# II. Riemann Integral of a Function of One Variable

## II.1. Motivation and definition of the Riemann integral.

**II.1.1. Physical motivation.** 1. Suppose that we have a spring or a thin rod of a finite length which need not be homogeneous. This can be caused e.g. by a varying cross–section of the rod or by varying density of the material that the rod is made of. We can assume that the rod covers the interval $\langle a,b\rangle$ on the $x$–axis and its longitudinal density (i.e. amount of mass per unit of length) is a function $y=\rho(x)$. We wish to evaluate the mass $M$ of the rod.

If function $\rho$ is constant then the problem is very easy – we can simply put $M$ equal to the product of the constant longitudinal density $\rho$ and the length of the interval $\langle a,b\rangle$, i.e. $M=\rho\cdot(b-a)$.

If $\rho$ is not constant then we can divide the rod (i.e. the interval $\langle a,b\rangle$) into many shorter parts (the subintervals $\langle x_0,x_1\rangle$, $\langle x_1,x_2\rangle$, $\langle x_2,x_3\rangle$, ..., $\langle x_{n-1},x_n\rangle$ where $x_0=a$ and $x_n=b$) and we can approximate function $\rho$ by a constant on each of the subintervals. A reasonable value of this constant is $\rho(\zeta_i)$ for some $\zeta_i\in\langle x_{i-1},x_i\rangle$ $(i=1,2,\dots,n)$. Then the approximate masses of the the shorter parts of the rod are

$$\rho(\zeta_1)\cdot\Delta x_1,\quad \rho(\zeta_2)\cdot\Delta x_2,\quad \dots,\quad \rho(\zeta_n)\cdot\Delta x_n$$

where $\Delta x_1=x_1-x_0$, $\Delta x_2=x_2-x_1$, ..., $\Delta x_n=x_n-x_{n-1}$. The approximate mass of the whole rod is

$$\sum_{i=1}^{n}\rho(\zeta_i)\cdot\Delta x_i\,.$$

We can naturally expect that this sum will approach the exact value of the total mass $M$ of the rod if $n\to+\infty$ and the numbers $\Delta x_i$ $(i=1,2,\dots,n)$ tend to zero.

2. Suppose that a car moves in a time interval $\langle a,b\rangle$ and its velocity is given by the function $y=v(t)$. We wish to compute the distance $d$ the car travels in the time interval $\langle a,b\rangle$.

If the velocity $v$ is constant then the distance is obviously $d=v\cdot(b-a)$.

If the velocity is varying then we can divide the time interval $\langle a,b\rangle$ into many shorter subintervals $\langle t_0,t_1\rangle$, $\langle t_1,t_2\rangle$, ..., $\langle t_{i-1},t_i\rangle$ (where $t_0=a$ and $t_n=b$) and we can approximate the velocity by a constant on each of these shorter subintervals. A natural value of this constant is $v(\zeta_i)$ for some $\zeta_i\in\langle t_{i-1},t_i\rangle$ $(i=1,2,\dots,n)$. The approximate distances moved in the time intervals $\langle t_0,t_1\rangle$, $\langle t_1,t_2\rangle$, ..., $\langle t_{n-1},t_n\rangle$ are

$$v(\zeta_1)\cdot\Delta t_1,\quad v(\zeta_2)\cdot\Delta t_2,\quad \dots,\quad v(\zeta_n)\cdot\Delta t_n$$

where $\Delta t_1=t_1-t_0$, $\Delta t_2=t_2-t_1$, ..., $\Delta t_n=t_n-t_{n-1}$. The approximate distance travelled in the whole time interval $\langle a,b\rangle$ is



$$\sum_{i=1}^{n}v(\zeta_i)\cdot\Delta t_i\,.$$

One can expect that this sum will approach the real distance that the car travels in the time interval $\langle a,b\rangle$ if $n\to+\infty$ and the numbers $\Delta t_i$ $(i=1,2,\dots,n)$ tend to zero.

**II.1.2. Geometric motivation.** Suppose that $f$ is a nonnegative and bounded function on an interval $\langle a,b\rangle$ and $D$ is the region between its graph and the $x$–axis. (See Fig 1.) An important question is how to define and evaluate the area of $D$.

If $f$ is a constant function on $\langle a,b\rangle$ then $D$ is a rectangle and its area is equal to the product $f\cdot(b-a)$.

If function $f$ is not constant then we can again subdivide the interval $\langle a,b\rangle$ into many short subintervals $\langle x_0,x_1\rangle$, $\langle x_1,x_2\rangle$, ..., $\langle x_{n-1},x_n\rangle$ (with $x_0=a$ and $x_n=b$) and we can approximate $f$ by a constant on each of these subintervals. A possible value of this constant is $f(\zeta_i)$ for some $\zeta_i\in\langle x_{i-1},x_i\rangle$ $(i=1,2,\dots,n)$. Thus, we can approximate the area of the region below the graph of $f$ on the subinterval $\langle x_{i-1},x_i\rangle$ by the area of the rectangle with the sides $f(\zeta_i)$ and $\Delta x_i$ $(\equiv x_i-x_{i-1})$. The approximate value of the area of the whole region $D$ is equal to the total area of all the rectangles:

$$\sum_{i=1}^{n}f(\zeta_i)\cdot\Delta x_i\,.$$

(See Fig. 1.) We can now define the area of $D$ as a limit of this sum for $n\to+\infty$ and the lengths $\Delta x_i$ of the subintervals $\langle x_{i-1},x_i\rangle$ $(i=1,2,\dots,n)$ tending to zero.

