III. Riemann Integral of a Function of Two and Three Variables

III.1. The double integral – motivation and definition. The two-dimensional Jordan measure and measurable sets in $\mathbf{E}_2$.

III.1.1. Physical motivation. Suppose that we have a thin plate covering the rectangle $R = \langle a, b\rangle \times \langle c, d\rangle$ in the xy-plane. The plate need not be homogeneous and so its planar density (i.e. the amount of mass per unit area – let us denote it $\rho(x, y)$) varies with the position of the point $[x, y]$. We wish to evaluate the mass M of the plate.

If density ρ is constant then $M = \rho \cdot (b - a) \cdot (d - c)$. (Why?)

In a general case when the density is not constant, we can subdivide rectangle R into many smaller pieces $R_1, \ldots, R_n$ by a network of lines parallel to the x- and y-axes. If rectangles $R_1, \ldots, R_n$ are "small enough" then ρ can be approximated by a constant on each of them. A reasonable value of this constant is $\rho(Z_i)$ where Z_i is some point from R_i. Then the approximate mass of the part of the plate covering rectangle R_i is $\rho(Z_i \cdot \Delta x_i \, \Delta y_i$ where Δx_i and Δy_i are the lengths of sides of R_i. The approximate mass of the whole plate is

$$\sum_{i=1}^{n} \rho(Z_i) \cdot \Delta x_i \, \Delta y_i \, .$$

It is now natural to expect that the exact value of the total mass M of the plate will be equal to the limit of this sum as $n \to +\infty$, and the numbers Δx_i and Δy_i $(i = 1, 2, \ldots, n)$ tend to zero.

III.1.2. Geometric motivation. Suppose that $z = f(x, y)$ is a nonnegative function on set $R \in \mathbf{E}_2$ and we wish to define and evaluate the volume V of the region between the graph of function f and the xy-plane. Suppose for simplicity that R is the rectangle $\langle a, b\rangle \times \langle c, d\rangle$.

If f is a constant function on R then the volume is $V = f \cdot m_2(R) = f \cdot (b - a) \cdot (d - c)$.

If f is not a constant function then we can use the same partition of R into n smaller pieces $R_1, \ldots, R_n$ as in paragraph III.1.1 and we can approximate the volume of the region between the graph of function f on rectangle R_i and the xy-plane by the number $f(Z_i) \cdot \Delta x_i \, \Delta y_i$ for some $Z_i \in R_i$ $(i = 1, 2, \ldots, n)$. The volume of the whole region between the graph of f on set R and the xy-plane can be approximated by the sum

$$\sum_{i=1}^{n} f(Z_i) \cdot \Delta x_i \, \Delta y_i \, .$$

The volume of the region between the graph of function f on R and the xy-plane can now be naturally defined as the limit for $n \to +\infty$ and the numbers Δx_i, Δy_i $(i = 1, 2, \ldots, n)$ tending to zero.

III.1.3. Rectangular region in $\mathbf{E}_2$ and its partition. If $\langle a, b\rangle$ is a closed interval on the x-axis and $\langle c, d\rangle$ is a closed interval on the y-axis then the Cartesian product $R = \langle a, b\rangle \times \langle c, d\rangle$ forms a rectangle in $\mathbf{E}_2$. We can subdivide this rectangle by a network of lines parallel to the x- and y-axes into n smaller rectangles $R_1, \ldots, R_n$. The system of these smaller rectangles is called the <u>*partition*</u> of rectangle R.

If this partition is named P and if the lengths of sides of smaller rectangles $R_1, \ldots, R_n$ are $\Delta x_1, \Delta y_1, \ldots, \Delta x_n, \Delta y_n$ then the number which is equal to the maximum of $\Delta x_1, \Delta y_1, \ldots, \Delta x_n, \Delta y_n$ is denoted by $\|P\|$ and it is called the <u>*norm*</u> of partition P.

