III.3.10. Remark. In fact, transformation (III.9) is not a one–to–one mapping of set D' onto set D in example III.3.8. The one–to–one correspondence is disturbed on the subset $D'_0 = \{[r,\varphi] \in \mathbf{E}_2;\ r=0,\ 0 \le \varphi < 2\pi\}$ of D'. (You can observe that D'_0 is a subset of the boundary of D'.) This is clear, because transformation (III.9) maps all points of D'_0 onto the point $[2,1]$ in D. Thus, the point $[2,1]$ in D has infinitely many opposite points in D' – i.e. all points of D'_0. However, since $m_2(D'_0) = 0$, this does not affect the existence and the value of the integral.

III.4. Some physical applications of the double integral.

Suppose that a two–dimensional thin plate coincides with a measurable set D in the xy–plane. The plane need not be homogeneous. It means that its planar density (amount of mass per unit of area) need not be constant. Let the planar density be given by function $\rho(x,y)$. The double integral enables us to define and evaluate some fundamental mechanical characteristics of the plate. Suppose that ρ is expressed in $[\mathrm{kg}\cdot\mathrm{m}^{-2}]$. Then we have:

Mass $\quad M = \iint_D \rho(x,y)\,dx\,dy \quad [\mathrm{kg}]$,

Static moment about the x–axis $\quad M_x = \iint_D y\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Static moment about the y–axis $\quad M_y = \iint_D x\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Center of mass $[x_m, y_m]$ $\quad x_m = \dfrac{M_y}{M},\quad y_m = \dfrac{M_x}{M} \quad [\mathrm{m}]$,

Moment of inertia about the x–axis $\quad J_x = \iint_D y^2\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the y–axis $\quad J_y = \iint_D x^2\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the origin $\quad J_0 = \iint_D (x^2+y^2)\cdot\rho(x,y)\,dx\,dy \quad [\mathrm{kg}\cdot\mathrm{m}^2]$.

Suggest a formula for the moment of inertia about a general straight line in $\mathbf{E}_2$ whose equation is $ax + by + c = 0$!

III.5. The volume integral – motivation and definition. The three–dimensional Jordan measure and measurable sets in $\mathbf{E}_3$.

The theory of the volume integral is almost identical with the theory of the double integral. The main difference lies in the simple fact that we have one more dimension. Thus, we can repeat almost everything that was written about the double integral. The same holds for the three–dimensional Jordan measure. This is why we

present the theory of the volume integral very briefly and we do not explain the details.

On the other hand, since one more dimension causes higher variety of possible domains of integration as well as integrated functions, you will see that the methods of evaluation of the volume integral, though their techniques are again based on Fubini's theorem and on the transformation to other coordinates, are usually technically more complicated than in the case of the double integral.

III.5.1. Physical motivation. Suppose that we have a body whose density is $\rho(x,y,z)$. We wish to evaluate the mass M of the body. Suppose for simplicity that the body has the form of the block $B = \langle a,b\rangle \times \langle c,d\rangle \times \langle r,s\rangle$.

If density ρ is constant then $M = \rho\cdot(b-a)\cdot(d-c)\times(s-r)$. (Why?)

However, in a general situation when the density is not constant, we can subdivide B into n smaller rectangular cells $B_1, \dots, B_n$ by planes parallel to the coordinate planes xy, xz and yz. If the cells B_1 are "small enough" then we can approximate ρ by a constant on each of them. A possible value of this constant is $\rho(Z_i)$ where Z_i is some point from the cell R_i. Then the approximate mass of cell R_i is $\rho(Z_i)\cdot\Delta x_i\,\Delta y_i\,\Delta z_i$ where Δx_i, Δy_i and Δz_i are the lengths of sides of R_i. The approximate mass of the whole body is

$$\sum_{i=1}^{n} \rho(Z_i)\cdot\Delta x_i\,\Delta y_i\,\Delta z_i.$$

The exact value of the mass M of the body is equal to the limit of this sum as $n \to +\infty$ and the numbers Δx_i, Δy_i, Δz_i $(i = 1,2,\dots,n)$ tend to zero.