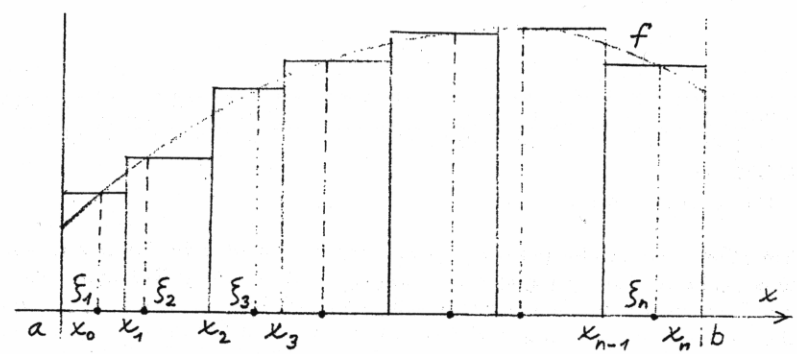


Fig. 1

One can observe that all the situations described in paragraphs II.1.1 and II.1.2 lead to the limit of a certain sum and the sum is the same in all the considered situations. We explain in the next paragraphs what we exactly understand under the limit of this sum, what we call it, how we denote it and how we can evaluate it.

**II.1.3. Partition of an interval.** Let $\langle a, b\rangle$ be a bounded closed interval. A system of points $x_0, x_1, \ldots, x_n$ such that $a = x_0 < x_1 < \ldots < x_n = b$ is called a *partition* of the interval $\langle a, b\rangle$. If this partition is named $P$ then we write

$$P: \quad a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b. \qquad \text{(II.1)}$$

The *norm* of partition $P$ is the number $\|P\| = \max_{i=1,\ldots,n}(x_i - x_{i-1})$. (Thus, $\|P\|$ is the length of the largest of the subintervals $\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{n-1}, x_n\rangle$ and it informs us how "fine" partition $P$ is.)

**II.1.4. Riemann sums and their limit.** Let $y = f(x)$ be a bounded function on the interval $\langle a, b\rangle$ and let $P$ be the partition of $\langle a, b\rangle$ given by (II.1). Denote by $\Delta x_i$ the length of the $i$–th subinterval $\langle x_{i-1}, x_i\rangle$ (i.e. $\Delta x_i = x_i - x_{i-1}$). Let $V$ be a system of points $\zeta_1 \in \langle x_0, x_1\rangle, \zeta_2 \in \langle x_1, x_2\rangle, \ldots, \zeta_n \in \langle x_{n-1}, x_n\rangle$. Then the *Riemann sum* of function $f$ on the interval $\langle a, b\rangle$ corresponding to partition $P$ and system $V$ is

$$s(f, P, V) = \sum_{i=1}^{n} f(\zeta_i) \cdot \Delta x_i.$$

We say that number $S$ is the *limit of the Riemann sums* $s(f, P, V)$ *as* $\|P\| \to 0+$ if to every given $\epsilon > 0$ there exists $\delta > 0$ such that for every partition $P$ of $\langle a, b\rangle$ and for every choice of $V$, $\|P\| < \delta$ implies $|s(f, P, V) - S| < \epsilon$. We write:

$$\lim_{\|P\| \to 0+} s(f, P, V) = S. \qquad \text{(II.2)}$$

**II.1.5. Riemann integral.** If the limit in (II.2) exists then function $f$ is called *integrable* in the interval $\langle a, b\rangle$ and $S$ is called the *Riemann integral* of function $f$ on $\langle a, b\rangle$. The integral is usually denoted as

$$\int_a^b f(x)\, dx \quad \text{or} \quad \int_a^b f\, dx.$$

The numbers $a$ and $b$ in this integral are called the *limits of integration*, $a$ being the *lower limit* and $b$ being the *upper limit*. The integrated function is called the *integrand*.

The Riemann integral is also often called the *definite integral*.

**II.1.6. The area of the region between the graph of a function and the $x$–axis.** It follows from paragraph II.1.2 and the definition of the Riemann integral that if $f$ is a nonnegative and integrable function on the interval $\langle a, b\rangle$ then the *area* of the region between the graph of $f$ and the $x$–axis can be defined as the value of the integral $\int_a^b f\, dx$.

By analogy, if function $f$ is nonpositive and integrable on the interval $\langle a, b\rangle$ then the area of the region bounded by the $x$–axis (from above) and the graph of $f$ (from below) can be defined as $-\int_a^b f\, dx$.

In a general case, when $f$ has both negative and positive values in the interval $\langle a, b\rangle$, the integral $\int_a^b f\, dx$ expresses the sum of the areas of all the parts of the



region between the graph of $f$ and the $x$–axis, but the contributions of the parts below the $x$–axis are taken negatively.

You will see in Chapter III that the area of more general sets than the regions below or above the graph of a function $y = f(x)$ can be defined by means of a so called two–dimensional measure $m_2$ and evaluated by means of a double integral.