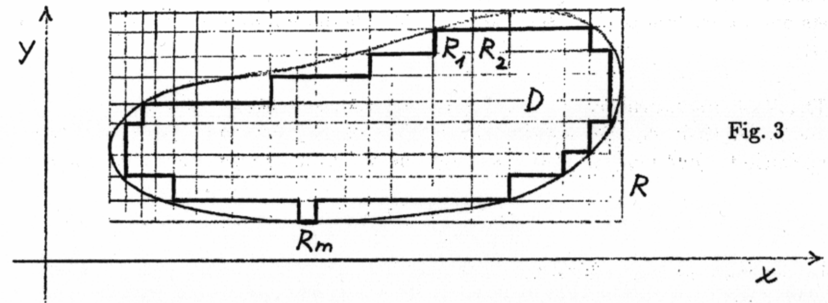

Fig. 3

III.1.4. Riemann sums and their limit. Let $z = f(x, y)$ be a bounded function on a bounded set $D \subset \mathbf{E}_2$. Let R be the smallest rectangle in $\mathbf{E}_2$ whose sides are parallel to the x- and y-axes and which contains D. Let P be a partition of R to smaller rectangles $R_1, \ldots R_n$ whose lengths of sides are $\Delta x_1, \Delta y_1, \ldots, \Delta x_n, \Delta y_n$. The smaller rectangles can be numbered so that those of them which are inside D are $R_1, \ldots, R_m$. (See Fig. 3.) Let V be a system of points $Z_i \in R_i$ $(i = 1, 2, \ldots, m)$. Then the <u>*Riemann sum*</u> of function f on set D corresponding to partition P and system V is

$$s(f, P, V) = \sum_{i=1}^{m} f(Z_i) \cdot \Delta x_i \, \Delta y_i \, .$$

We say that number S is the <u>*limit of the Riemann sums*</u> $s(f, P, V)$ <u>*as*</u> $\|P\| \to 0+$ if to every given $\epsilon > 0$ there exists $\delta > 0$ such that for every partition P of R and for every choice of V, $\|P\| < \delta$ implies $|s(f, P, V) - S| < \epsilon$. We write:

$$\lim_{\|P\| \to 0+} s(f, P, V) = S. \qquad \text{(III.1)}$$

III.1.5. The double integral. If the limit in (III.1) exists, then function f is called <u>*integrable*</u> in set D and S is called the <u>*double integral*</u> of function f on D. The integral is usually denoted as

$$\iint_D f(x, y) \, dx \, dy \qquad \text{or} \qquad \iint_D f \, dx \, dy.$$

III.1.6. Remark. It follows from paragraph III.1.2 and from the definition of the double integral that if the function $y = f(x, y)$ is nonnegative and integrable on set $D \in \mathbf{E}_2$ then the integral $\iint_D f\,dx\,dy$ defines and evaluates the volume of the region between the graph of f on D and the xy-plane. However, you will see in Sections III.5 – III.7 that the volumes of even more general regions in $\mathbf{E}_3$ can also be defined and evaluated by means of volume integrals.

The notion of a bounded set in $\mathbf{E}_2$ is too general for practical purposes. For example, it can be shown that there exist bounded sets $D \in \mathbf{E}_2$ such that even the constant function is not integrable on D. In order to distinguish between these "bad" sets and other "reasonable" sets, we introduce the notion of a so called measurable set.

III.1.7. A measurable set in $\mathbf{E}_2$ and its Jordan measure. Suppose that D is a bounded set in $\mathbf{E}_2$. We say that this set is *measurable* (in the sense of Jordan) if the constant function $f(x, y) = 1$ is integrable on D. In this case, we call the number

$$m_2(D) = \iint_D dx\,dy$$

the *two-dimensional Jordan measure* of set D.

$m_2(D)$ has a very simple geometric interpretation – it defines and evaluates the *area of set D*.

It is important to have a criterion which enables us easily to recognize some simple measurable sets. We will give such a criterion in paragraph III.1.10. However, we first list some sets whose measure is zero.

III.1.8. Some sets whose two–dimensional Jordan measure is zero. It can be proved for example that the following sets in $\mathbf{E}_2$ have the measure equal to zero:

a) Sets consisting of a finite number of points.

b) Graphs of continuous functions $y = \varphi(x)$ or $x = \psi(y)$ on closed bounded intervals.

c) So called simple smooth curves, respectively simple piecewise–smooth curves (see Section IV.1).

The next theorem is quite obvious, and it also concerns sets of measure zero.