III.5.2. A block in $\mathbf{E}_3$ and its partition. If $\langle a,b\rangle$ is a bounded closed interval on the x–axis, $\langle c,d\rangle$ is a bounded closed interval on the y–axis and $\langle r,s\rangle$ is a closed bounded interval on the z–axis then the set $B = \langle a,b\rangle \times \langle c,d\rangle \times \langle r,s\rangle$ forms a block in $\mathbf{E}_3$. We can subdivide this block to n rectangular cells $B_1, \dots, B_n$ by planes parallel to the xy–plane, xz–plane and yz–plane. The system of these cells is called the *partition* of B.

If this partition is named P and if the lengths of sides of smaller cells $B_1, \dots, B_n$ are $\Delta x_1, \Delta y_1, \Delta z_1, \dots, \Delta x_n, \Delta y_n, \Delta z_n$ then the maximum of all these lengths is denoted by $\|P\|$ and it is called the *norm* of partition P.

III.5.3. Riemann sums and their limit. Let $u = f(x,y,z)$ be a bounded function on a bounded set $D \subset \mathbf{E}_3$. Let B be the smallest block in $\mathbf{E}_2$ whose sides are parallel to the xy–, xz– and yz–planes and which contains D. Let P be a partition of B into smaller rectangular cells $B_1, \dots B_n$ whose lengths of sides are $\Delta x_1, \Delta y_1, \Delta z_1, \dots, \Delta x_n, \Delta y_n, \Delta z_n$. These smaller cells can be numbered so that those of them which are inside D are $B_1, \dots, B_m$. Let V be a system of points $Z_i \in B_i$ $(i = 1,2,\dots,m)$. Then the *Riemann sum* of function f on set D corresponding to partition P and system V is

$$s(f,P,V) = \sum_{i=1}^{m} f(Z_i)\cdot\Delta x_i\,\Delta y_i\,\Delta z_i.$$

We say that number S is the limit of the Riemann sums $s(f,P,V)$ as $\|P\| \to 0+$ if to every given $\epsilon > 0$ there exists $\delta > 0$ such that for every partition P of B and for every choice of V, $\|P\| < \delta$ implies $|s(f,P,V) - S| < \epsilon$. We write:

$$\lim_{\|P\| \to 0+} s(f,P,V) = S. \tag{III.10}$$

III.5.4. The volume integral. If the limit in (III.10) exists, then function f is called integrable in set D and S is called the volume integral of function f on D. The integral is usually denoted as

$$\iiint_D f(x,y)\,dx\,dy\,dz \quad \text{or} \quad \iiint_D f\,dx\,dy\,dz.$$

You may also find another name for the volume integral in literature: the triple integral.

III.5.5. A measurable set in E_3 and its Jordan measure. Suppose that D is a bounded set in $\mathbf{E}_3$. We say that D is measurable (in the sense of Jordan) if the constant function $f(x,y,z) = 1$ is integrable on D. In this case, we call the number

$$m_3(D) = \iiint_D dx\,dy\,dz$$

the three-dimensional Jordan measure of set D.

$m_3(D)$ has an important geometric meaning – it defines and evaluates the volume of set D.

III.5.6. Some sets whose three–dimensional Jordan measure is zero. The following sets in $\mathbf{E}_3$ have the measure equal to zero:

a) Sets consisting of a finite number of points or bounded curves.

b) Graphs of continuous functions $z = \varphi(x,y)$ or $y = \psi(x,z)$ or $x = \eta(y,z)$ defined on bounded measurable sets in $\mathbf{E}_2$.

c) So called simple smooth surfaces, respectively simple piecewise–smooth surfaces (see Section V.1).

Usually, if M is a set in $\mathbf{E}_k$ (with $k = 1$, 2 or 3) and we say that M has measure zero, we mean that the k–dimensional measure of M is zero, i.e. $m_k(M) = 0$.