**II.1.7. Extension of the definition of the Riemann integral.** If function $f$ is integrable in the interval $\langle a, b\rangle$ then we put

$$\int_b^a f\, dx = -\int_a^b f\, dx.$$

Specially, we also put $\displaystyle\int_a^a f\, dx = 0.$

**II.1.8. The mean value of function $f$ on an interval.** Let function $f$ be integrable in the interval $\langle a, b\rangle$. The number

$$\mu = \frac{1}{b-a} \int_a^b f\, dx$$

is called the *mean value* (or the *average value*) of function $f$ on the interval $\langle a, b\rangle$.

The mean value has the following geometric interpretation: Suppose for simplicity that function $f$ is nonnegative on the interval $\langle a, b\rangle$. Then the mean value $\mu$ is such a number that the region between the graph of $f$ and the $x$–axis has the same area as the rectangle with the sides $b - a$ and $\mu$. It is clear that

$$\inf_{x \in \langle a, b\rangle} f(x) \le \mu \le \sup_{x \in \langle a, b\rangle} f(x). \qquad \text{(II.3)}$$

## II.2. Integrability (existence of the Riemann integral) – sufficient conditions.

The two statements "the Riemann integral $\int_a^b f(x)\, dx$ exists" and "function $f$ is integrable in the interval $\langle a, b\rangle$" say exactly the same.

Most of the functions you will use in various applications will be integrable. Nevertheless, you should be aware that there also exist "bad" functions such that the limit of the Riemann sums (II.2) does not exist. Thus, the Riemann integral of these functions also does not exist. These functions are called *non–integrable*. The next theorem and Remark II.2.2 give sufficient conditions for the integrability of function $f$ (i.e. for the existence of the Riemann integral of $f$).

**II.2.1. Existence theorem for the Riemann Integral.** *Let function $f$ be continuous on the interval $\langle a, b\rangle$. Then $f$ is integrable in $\langle a, b\rangle$.*

**II.2.2. Remark.** This theorem can be generalized:

*Let function $f$ be bounded and piecewise–continuous on the interval $\langle a, b\rangle$. Then it is integrable in $\langle a, b\rangle$.*

(A function $f$ is said to be <u>*piecewise-continuous*</u> in the interval $\langle a,b\rangle$ if $\langle a,b\rangle$ can be divided into a finite number of subintervals such that $f$ is continuous in the interior of each of them.)

## II.3. Important properties of the Riemann integral.

**II.3.1. Theorem. (The domination inequality for the Riemann integral.)** *If functions $f$ and $g$ are both integrable in the interval $\langle a,b\rangle$ and $g(x)\le f(x)$ for all $x\in\langle a,b\rangle$ then*

$$\int_a^b g\,dx \le \int_a^b f\,dx.$$

*Specially, if $f(x)\ge 0$ for all $x\in\langle a,b\rangle$ then* $\displaystyle\int_a^b f\,dx \ge 0.$

**II.3.2. Theorem. (Boundedness of the Riemann integral.)** *If function $f$ is integrable in the interval $\langle a,b\rangle$ and $m\le f(x)\le M$ for all $x\in\langle a,b\rangle$ then*

$$m\cdot(b-a) \le \int_a^b f(x)\,dx \le M\cdot(b-a).$$

Both theorems II.3.1 and II.3.2 easily follow from the definition of the Riemann integral. Theorem II.3.1 tells us that if function $f$ dominates function $g$ on $\langle a,b\rangle$ and the functions $f$ and $g$ are both integrable in $\langle a,b\rangle$ then also the integral of $f$ dominates the integral of $g$ on $\langle a,b\rangle$. The inequality in II.3.2 shows that the value of the Riemann integral can be estimated by means of the lower bound and the upper bound of function $f$.

**II.3.3. Theorem. (Linearity of the Riemann integral.)** *If functions $f$ and $g$ are integrable in $\langle a,b\rangle$ and $\alpha\in\mathbf{R}$ then*

$$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx \quad\text{and}\quad \int_a^b \alpha\cdot f\,dx = \alpha\cdot\int_a^b f\,dx.$$

(This property is already known from the theory of the indefinite integral.)

**II.3.4. Theorem. (Additivity of the Riemann integral with respect to the interval.)** *If the integrals $\int_a^c f\,dx$ and $\int_c^b f\,dx$ exist then*

$$\int_a^c f\,dx + \int_c^b f\,dx = \int_a^b f\,dx.$$

**II.3.5. Theorem.** a) *If function $f$ is integrable in the interval $\langle a,b\rangle$ and if $\langle c,d\rangle\subset\langle a,b\rangle$ then $f$ is also integrable in $\langle c,d\rangle$.*

b) *If functions $f$ and $g$ are both integrable in the interval $\langle a,b\rangle$ then the product $f\cdot g$ is also integrable in $\langle a,b\rangle$.*



c) *If function $f$ is integrable in the interval $\langle a,b\rangle$ and function $g$ differs from $f$ in at most a finite number of points then function $g$ is also integrable in $\langle a,b\rangle$ and*

$$\int_a^b g\,dx = \int_a^b f\,dx.$$

Item a) is an immediate consequence of the definition of the Riemann integral.

Item b) is a statement about the integrability of a function which is the product of two other functions. However, bear in mind that the fact that "the integrability of $f$ and $g$ implies the integrability of $f\cdot g$" <u>does not mean</u> that $\int_a^b f\cdot g\,dx = \left(\int_a^b f\,dx\right)\cdot\left(\int_a^b g\,dx\right)$!