III.1.9. Theorem. a) *If $N_1, N_2, \ldots, N_n$ are sets whose measure is zero then $m_2\left(\cup_{i=1}^n N_i\right) = 0$.*

b) *If $M \subset N$ and $m_2(N) = 0$ then $m_2(M) = 0$.*

III.1.10. Theorem. (Sufficient and necessary condition for measurability of a set in $\mathbf{E}_2$.) *A bounded set $D \subset \mathbf{E}_2$ is measurable if and only if $m_2(\partial D) = 0$ (where ∂D is the boundary of D).*

III.2. Existence and important properties of the double integral.

The two statements "f is integrable on set D" and "the double integral $\iint_D f\,dx\,dy$ exists" say exactly the same.

III.2.1. Existence theorem for the double Integral. *Let D be a measurable set in $\mathbf{E}_2$ and let f be a bounded function on D whose set of discontinuities has measure m_2 equal to zero. Then f is integrable on D.*

In particular, if D is a measurable set and f is a bounded continuous function on D then f is integrable on D.

III.2.2. Important properties of the double integral. The double integral has many properties which are exactly the same as the properties of the one–dimensional Riemann integral explained in paragraphs III.3.1 – III.3.5. Let us mention only several of them:

a) **(Linearity of the double integral.)** *If functions f and g are integrable on set $D \subset \mathbf{E}_2$ and $\alpha \in \mathbf{R}$ then*

$$\iint_D (f+g)\,dx\,dy = \iint_D f\,dx\,dy + \iint_D g\,dx\,dy,$$

$$\iint_D \alpha \cdot f\,dx\,dy = \alpha \cdot \iint_D f\,dx\,dy.$$

b) **(Additivity of the double integral with respect to the set.)** *If D_1 and D_2 are measurable sets such that $m_2\left(D_1 \cap D_2\right) = 0$ (i.e. D_1 and D_2 are not overlapping) and if f is integrable on D_1 and on D_2 then*

$$\iint_{D_1} f\,dx\,dy + \iint_{D_2} f\,dx\,dy = \iint_{D_1 \cup D_2} f\,dx\,dy.$$

c) *If function f is integrable on set $D \in \mathbf{E}_2$ and function g differs from f at most on a set whose measure is zero then g is also integrable on D and*

$$\iint_D g\,dx\,dy = \iint_D f\,dx\,dy.$$

d) *If $D \subset \mathbf{E}_2$ and $m_2(D) = 0$ then $\iint_D f\,dx\,dy = 0$ for every function f.*

Proposition c) shows that the behaviour of the integrated function on a set of measure zero does not affect the existence and the value of the double integral. Thus, from the point of view of integration on set D, whose boundary has measure zero, it is not important whether D is considered open (i.e. without its boundary) or closed (i.e. with its boundary).

III.3. Evaluation of the double integral – Fubini's theorem and transformation to the polar coordinates.

Fubini's theorem transforms the evaluation of a double integral to the computation of two single (= one–dimensional) integrals. It can be applied if the domain of integration is a so called elementary region.

III.3.1. Elementary region in $\mathbf{E}_2$. a) Let $y = \phi_1(x)$ and $y = \phi_2(x)$ be continuous functions on the interval $\langle a, b\rangle$ and let $\phi_1(x) \leq \phi_2(x)$ for all $x \in \langle a, b\rangle$. Then the set

$$D = \{[x,y] \in \mathbf{E}_2;\ a \leq x \leq b,\ \phi_1(x) \leq y \leq \phi_2(x)\}$$

is called the *elementary region relative to the x–axis*.

b) Let $x = \psi_1(y)$ and $x = \psi_2(y)$ be continuous functions on the interval $\langle c, d\rangle$ and let $\psi_1(y) \leq \psi_2(y)$ for all $y \in \langle c, d\rangle$. Then the set

$$D = \{[x,y] \in \mathbf{E}_2;\ c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}$$

is called the *elementary region relative to the y–axis*.