III.5.7. Theorem. a) *If $N_1, N_2, \dots, N_n$ are sets in $\mathbf{E}_3$ whose measure is zero then* $m_3\left(\cup_{i=1}^n N_i\right) = 0$.

b) *If $M \subset N$ and $m_3(N) = 0$ then $m_3(M) = 0$.*

III.5.8. Theorem. (A sufficient and necessary condition for measurability of a set in E_3.) *A bounded set $D \subset \mathbf{E}_3$ is measurable if and only if $m_3(\partial D) = 0$ (where ∂D is the boundary of D).*

III.6. Existence and important properties of the volume integral.

The two statements "f is integrable on set D" and "the volume integral $\iint_D f\,dx\,dy$ exists" say exactly the same.

III.6.1. Existence theorem for the volume Integral. *Let D be a measurable set in $\mathbf{E}_3$ and let f be a bounded function on D whose set of discontinuities has measure m_3 equal to zero. Then f is integrable on D.*

Specifically, if D is a measurable set in $\mathbf{E}_3$ and f is a bounded continuous function on D then f is integrable on D.

III.6.2. Important properties of the volume integral.

a) **(Linearity of the volume integral.)** *If functions f and g are integrable on set $D \subset \mathbf{E}_3$ and $\alpha \in \mathbf{R}$ then*

$$\iiint_D (f+g)\,dx\,dy\,dz = \iiint_D f\,dx\,dy\,dz + \iiint_D g\,dx\,dy\,dz,$$

$$\iiint_D \alpha \cdot f\,dx\,dy\,dz = \alpha \cdot \iiint_D f\,dx\,dy\,dz.$$

b) **(Additivity of the volume integral with respect to the set.)** *If D_1 and D_2 are measurable non–overlapping sets in $\mathbf{E}_3$ (i.e. $m_3\left(D_1 \cap D_2\right) = 0$) and f is integrable on D_1 and D_2 then*

$$\iiint_{D_1} f\,dx\,dy\,dz + \iiint_{D_2} f\,dx\,dy\,dz = \iiint_{D_1 \cup D_2} f\,dx\,dy\,dz.$$

c) *If function f is integrable on set $D \in \mathbf{E}_3$ and function g differs from f at most on a set whose measure is zero, then g is also integrable on D and*

$$\iiint_D g\,dx\,dy\,dz = \iiint_D f\,dx\,dy\,dz.$$

d) *If $D \subset \mathbf{E}_3$ and $m_3(D) = 0$ then* $\iiint_D f\,dx\,dy\,dz = 0$ *for every function f.*

Thus, the behaviour of the integrated function on a set of measure zero does not affect the existence and the value of the volume integral.

III.7. Evaluation of the volume integral – Fubini's theorem and transformation to the cylindrical and to the spherical coordinates.

Fubini's theorem for the volume integral transforms the evaluation of the integral to the computation of one single and one double integral. It can be applied if the domain of integration D is a so called elementary region:

III.7.1. Elementary region in $\mathbf{E}_3$. a) Let D_{xy} be a measurable closed set in $\mathbf{E}_2$ and $z = \phi_1(x,y)$ and $z = \phi_2(x,y)$ be continuous functions on D_{xy} such that $\phi_1(x,y) \leq \phi_2(x,y)$ for all $[x,y] \in D_{xy}$. Then the set

$$D = \{[x,y,z] \in \mathbf{E}_3;\ [x,y] \in D_{xy},\ \phi_1(x,y) \leq z \leq \phi_2(x,y)\}$$

is called the *elementary region relative to the xy–plane*. (See Fig. 5.)

We can also define quite analogously an elementary region relative to the xz–plane and an elementary region relative to the yz–plane. Try to write down these definitions for yourself!

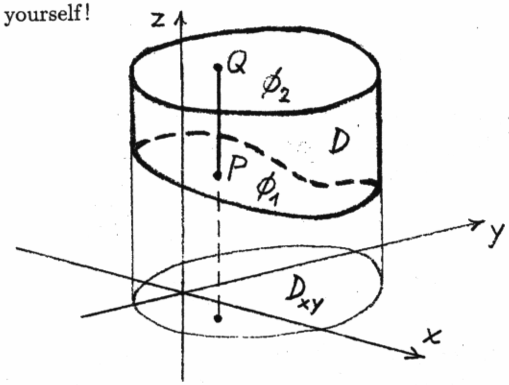

Fig. 5

Elementary regions are measurable sets in $\mathbf{E}_3$. The idea of integrating function $f(x,y,z)$ on the elementary region relative to the xy–plane is the following: Imagine that we cut the region into infinitely many vertical line segments. One of them is the line segment PQ in Fig. 5. We first integrate f on each such segment as a function of one variable z – we obtain $F(x,y) = \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz$. This depends on x and y because the position of the line segment PQ depends on x and y. Then we integrate $F(x,y)$ as a function of x and y on set D_{xy}. Thus, we obtain formula (III.11) (see the next paragraph III.7.2).