Item c) tells us that the change of function values of $f$ in at most a finite number of points does not affect the existence or the value of the integral $\int_a^b f\,dx$. In other words: <u>The existence and the value of the integral $\int_a^b f\,dx$ do not depend on function values of $f$ in a finite number of points.</u> Thus, function $f$ need not even be defined in a finite number of points of the interval $\langle a,b\rangle$ and this has no influence on the existence and the value of $\int_a^b f\,dx$. Specially, it plays no role whether the integral $\int_a^b f\,dx$ is considered on a closed or on an open interval!

**II.3.6. Theorem. (The Riemann integral as a function of its upper limit.)** *Suppose that function $f$ is integrable in the interval $\langle a,b\rangle$. Then*

a) *the function $F(x)=\int_a^x f(t)\,dt$ is continuous in $\langle a,b\rangle$,*

b) *the equality*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \tag{II.4}$$

*holds in all points $x\in(a,b)$ in which $f$ is continuous.*

Function $G(x)=\int_x^b f(t)\,dx$ (with the variable lower limit) is also continuous in $\langle a,b\rangle$. However, it satisfies the equality in b) with the change of the sign:

$$\frac{d}{dx}\int_x^b f(t)\,dt = -f(x). \tag{II.5}$$

(This is a consequence of the equation $G(x)=\int_a^b f(t)\,dt - F(x)$.)

Equalities (II.4) and (II.5) can also be modified for the boundary points of the interval $\langle a,b\rangle$ so that if function $f$ is right–continuous at point $a$ (respectively left–continuous at point $b$) then $F'_+(a)=f(a)$ (respectively $F'_-(b)=f(b)$).

The validity of statement a) follows (at least intuitively) from the geometric interpretation of the Riemann integral (see paragraphs II.1.2 and II.1.6). Formula (II.4) can be proved in this way:

$$F'(x)=\lim_{h\to0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to0}\frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] =$$

$$= \lim_{h\to0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to0}\mu(h)$$

where $\mu(h)$ is the mean value of function $f$ on the interval with the end points $x$ and $x+h$. The continuity of $f$ at point $x$ and (II.3) imply that $\mu(h) \to f(x)$ if $h \to 0$. This proves (II.4).

**II.3.7. Remark.** It follows from Theorem II.3.6 that if function $f$ is continuous in interval $J$ and $c \in J$ then the function $F(x) = \int_c^x f(t)\, dt$ is an antiderivative to function $f$ in $J$.

**II.3.8. Remark.** Formula (II.4) can be generalized. If function $f$ is continuous in interval $I$ and $a(x)$, $b(x)$ are differentiable functions of variable $x$ in interval $J$ with their values in $I$ then

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\, dt = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

for $x \in J$.

## II.4. Evaluation of the Riemann integral.

We come to one of the fundamental topics of this chapter – to the question how to evaluate the integral $\int_a^b f\, dx$. Due to its importance, the next theorem is called the Fundamental Theorem of Integral Calculus:

**II.4.1. Theorem.** *If function $f$ is continuous in the interval $\langle a,b\rangle$ and $F$ is an antiderivative to $f$ in $\langle a,b\rangle$ then*

$$\int_a^b f\, dx = F(b) - F(a). \tag{II.6}$$

The formula (II.6) is called the *Newton–Leibnitz formula*. The difference $F(b)-F(a)$ is often written in a shorter form: $F(b)-F(a) = [F]_a^b$

The proof of the Fundamental Theorem of Integral Calculus is easy: The function $G(x) = \int_a^x f(t)\, dt$ is also an antiderivative to $f$ in $\langle a,b\rangle$. Thus, there exists a constant $c$ such that $F = G + c$ on $\langle a,b\rangle$. This means that $F(a) = G(a) + c = c$ (because $G(a) = 0$) and $F(b) = G(b) + c = G(b) + F(a)$. This yields: $\int_a^b f(t)\, dt = G(b) = F(b) - F(a)$.

The Newton–Leibnitz formula connects the indefinite and the definite integral: When you know the indefinite integral of $f$ on the interval $\langle a,b\rangle$ then you also know all antiderivatives to $f$ on $\langle a,b\rangle$. You can choose any of them and use it in the Newton–Leibnitz formula to obtain the value of the definite integral of $f$ on $\langle a,b\rangle$. The fact that the indefinite integral and the antiderivative are so important in the calculation of the definite integral was one of the main reasons why you have learned to compute indefinite integrals.

You already know that all antiderivatives to function $f$ on the interval $\langle a,b\rangle$ differ at most in an additive constant. Thus, if you choose e.g. an antiderivative



$F + k$ (where $k$ is a constant) instead of $F$ and you use it in the Newton–Leibnitz formula, you get

$$\int_a^b f\, dx = [F+k]_a^b = (F(b)+k) - (F(a)+k) = F(b) - F(a).$$

The result is the same as in formula (II.6). Hence you can see that it is not important which one of the infinitely many antiderivatives to $f$ on $\langle a,b\rangle$ you use.

**II.4.2. Example.** $\int_0^\pi \sin x\, dx = [-\cos x]_0^\pi = (-\cos\pi) - (-\cos 0) = 2.$

The next two theorems show that the method of integration by parts and the method of substitution, known from the theory of indefinite integral, can also be directly applied to the Riemann integral.