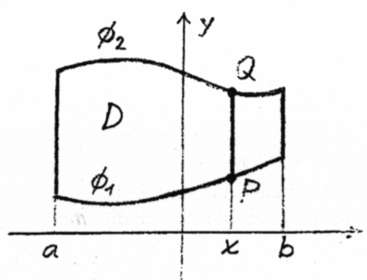

Fig. 4a

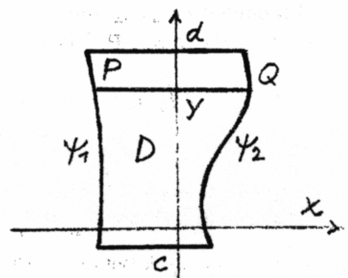

Fig. 4b

Elementary regions are measurable sets in $\mathbf{E}_2$. Let us now explain the idea of integrating of function $z = f(x,y)$ on the elementary region relative to the x–axis (see Fig. 4a). Imagine that we can cut the region into infinitely many infinitely narrow vertical stripes. One such stripe is the line segment PQ on Fig. 4a. We first integrate f on each such segment as a function of one variable y – we obtain $F(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy$. Certainly, this depends on x because the position of the line segment PQ depends on x. Then we integrate $F(x)$ as a function of x from a to b. Thus, we obtain formula (III.2) (see the next paragraph III.3.2).

The next theorem precisely formulates the assumptions under which we can apply the described method, and it also treats the case when D is an elementary region relative to the y–axis.

III.3.2. Fubini's theorem for the double integral. a) *Let D be the elementary region relative to the x–axis from paragraph III.3.1. Let the function $z = f(x,y)$ be continuous on D. Then*

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right) dx. \qquad (III.2)$$

b) *Let D be the elementary region relative to the y–axis from paragraph III.3.1. Let function $z = f(x,y)$ be continuous on D. Then*

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right) dy. \qquad (III.3)$$

III.3.3. Example Evaluate the integral $\iint_D (2x + 3y + 5)\,dx\,dy$ where D is the region bounded by the curves $y = \frac{1}{2}x$, $y = 1/x$ and $x = \sqrt{2}$.

The given curves divide the xy–plane into various regions (Sketch a figure!) but only one of them is bounded and this is D. It can be described as the set of all points $[x,y] \in \mathbf{E}_2$ such that $\sqrt{2}/2 \leq x \leq \sqrt{2}$ and $1/x \leq y \leq 2x$.

D is measurable (because it is bounded and its boundary has the measure equal to zero – see Theorem III.1.11). Function f is continuous on D. Thus, using the Fubini theorem III.3.2, we obtain:

$$\iint_D (2x + 3y + 5)\,dx\,dy = \int_{\sqrt{2}/2}^{\sqrt{2}} \left(\int_{1/x}^{2x} (2x + 3y + 5)\,dy\right) dx =$$

$$= \int_{\sqrt{2}/2}^{\sqrt{2}} \left[2xy + \tfrac{3}{2}y^2 + 5y\right]_{y=1/x}^{y=2x} dx = \int_{\sqrt{2}/2}^{\sqrt{2}} \left(4x^2 + 6x^2 + 10x - 2 - \frac{3}{2x^2} - \frac{5}{x}\right) dx =$$

$$= \left[\frac{10x^3}{3} + 5x^2 - 2x + \frac{3}{2x} - 5\ln x\right]_{\sqrt{2}/2}^{\sqrt{2}} = \tfrac{79}{12}\sqrt{2} + 7.5 - 5\ln 2.$$

You remember that a powerful method for computation of a one–dimensional integral is the method of substitution. This method can also be used when we evaluate a double integral. When applying it, we usually say that we transform the integral to new coordinates. The most–used new coordinates in $\mathbf{E}_2$ are so called polar (or generalized polar) coordinates.

III.3.4. Polar coordinates in $\mathbf{E}_2$. The position of a point $X \in \mathbf{E}_2$ is uniquely given by its *polar coordinates* r, φ whose geometric meaning is the following: r is the distance of X from the origin O and φ is the angle between the positive part of the x–axis and the line segment OX. φ is measured from the x–axis towards the line segment OX. (Sketch a figure!) The relation between the Cartesian coordinates x, y and the polar coordinates r, φ is given by the equations

$$x = r\cos\varphi, \quad y = r\sin\varphi. \qquad (III.4)$$

III.3.5. Transformation of the double integral to the polar coordinates. Suppose that we have to evaluate the integral $\iint_D f(x,y)\,dx\,dy$. We can use the equations (III.4) and replace x, y by $r\cos\varphi$, respectively $r\sin\varphi$. However, we must also

a) change D (analogously to the change of the limits in the one–dimensional integral if we apply the method of substitution – see Theorem II.4.6),

b) substitute for the term $dx\,dy$ in the integral (analogously to the equation $dx = g'(t)\,dt$ if we use the substitution $x = g(t)$ in a one–dimensional integral).

