = 0 \quad \Rightarrow \quad x = 0, \frac{6}{7}.$$

The interior point of $[0;9]$ is $x = \frac{6}{7}$. After evaluation of the y-coordinate ($y = 3x = \frac{18}{7}$) we denote $A_2 \equiv [\frac{6}{7}, \frac{18}{7}]$. We can compute the function value of f at $A_2 : f(A_2) = F_2(\frac{6}{7}) \doteq -2.938775$.

Bb) Now we evaluate the function values $f(0,0) = F_1(0) = F_2(0)$, and $f(9,27) = F_1(9) = F_2(9)$:

$$x = 0 \ \Rightarrow \ y = \frac{x^2}{3} = 3x = 0 \ \Rightarrow \ A_3 \equiv [0,0], \ \ f(A_3) = 0$$

$$x = 9 \ \Rightarrow \ y = \frac{x^2}{3} = 3x = 27 \ \Rightarrow \ A_4 \equiv [9,27], \ \ f(A_4) = 5832.$$

If we compare the function values of f at points $A_0, A_1, \ldots, A_4$ we get: function f has the global minimum on G at the point $A_0 \equiv [2,2]$ and the global maximum on G at the point $A_4 \equiv [9,27]$. (Point A_0 is an interior point of G, point A_4 is a boundary point of G.)

II.8. Exercises.

1. Find the function's domain and range.

$f(x,y) = e^{16-x^2-y^2}$	$f(x,y) = \dfrac{1}{x(y+3)}$	$f(x,y) = \ln(e^2 + x^2 + y^2)$
$f(x,y) = \sqrt[4]{y-x}$	$f(x,y) = \sqrt{y-x^2}$	$f(x,y) = \cos(3x^2 - 2y + 5)$

$f(x,y,z)=\sqrt{x^2+y^2+z^2-1}$ $\qquad f(x,y,z)=yz\ln x$

$f(x,y,z)=\dfrac{1}{x^2+y^2+z^2}$ $\qquad f(x,y,z)=\arctan(x+y+z)$

2. Do the following limits exist? If yes, evaluate them.

$$\lim_{[x,y]\to[0,0]}\frac{3x^2-y^2+5}{x^2+y^2+2}\qquad \lim_{[x,y]\to[0,0]}\frac{x^4}{x^4+y^2}\qquad \lim_{[x,y]\to[0,0]}\frac{e^y\sin x}{x}$$

$$\lim_{[x,y]\to[0,0]}\frac{xy}{|xy|}\qquad \lim_{[x,y]\to[2,2]}\frac{x+y-4}{\sqrt{x+y}-2}\qquad \lim_{[x,y]\to[0,0]}\frac{x+y}{x-y}$$

$$\lim_{[x,y]\to[0,0]}\frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}\qquad \lim_{\substack{[x,y]\to[2,-4]\\ y\neq-4,\ x\neq x^2}}\frac{y+4}{x^2y-xy+4x^2-4x}$$

3. At what points $[x,y]$ in the plane are the functions continuous?

$f(x,y)=\dfrac{x+y}{x-y}$ $\qquad f(x,y)=\dfrac{x^2+y^4+1}{x^2+x-12}$ $\qquad f(x,y)=\dfrac{1}{x^2-2y}$

$f(x,y)=\ln\dfrac{y}{x}$ $\qquad f(x,y)=\cos(x^2+xy)$ $\qquad f(x,y)=e^{\frac{1}{x+y}}$

4. At what points $[x,y,z]$ in space are the functions continuous?

$f(x,y,z)=\dfrac{1}{x^2+z^2-4}$ $\qquad f(x,y,z)=\ln xyz$ $\qquad f(x,y,z)=e^z\sin(x+y)$

$f(x,y,z)=\dfrac{x+y}{x-y}$ $\qquad f(x,y,z)=\ln\dfrac{1}{xyz}$ $\qquad f(x,y,z)=\dfrac{1}{|xy|+|z|}$

$f(x,y,z)=\dfrac{1}{\ln\sqrt{x^2+y^2+z^2}}$ $\qquad f(x,y,z)=\dfrac{y+4}{x^2y-xy+4x^2-4x}$

5. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$f(x,y)=x^2-7xy+13y^2$ $\qquad f(x,y)=(x+2)^2(y+3)$ $\qquad f(x,y)=x^2(3y-5)^7$

$f(x,y)=x\sin(xy)$ $\qquad f(x,y)=\ln(x^2y)$ $\qquad f(x,y)=\dfrac{2x}{x-\sin y}$

$f(x,y)=\dfrac{x+y}{x-y}$ $\qquad f(x,y)=\ln(x^2-2y)$ $\qquad f(x,y)=\sqrt{x^2+y^2}$

$f(x,y)=e^x\ln y$ $\qquad f(x,y)=\dfrac{1}{\tan(\frac{y}{x})}$ $\qquad f(x,y)=ye^{x^2y}$

6. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

$f(x,y,z)=\dfrac{x^5y^2}{z^3}$ $\qquad f(x,y,z)=x-\sqrt{y^2+z^2}$ $\qquad f(x,y,z)=\arctan(x+y+z)$

$f(x,y,z)=xy+yz+zx$ $\qquad f(x,y,z)=\sqrt{x^2+y^2+z^2}$ $\qquad f(x,y,z)=\dfrac{1}{\sqrt{(x^2+y^2+z^2)}}$

$f(x,y,z)=x^2\sin^2y\cos z^2$ $\qquad f(x,y,z)=\dfrac{x^2}{\sqrt{y^2+z^2}}$ $\qquad f(x,y,z)=\dfrac{e^z+\ln y^2}{\sqrt{x}}$

7. Find the second order partial derivatives of the following functions.

$f(x,y)=x^2y+\cos y+y\sin x$ $\qquad f(x,y)=xe^y+y+x^5y^4-13$

$f(x,y)=e^{x+3y}+x\ln y+y\ln x+3$ $\qquad f(x,y)=y+x^2y+4y^3x-\ln(y^2+x)$

$f(x,y)=y^2+y(\sin x-x^4)$ $\qquad f(x,y)=x^2+5xy+\sin(xy)+xe^{\frac{y^2}{2}}$

8. Evaluate $\operatorname{grad} f$ at point M and directional derivative $\frac{\partial f}{\partial \vec{s}}(M)$.

$f(x,y)=x^2+2xy-3y^2,\quad M=[1,1],\quad \vec{s}=(3,4)$

$f(x,y,z)=x^2+2y^2-3z^3-17,\quad M=[1,1,1],\quad \vec{s}=(1,1,1)$

$f(x,y,z)=\cos(xy)+e^{yz}+\ln(zx),\quad M=[1,0,0.5],\quad \vec{s}=(1,2,2)$

9. Show that the following equations $F(x,y,z)=0$ define implicit functions $f: z=f(x,y)$ in the neighbourhoods of the given points $M\equiv[M_1,M_2,M_3]$, and find its partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ at $[M_1,M_2]$.

$F(x,y,z)=z^3-xy+yz+y^3-2=0,\quad M=[1,1,1]$

$F(x,y,z)=x^2-2y^2+z^2-4x+2z-5=0,\quad M=\left[-1,\sqrt{\frac{3}{2}},1\right]$

$F(x,y,z)=xz^2-x^2y+y^2z+2x-y=0,\quad M=[0,1,1]$

$F(x,y,z)=\sin(x+y)+\sin(y+z)+\sin(x+z)=0,\quad M=[\pi,\pi,\pi]$

10. Find the equations for the tangent planes and normal lines at points M on given surfaces $F(x,y,z)=0$.

$F(x,y,z)=x^2+y^2+z^2-3=0,\quad M=[1,1,1]$

$F(x,y,z)=\cos(\pi x)-x^2y+e^{xz}+yz-4=0,\quad M=[0,1,2]$

11. Find all the local maxima and local minima of the following functions.

$f(x,y)=2xy-5x^2-2y^2+4x+4y-4$ $\qquad f(x,y)=x^2+xy+3x+2y+5$

$f(x,y)=5xy-7x^2+3x-6y+2$ $\qquad f(x,y)=x^2-4xy+y^2+6y+2$

$f(x,y)=2x^2+3xy+4y^2-5x+2y$ $\qquad f(x,y)=x^2-y^2-2x+4y+6$

$f(x,y)=9x^3+3\dfrac{y^3}{3}-4xy$ $\qquad f(x,y)=8x^3+y^3+6xy$

$f(x,y)=x^3+y^3+3x^2-3y^2-8$ $\qquad f(x,y)=2x^3+2y^3-9x^2+3y^2-12y$

$f(x,y)=4xy-x^4-y^4-11$ $\qquad f(x,y)=x^4+y^4+4xy+7$

12. Find all the global maxima and global minima of the functions on the given subsets.

$f(x,y)=2x^2-4x+y^2-4y+2,\quad G=\{[x,y]:x\geq0,\ y\leq2,\ y\geq2x\}$

$f(x,y)=x^2-xy+y^2+7,\quad G=\{[x,y]:x\geq0,\ y\leq4,\ y\geq x\}$

$f(x,y)=x^2+xy+y^2-6x+2,\quad G=\{[x,y]:0\leq x\leq5,\ -3\leq y\leq3\}$

$f(x,y)=x^2+xy+y^2-6x,\quad G=\{[x,y]:0\leq x\leq5,\ -3\leq y\leq0\}$

$f(x,y)=48xy-32x^3-24y^2,\quad G=\{[x,y]:0\leq x\leq1,\ 0\leq y\leq1\}$

$f(x,y)=x^2-y^2,\quad G=\{[x,y]:x\geq-1,\ y\geq-1,\ x+2y\leq2\}$