**II.4.3. Theorem. (Integration by parts for the Riemann integral.)** *Let the functions $u$ and $v$ have continuous derivatives in the interval $\langle a,b\rangle$. Then*

$$\int_a^b u'\cdot v\, dx = [u\cdot v]_a^b - \int_a^b u\cdot v'\, dx. \tag{II.7}$$

**II.4.4. Example.** $\int_0^2 e^{2x}\cdot x\, dx = {}^{*)}\ \left[\tfrac{1}{2}e^{2x}\cdot x\right]_0^2 - \int_0^2 \tfrac{1}{2}e^{2x}\, dx =$

$$= \tfrac{1}{2}e^4\cdot 2 - \tfrac{1}{2}e^0\cdot 0 - \left[\tfrac{1}{4}e^{2x}\right]_0^2 = e^4 - \tfrac{1}{4}e^4 + \tfrac{1}{4}e^0 = \tfrac{3}{4}e^4 + \tfrac{1}{4}.$$

*) We have put $u'(x) = e^{2x}$, $u(x) = \frac{1}{2}e^{2x}$, $v(x) = x$ and $v'(x) = 1$.

**II.4.5. Theorem. (Integration by substitution for the Riemann integral.)** *Let function $g$ have a continuous derivative in the interval $\langle a,b\rangle$ and let $g$ map $\langle a,b\rangle$ into interval $J$. Let function $f$ be continuous in $J$. Then*

$$\int_a^b f(g(x))\cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(s)\, ds. \tag{II.8}$$

Formula (II.8) can be used in two situations: you wish to evaluate the integral on the left hand side and you transform it to the integral on the right hand side (if the integral on the right hand side is simpler) OR vice versa.

**II.4.6. Example.** Let us evaluate $\int_0^{\pi/2} \sin^2 x\cdot\cos x\, dx$.

If we put $\langle a,b\rangle = \langle 0,\pi/2\rangle$, $s = g(x) = \sin x$, $f(s) = s^2$, $J = (-\infty,+\infty)$, we can see that all the assumptions of Theorem II.4.5 are satisfied. Moreover, $g(0) = \sin 0 = 0$ and $g(\pi/2) = \sin(\pi/2) = 1$. Applying formula (II.8), we obtain:

$$\int_0^{\pi/2} \sin^2 x\cdot\cos x\, dx = \int_0^1 s^2\, ds = \left[\tfrac{1}{3}s^3\right]_0^1 = \tfrac{1}{3}.$$

**II.4.7. Example.** Let us evaluate $\int_0^2 \sqrt{4-x^2}\,dx$.

We can take this integral for the integral on the right hand side of (II.8) (with the variable denoted by $x$ instead of $s$). The function $f(x)=\sqrt{4-x^2}$ is continuous on $\langle 0,2\rangle$ and so the integral exists. Put $x=g(t)=2\sin t$, $dx=g'(t)\,dt=2\cos t\,dt$. We have $g(a)=2\sin a=0$ and $g(b)=2\sin b=2$. Thus, we can choose $a=0$ and $b=\pi/2$. Then all the assumptions of Theorem II.4.5 are satisfied and we obtain:

$$\int_0^2 \sqrt{4-x^2}\,dx = \int_0^{\pi/2} \sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt = \int_0^{\pi/2} 4\cos^2 t\,dt =$$
$$= \int_0^{\pi/2} 2(1+\cos 2t)\,dt = \left[2t+\sin 2t\right]_0^{\pi/2} = \pi.$$

**II.4.8. Remark.** Suppose that you have to evaluate a Riemann integral on the interval $\langle a,b\rangle$ and you wish to use integration by parts or a substitution. Then you have two possibilities:

1) You can use Theorem II.4.3 or Theorem II.4.5. You transform the integral to other (simpler) integrals and you deal with the upper and the lower limits of all the integrals during the computation. This approach is explained in examples II.4.5, II.4.6 and II.4.7.

2) You can first compute the integral as an indefinite integral on the interval $\langle a,b\rangle$ and then you apply the Newton–Leibnitz formula (II.6) on $\langle a,b\rangle$.

   To show what we exactly mean by this, let us compute the integral from example II.4.6 once again, this time by the method we are just explaining. Thus, let us start with the indefinite integral $\int \sin^2 x\cos x\,dx$. We can use the substitution $s=\sin x$. Then $ds=\cos x\,dx$ and

$$\int \sin^2 x\cdot\cos x\,dx = \int s^2\,ds = \tfrac{1}{3}s^3+c = \tfrac{1}{3}\sin^3 x + c.$$

   Formula (II.6) now gives: $\int_0^{\pi/2} \sin^2 x\cos x\,dx = \left[\tfrac{1}{3}\sin^3 x\right]_0^{\pi/2} = \tfrac{1}{3}$.

   As you will observe after having solved a larger number of examples, approach 1), based on direct application of integration by parts or integration by substitution to definite integrals, is usually technically simpler and less laborious.

## II.5. Numerical integration.

You will remember from the theory of the indefinite integral that an antiderivative to a given function $f$ often exists, but it cannot be obtained by standard methods of integration and it cannot be expressed in a "closed form" (i.e. by a formula prescribing a finite number of operations). Analogously, it often happens that the the Riemann integral $\int_a^b f\,dx$ exists, but it cannot be evaluated by a standard integration based on the Newton–Leibnitz formula. However, there exist approximate methods (also called numerical methods) which enable us to evaluate the integral approximately, with an error as small as we wish. We shall explain two such methods in this section.