Set D corresponds to some other set D' in the polar coordinates. Optimally, every point $[x, y] \in D$ should have just one opposite point $[r, \varphi] \in D'$ (such that $x = r\cos\varphi$ and $y = r\sin\varphi$). However, since the sets of measure zero play no role, the one–to–one correspondence between the points $[x, y] \in D$ and the points $[r, \varphi] \in D'$ can be disturbed on a set of measure zero, both on the side of D and on the side of D'.

It can be proved that $dx\,dy$ must be substituted in this way:

$$dx\,dy = r\,dr\,d\varphi. \tag{III.5}$$

The factor r on the right hand side is a so called "Jacobian" (the abbreviation of the "Jacobi determinant") and you will find more about it in Section III.9.

The transformation of the double integral to the polar coordinates has a sense if it leads either to the simplification of the integrand (see example III.3.6) or to the simplification of the domain of integration (see example III.3.7). It follows from the geometric sense of the polar coordinates that $r \geq 0$ and φ can be taken from an interval whose length does not exceed 2π (i.e. the interval $(0, 2\pi)$).

III.3.6. Example. Evaluate the integral $\iint_D (x+y)\,dx\,dy$ where $D = \{[x, y] \in \mathbf{E}_2;\ x > 0,\ y > 0,\ x^2 + y^2 < 4\}$.

D is the intersection of the disk (with the center at the origin and radius 2) and the first quadrant. It corresponds to the domain $D' = \{[r, \varphi] \in \mathbf{E}_2;\ 0 < r < 2,\ 0 < \varphi < \pi/2\}$ in the polar coordinates. Thus, using the transformation (III.4), (III.5), we obtain:

$$\iint_D (x+y)\,dx\,dy = \iint_{D'} (r\cos\varphi + r\sin\varphi)\,r\,dr\,d\varphi =$$

$$= {}^{1)} \int_0^2 \left(\int_0^{\pi/2} r^2 (\cos\varphi + \sin\varphi)\,d\varphi \right) dr = \int_0^2 r^2 \left[\sin\varphi - \cos\varphi\right]_0^{\pi/2} dr =$$

$$= \int_0^2 2r^2\,dr = \tfrac{16}{3}.$$

[1]) We have applied Fubini's theorem.

III.3.7. Example. Evaluate the integral $\iint_D (x^2+y^2)^{-1/2}\,dx\,dy$ where D is a triangle with the vertices $[0, 0]$, $[1, 0]$, $[1, 1]$.

D can also be described as the set of all points $[x, y]$ such that $0 < x < 1$ and $0 < y < x$. Transforming these inequalities to the polar coordinates, we obtain

$$0 < r\cos\varphi < 1, \quad 0 < r\sin\varphi < r\cos\varphi. \tag{III.6}$$

The second inequality implies: $0 < \sin\varphi < \cos\varphi$ which means that $0 < \varphi < \pi/4$. The first inequality in (III.6) implies: $0 < r < 1/\cos\varphi$. Hence D corresponds to the set $D' = \{[r, \varphi] \in \mathbf{E}_2; 0 < \varphi < \pi/4,\ 0 \leq r \leq 1/\cos\varphi\}$ in the polar coordinates. Thus,

using the transformation (III.4), (III.5) and afterwards applying Fubini's theorem, we obtain:

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy = \iint_{D'} \frac{1}{r}\,r\,dr\,d\varphi = \int_0^{\pi/4} \left(\int_0^{1/\cos\varphi} dr \right) d\varphi =$$

$$= \int_0^{\pi/4} \frac{1}{\cos\varphi}\,d\varphi = \int_0^{\pi/4} \frac{\cos\varphi}{1-\sin^2\varphi}\,d\varphi = {}^{2)} \int_0^{\sqrt{2}/2} \frac{dt}{1-t^2} =$$

$$= \tfrac{1}{2} \int_0^{\sqrt{2}/2} \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt = \tfrac{1}{2} \left[-\ln(1-t) + \ln(1+t) \right]_0^{\sqrt{2}/2} = \tfrac{1}{2} \ln \frac{2+\sqrt{2}}{2-\sqrt{2}}.$$

[2]) We have used the substitution $\sin\varphi = t$, $\cos\varphi\,d\varphi = dt$.