Both these methods usually require the performance of a higher number of arithmetic operations in order to reach a higher accuracy (i.e. a smaller error). Therefore approximate methods are usually used on computers.

Both the methods are based on the partition

$$P:\quad a=x_0<x_1<x_2\ldots<x_{n-1}<x_n=b \qquad \text{(II.9)}$$

of the interval $\langle a,b\rangle$ to $n$ subintervals $\langle x_{k-1},x_k\rangle$ $(k=1,2,\ldots,n)$ of equal length $h$. Thus,

$$h=\frac{b-a}{n} \quad\text{and}\quad x_k=a+k\cdot h \;\; (k=1,2,\ldots,n).$$

We shall denote $y_k=f(x_k)$.

**II.5.1. The trapezoidal method.** Suppose that we approximate function $f$ by a linear function on each of the subintervals $\langle x_{k-1},x_k\rangle$. A linear function is uniquely determined by the requirement that its graph (a straight line) passes through two chosen points. Let these points be $[x_{k-1},y_{k-1}]$ and $[x_k,y_k]$. Then the linear function has the equation $y=y_{k-1}+(y_k-y_{k-1})/h\cdot(x-x_{k-1})$. We can easily integrate it on the interval $\langle x_{k-1},x_k\rangle$ and we obtain $I_k=h\cdot(y_{k-1}+y_k)/2$. $I_k$ is the area of the trapezoid (see Fig. 2). When we sum all the numbers $I_1, I_2, \ldots, I_n$, we get

$$T_n = \frac{h}{2}\cdot\left[y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n\right]. \qquad \text{(II.10)}$$

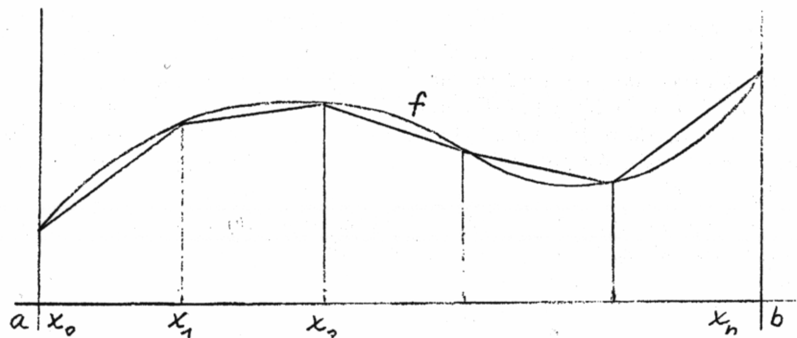


Fig. 2

$T_n$ is an approximate value of the Riemann integral $\int_a^b f\,dx$. The geometric sense of $T_n$ is seen on Fig. 2 – it is the sum of the areas of $n$ trapezoids constructed on the intervals $\langle x_0,x_1\rangle$, $\langle x_1,x_2\rangle$ …, $\langle x_{n-1},x_n\rangle$.

As to the accuracy of the approximation, it generally holds that the finer the partition of $\langle a,b\rangle$, the better are the results. In other words, the accuracy of the approximation increases with increasing $n$ (i.e. decreasing $h$). It can be proved that if $f''$ is continuous on $\langle a,b\rangle$ and $M$ is an upper bound for the values of $|f''|$ on $\langle a,b\rangle$ then the following error estimate holds:

$$\left| T_n - \int_a^b f\,dx \right| \le \frac{b-a}{12}\,h^2 M. \tag{II.11}$$

**II.5.2. Simpson's method.** Suppose now that $n$ is an even number. We can approximate function $f$ by a quadratic function on each of the subintervals $\langle x_0, x_2\rangle$, $\langle x_2, x_4\rangle$, ... , $\langle x_{n-2}, x_n\rangle$. A quadratic function on a subinterval $\langle x_{k-2}, x_k\rangle$ ($k = 2, 4, \dots, n$) is uniquely defined by the requirement that its graph (a parabola) passes through three chosen points – let it be the points $[x_{k-2}, y_{k-2}]$, $[x_{k-1}, y_{k-1}]$, $[x_k, y_k]$. The integral of this quadratic function on $\langle x_{k-2}, x_k\rangle$ can be relatively easily evaluated – you can check that it is $I_k = h\cdot(y_{k-2} + 4y_{k-1} + y_k)/3$. Summing all the numbers $I_2, I_4, \dots, I_n$, we obtain

$$S_n = \frac{h}{3}\cdot\left[y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 2y_{n-2} + 4y_{n-1} + y_n\right]. \tag{II.12}$$

Provided that the fourth derivative $f^{(4)}$ of function $f$ is continuous on $\langle a, b\rangle$ and $M$ is an upper bound for the values of $|f^{(4)}|$ on $\langle a, b\rangle$, the following error estimate holds:

$$\left| S_n - \int_a^b f\,dx \right| \le \frac{b-a}{180}\,h^4 M. \tag{II.13}$$

## II.6. Improper Riemann integral.

A fundamental assumption in the definition of the Riemann integral $\int_a^b f\,dx$ was the boundedness of the interval $\langle a, b\rangle$ and the boundedness of function $f$ on $\langle a, b\rangle$. However, we often need to work with integrals whose domain of integration (the interval) or the integrand (the function) are unbounded. Such integrals, where either the interval or the integrand (or both) are unbounded, are called improper Riemann integrals. We will explain the definition of the improper Riemann integral in this section.