III.3.8. Generalized polar coordinates in $\mathbf{E}_2$. We shall denote these coordinates again by r, φ. They are analogous to the polar coordinates, though their origin need not be the same as the origin of the Cartesian coordinates and they are not "isotropic", i.e. the rate of change of r is different in the x–direction and in the y–direction. The relation between the Cartesian coordinates x, y and the generalized polar coordinates r, φ is

$$x = x_0 + ar\cos\varphi, \quad y = y_0 + br\sin\varphi \tag{III.7}$$

where $[x_0, y_0]$ is a given point in $\mathbf{E}_2$ and a, b are positive constants.

By analogy with (III.5), it can be proved that if we transform a double integral to the generalized polar coordinates then $dx\,dy$ must be substituted in this way:

$$dx\,dy = rab\,dr\,d\varphi. \tag{III.8}$$

The factor rab on the right hand side is again the "Jacobian", and it will be explained in Section III.9.

The transformation of a double integral to the generalized polar coordinates usually simplifies the integral if the integrand depends on x and y through the expression $(x-x_0)^2/a^2 + (y-y_0)^2/b^2$ or if the domain of integration is the interior of an ellipse $(x-x_0)^2/a^2 + (y-y_0)^2/b^2 = 1$ or a sector of an ellipse.

III.3.9. Example. Evaluate the integral $\iint_D x\,dx\,dy$ where $D = \{[x, y] \in \mathbf{E}_2;\ (x-2)^2 + (y-1)/4^2 \leq 1\}$.

We can observe that if we use the transformation

$$x = 2 + r\cos\varphi, \quad y = 1 + 2r\sin\varphi \tag{III.9}$$

then the points $[x, y]$ fill up D if and only if the points $[r, \varphi]$ fill up the set $D' = \{[r, \varphi] \in \mathbf{E}_2;\ 0 \leq r \leq 1,\ 0 \leq \varphi < 2\pi\}$. Using transformation (III.9), equality $dx\,dy = 2r\,dr\,d\varphi$ (following from (III.8)) and also applying Fubini's theorem, we get:

$$\iint_D x\,dx\,dy = \iint_{D'} (2 + r\cos\varphi)\,dr\,d\varphi = \int_0^{2\pi} \left(\int_0^1 (2 + r\cos\varphi)\,dr \right) d\varphi =$$

$$= \int_0^{2\pi} \left[2r + \tfrac{1}{2} r^2 \cos\varphi\right]_0^1 d\varphi = \int_0^{2\pi} (2 + \tfrac{1}{2}\cos\varphi)\,d\varphi = 4\pi.$$

III.3.10. Remark. In fact, transformation (III.9) is not a one–to–one mapping of set D' onto set D in example III.3.8. The one–to–one correspondence is disturbed on the subset $D'_0 = \{[r,\varphi] \in \mathbf{E}_2;\ r = 0,\ 0 \le \varphi < 2\pi\}$ of D'. (You can observe that D'_0 is a subset of the boundary of D'.) This is clear, because transformation (III.9) maps all points of D'_0 onto the point $[2,1]$ in D. Thus, the point $[2,1]$ in D has infinitely many opposite points in D' – i.e. all points of D'_0. However, since $m_2(D'_0) = 0$, this does not affect the existence and the value of the integral.

III.4. Some physical applications of the double integral.

Suppose that a two–dimensional thin plate coincides with a measurable set D in the xy–plane. The plane need not be homogeneous. It means that its planar density (amount of mass per unit of area) need not be constant. Let the planar density be given by function $\rho(x,y)$. The double integral enables us to define and evaluate some fundamental mechanical characteristics of the plate. Suppose that ρ is expressed in $[\mathrm{kg}\cdot\mathrm{m}^{-2}]$. Then we have:

Mass $\quad M = \iint_D \rho(x,y)\,dx\,dy \quad [\mathrm{kg}]$,

Static moment about the x–axis $\quad M_x = \iint_D y\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Static moment about the y–axis $\quad M_y = \iint_D x\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Center of mass $[x_m, y_m]$ $\quad x_m = \dfrac{M_y}{M},\quad y_m = \dfrac{M_x}{M} \quad [\mathrm{m}]$,

Moment of inertia about the x–axis $\quad J_x = \iint_D y^2\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the y–axis $\quad J_y = \iint_D x^2\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the origin $\quad J_0 = \iint_D (x^2+y^2)\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}^2]$.