Suppose that function $f$ is defined in the interval $\langle a, b)$ and that it is integrable on each interval $\langle a, t\rangle$ (for $a \le t < b$). If the limit

$$\lim_{t\to b-}\int_a^t f(x)\,dx$$

exists, then its value is called an *improper Riemann integral* with a *singular upper limit*.

The improper Riemann integral of function $f$ is denoted in the same way as the "usual" Riemann integral, i.e. $\int_a^b f\,dx$. Thus, we can write:

$$\int_a^b f(x)\,dx = \lim_{t\to b-}\int_a^t f(x)\,dx.$$

The improper Riemann integral with a *singular lower limit* can be defined quite analogously. The definition can even be extended to the case when both the limits are singular: If the two integrals $\int_a^c f\,dx$ and $\int_c^b f\,dx$ exist (the first one as an improper integral with a singular lower limit and the second one as an improper integral with a singular upper limit) and their sum is defined (i.e. it is not for example $-\infty + \infty$) then we put $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$.



If function $f$ is integrable (in the sense of paragraph II.1.5) on the interval $\langle a, b\rangle$ then the improper Riemann integral of $f$ on $\langle a, b)$ coincides with the "usual" Riemann integral of $f$ on $\langle a, b\rangle$. Thus, the improper Riemann integral represents an extension of the definition of the "usual" Riemann integral.

The value of the improper Riemann integral $\int_a^b f\,dx$ can either be finite (we say that the integral $\int_a^b f\,dx$ *converges*) or it can be infinite (the integral $\int_a^b f\,dx$ *diverges*).

**II.6.1. Example.** Let us evaluate the improper Riemann integral $\displaystyle\int_1^{+\infty}\frac{1}{x}\,dx$.

The function $f(x) = 1/x$ is continuous on the interval $\langle 1, +\infty)$ and it has an antiderivative $F(x) = \ln x$. Thus,

$$\int_1^t \frac{1}{x}\,dx = F(t) - F(1) = \ln t - \ln 1 = \ln t.$$

Since $\displaystyle\lim_{t\to+\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to+\infty}\ln t = +\infty$, we get: $\displaystyle\int_1^{+\infty}\frac{1}{x}\,dx = +\infty$.

## II.7. Historical remark.

Both differential calculus (i.e. limits, derivatives, their applications, etc.) and integral calculus (i.e. integrals) are together called ***calculus***. Many aspects of finding and analytically describing tangent lines were worked out by René Descartes (1596–1650), Bonaventura Cavalieri (1598–1647), Pierre de Fermat (1601–1665) and others. However, we usually consider Sir Isaac Newton (1642–1727) and Baron Gottfried Wilhelm Leibnitz (1646–1716) to be the inventors of calculus. They were the first to understand that the process of finding tangents and the process of finding areas are mutually inverse. Since they lived in the same time, the question of priority over the invention of calculus has lead to the bitter controversies. Leibnitz was accused of copying Newton's work and the Royal Society of London did not exonerate him from this charge after investigating the matter. Present–day historians and mathematicians consider that Leibnitz's and Newton's inventions were simultaneous, but independent. Nevertheless, the disputation caused a split in the mathematical world for one and half centuries. The followers of Newton, mostly British, pursued his methods while Leibnitz's pupils, mostly French, Germans and Swiss, followed his approach. Due to Leibnitz's superior notation and his simpler mathematical language, his followers were able to be more successful than their British counterparts in the further development of calculus.

The original historical definition of the definite integral was different from the definitions you can find in present–day literature. This is especially due to the fact that the concept accepted at the time of Newton and Leibnitz is not quite correct from the present–day point of view. However, since this concept is very simple, let us explain it.

Suppose that $f$ is a function defined and bounded on the interval $\langle a,b\rangle$. We divide the interval $\langle a,b\rangle$ into infinitely many "infinitely small" parts. A typical "infinitely small" part is an interval $\langle x, x+dx\rangle$, where $dx$ is an "infinitely small" positive number. The product $f(x)\cdot dx$ has the following geometric interpretation: The region between the graph of $f$ and the interval $\langle x, x+dx\rangle$ on the $x$–axis can be taken as an "infinitely narrow" rectangle and if $f(x) > 0$ then $f(x)\cdot dx$ is the area of this rectangle. The sum of all "infinitely small" numbers $f(x)\cdot dx$ (for all $x \in \langle a,b\rangle$) was called the *definite integral* of function $f$ on the interval $\langle a,b\rangle$.

The incorrectness of this approach can immediately be seen – the notion of an "infinitely small" positive real number $dx$ is wrong: such a number does not exist! If you do not believe this, then imagine that you have such a number. Is it e.g. $10^{-6}$? No, because you can find many positive numbers less than $10^{-6}$. And what about $dx = 10^{-20}$? Even this $dx$ is not "infinitely small" because there exist many other positive numbers, less than $10^{-20}$. You can see that the concept of an "infinitely small positive number" logically leads to the contradiction. Mathematics cannot allow itself to work with notions which are not defined precisely. (Overlooking this rule has often in the past lead to surprising contradictions or confusions in mathematical theories and models.) This motivated Georg F.B.Riemann (1826–1866) to study the definite integral in detail and put it on solid logical foundations.

Nevertheless, in spite of the logical incorrectness of the concept of an infinitely small positive number $dx$, the idea often still appears in various applications, and we have not completely abandoned it. We will use it again in paragraphs IV.2.1, IV.4.1 and V.2.1 which have a motivating character and whose main purpose it to show that the following definitions of various types of integrals are reasonable and that the integrals have some physical sense.

## II.8. Exercises.

**1.** Do the following Riemann integrals exist?

$$\int_{-2}^{1} \frac{x+1}{x^2-x-6}\,dx \qquad \int_{1}^{2} \frac{\ln x}{x}\,dx \qquad \int_{0}^{1} \frac{\sin x}{x}\,dx$$

$$\int_{-1}^{5} e^{-x}\,dx \qquad \int_{-2.5}^{3} \frac{x}{\ln(x+3)}\,dx \qquad \int_{-2}^{-1} \frac{x^2+1}{x^3-2x^2+x}\,dx$$

**2.** Evaluate the following integrals.

$$\int_{-1}^{1} (3x^2-4x+7)\,dx \qquad \int_{0}^{1} (8t^3-12t^2+5)\,dt \qquad \int_{1}^{2} \frac{4}{s^2}\,ds$$

$$\int_{1}^{27} x^{-4/3}\,dx \qquad \int_{0}^{2} s\sqrt{4x+1}\,dx \qquad \int_{0}^{1} \frac{36\,du}{(2u+1)^3}$$

$$\int_{1}^{2} \left(w+\frac{1}{w^2}\right) dw \qquad \int_{0}^{1/2} x^3\,(1+9x^4)^{-3/2}\,dx \qquad \int_{0}^{\pi} \sin 5r\,dr$$



$$\int_{0}^{\pi} \cos 3\varphi\,d\varphi \qquad \int_{0}^{3\pi} \cos^2\left(\frac{x}{6}\right) dx \qquad \int_{0}^{\pi} \tan^2\left(\frac{\theta}{3}\right) d\theta$$

$$\int_{0}^{\pi/2} 5\,(\sin x)^{3/2}\,\cos x\,dx \qquad \int_{\pi/2}^{\pi/2} 15\,(\sin 3x)^4\,\cos 3x\,dx \qquad \int_{-1}^{1} 2x\,\sin(1-x^2)\,dx$$

$$\int_{2}^{3} \frac{dv}{v^3-2v^2+v} \qquad \int_{1}^{4} \frac{1+\sqrt{z}}{z^2}\,dz \qquad \int_{-1}^{1} \frac{a^2}{a^2+y^2}\,dy \quad (a \neq 0)$$

$$\int_{0}^{\pi/2} x\,\cos x\,dx \qquad \int_{-3\pi/2}^{-2\pi} \cos^6 x\,dx \qquad \int_{0}^{1} \ln(a+x)\,dx \quad (a > 0)$$

$$\int_{0}^{\pi/2} \frac{dx}{2+\cos x}\,dx \qquad \int_{0}^{2} x^2\sqrt{4-x^2}\,dx \qquad \int_{0}^{1} \frac{dx}{1+\sqrt{x}}$$

$$\int_{0}^{\ln 2} \sqrt{e^x-1}\,dx \qquad \int_{e}^{e^2} \frac{dx}{x\ln x} \qquad \int_{0}^{\pi/8} \sin^3(4x)\,dx$$

**3.** Find the area of the region between the graph of $f$ and the $x$–axis.

$f(x) = x^2-4x+3, \quad 0 \le x \le 3$ $\qquad$ $f(x) = 1-(x^2/4), \quad -2 \le x \le 3$

$f(x) = 5-5x^{2/3}, \quad -1 \le x \le 8$ $\qquad$ $f(x) = 1-\sqrt{x}, \quad 0 \le x \le 4$

**4.** Find the average value of

$f(x) = \sqrt{3x}$ over $\langle 0,3\rangle$ $\qquad$ $f(x) = \sqrt{ax}$ over $\langle 0,a\rangle$

$f(x) = mx+b$ over $\langle -1,1\rangle$ $\qquad$ $f(x) = mx+b$ over $\langle -k,k\rangle$

**5.** Evaluate the following improper integrals.

$$\int_{0}^{+\infty} \frac{dx}{1+x^2} \qquad \int_{-\infty}^{+\infty} \frac{dx}{4+x^2} \qquad \int_{-\infty}^{-2} \frac{dx}{x^2}$$

$$\int_{2}^{+\infty} \frac{dx}{x^2-1} \qquad \int_{0}^{+\infty} y^2\,e^{-x}\,dx \qquad \int_{0}^{5} \frac{1}{\sqrt{x}}\,dx$$

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+4x+9}\,dx \qquad \int_{0}^{\infty} x\,e^{-x^2}\,dx \qquad \int_{1}^{5} (x-1)^a\,dx\,; \quad a > -1$$

**6.** Evaluate $F'(x)$ if function $F$ is defined by the following integrals.

$$F(x) = \int_{1/x}^{\sqrt{x}} \cos(t^2)\,dt\,; \quad x > 0 \qquad F(x) = \int_{0}^{2x} \frac{\sin x}{x}\,dx$$

$$F(x) = \int_{x}^{0} \sqrt{1+t^4}\,dt \qquad F(x) = \int_{x^2}^{x^3} \ln t\,dt\,; \quad x > 0$$