Suggest a formula for the moment of inertia about a general straight line in $\mathbf{E}_2$ whose equation is $ax + by + c = 0$!

III.5. The volume integral – motivation and definition. The three–dimensional Jordan measure and measurable sets in $\mathbf{E}_3$.

The theory of the volume integral is almost identical with the theory of the double integral. The main difference lies in the simple fact that we have one more dimension. Thus, we can repeat almost everything that was written about the double integral. The same holds for the three–dimensional Jordan measure. This is why we

present the theory of the volume integral very briefly and we do not explain the details.

On the other hand, since one more dimension causes higher variety of possible domains of integration as well as integrated functions, you will see that the methods of evaluation of the volume integral, though their techniques are again based on Fubini's theorem and on the transformation to other coordinates, are usually technically more complicated than in the case of the double integral.

III.5.1. Physical motivation. Suppose that we have a body whose density is $\rho(x,y,z)$. We wish to evaluate the mass M of the body. Suppose for simplicity that the body has the form of the block $B = \langle a,b\rangle \times \langle c,d\rangle \times \langle r,s\rangle$.

If density ρ is constant then $M = \rho\cdot(b-a)\cdot(d-c)\times(s-r)$. (Why?)

However, in a general situation when the density is not constant, we can subdivide B into n smaller rectangular cells $B_1, \dots, B_n$ by planes parallel to the coordinate planes xy, xz and yz. If the cells B_1 are "small enough" then we can approximate ρ by a constant on each of them. A possible value of this constant is $\rho(Z_i)$ where Z_i is some point from the cell R_i. Then the approximate mass of cell R_i is $\rho(Z_i)\cdot\Delta x_i\,\Delta y_i\,\Delta z_i$ where Δx_i, Δy_i and Δz_i are the lengths of sides of R_i. The approximate mass of the whole body is

$$\sum_{i=1}^{n} \rho(Z_i)\cdot\Delta x_i\,\Delta y_i\,\Delta z_i.$$

The exact value of the mass M of the body is equal to the limit of this sum as $n \to +\infty$ and the numbers Δx_i, Δy_i, Δz_i $(i = 1,2,\dots,n)$ tend to zero.

III.5.2. A block in $\mathbf{E}_3$ and its partition. If $\langle a,b\rangle$ is a bounded closed interval on the x–axis, $\langle c,d\rangle$ is a bounded closed interval on the y–axis and $\langle r,s\rangle$ is a closed bounded interval on the z–axis then the set $B = \langle a,b\rangle \times \langle c,d\rangle \times \langle r,s\rangle$ forms a block in $\mathbf{E}_3$. We can subdivide this block to n rectangular cells $B_1, \dots, B_n$ by planes parallel to the xy–plane, xz–plane and yz–plane. The system of these cells is called the *partition* of B.

If this partition is named P and if the lengths of sides of smaller cells $B_1, \dots, B_n$ are Δx_1, Δy_1, Δz_1, …, Δx_n, Δy_n, Δz_n then the maximum of all these lengths is denoted by $\|P\|$ and it is called the *norm* of partition P.

III.5.3. Riemann sums and their limit. Let $u = f(x,y,z)$ be a bounded function on a bounded set $D \subset \mathbf{E}_3$. Let B be the smallest block in $\mathbf{E}_2$ whose sides are parallel to the xy–, xz– and yz–planes and which contains D. Let P be a partition of B into smaller rectangular cells $B_1, \dots B_n$ whose lengths of sides are Δx_1, Δy_1, Δz_1, …, Δx_n, Δy_n, Δz_n. These smaller cells can be numbered so that those of them which are inside D are $B_1, \dots, B_m$. Let V be a system of points $Z_i \in B_i$ $(i = 1,2,\dots,m)$. Then the *Riemann sum* of function f on set D corresponding to partition P and system V is

$$s(f,P,V) = \sum_{i=1}^{m} f(Z_i)\cdot\Delta x_i\,\Delta y_i\,\Delta z_i.$$