III.7.2. Fubini's theorem for the volume integral. *Let D be the elementary region relative to the xy–plane from paragraph III.7.1. Let function $u = f(x,y,z)$ be continuous on D. Then*

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iint_{D_{xy}} \left(\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz \right) dx\,dy. \qquad \text{(III.11)}$$

Formulate for yourself analogous theorems for the integration on the elementary region relative to the xz–plane and the elementary region relative to the yz–plane!

III.7.3. Example Evaluate the integral $\iiint_D (z+2)\,dx\,dy\,dz$ where D is the region in $\mathbf{E}_3$ bounded by the surfaces $x^2+y^2 = 2$, $z = -2-x$, $z = 2+y$.

The given surfaces divide $\mathbf{E}_3$ into more regions, but only one of them is bounded and this is D. It is a part of the cylinder $x^2+y^2 \leq 2$ bounded by the plane $z = -2-x$

from below and by the plane $z = 2+y$ from above. Thus, D is the elementary region relative to the xy–plane with $D_{xy} = \{[x,y] \in \mathbf{E}_2;\ x^2+y^2 \leq 2\}$ and $\phi_1(x,y) = -2-x$, $\phi_2(x,y) = 2+y$. Applying Theorem III.7.2, we obtain

$$\iiint_D (z+2)\,dx\,dy\,dz = \iint_{D_{xy}} \left(\int_{-2-x}^{2+y} (z+2)\,dz \right) dx\,dy =$$

$$= \iint_{x^2+y^2<2} \left[z^2/2 + 2z\right]_{-2-x}^{2+y} dx\,dy =$$

$$= {}^{1)} \int_0^{\sqrt{2}} \left(\int_0^{2\pi} \left(\tfrac{1}{2} r^2 \sin^2\varphi - \tfrac{1}{2} r^2 \cos^2\varphi + 4r \sin\varphi - 4r \cos\varphi + 8\right) r\,d\varphi \right) dr =$$

$$= \int_0^{\sqrt{2}} 16\pi r\,dr = 16\pi.$$

[1]) We transform the double integral on D_{xy} to the polar coordinates.

III.7.4. Remark. Fubini's theorem III.7.2 transforms the volume integral to the composition of the two integrals – the outside double integral and the inside single integral. It is sometimes quite useful to do this conversely, i.e. to transform the volume integral to the outside single integral and the inside double integral. We allow ourselves to omit the corresponding theory (because it is quite analogous to the contents of paragraphs III.7.1 and III.7.2) and we show this procedure in the following example.

III.7.5. Example. Evaluate the volume V of the oblique cone $C = \{[x,y,z] \in \mathbf{E}_3;\ 0 < z < 5,\ (x-2z)^2 + y^2 < z^2\}$.

The volume of C is

$$V = \iiint_C dx\,dy\,dz = \int_0^5 \left(\iint_{(x-2z)^2+y^2<z^2} dx\,dy \right) dz =$$

$$= {}^{2)} \int_0^5 \left(\int_0^z \left(\int_0^{2\pi} rz\,d\varphi \right) dr \right) dz = \int_0^5 \pi z^3\,dz = \frac{625\,\pi}{4}.$$

[2]) The inside double integral is transformed from the Cartesian coordinates x, y to the generalized polar coordinates r, φ by the equations $x = 2z + z\cos\varphi$, $y = z\sin\varphi$.

III.7.6. Cylindrical coordinates in $\mathbf{E}_3$. The cylindrical coordinates of the point $X = [x,y,z] \in \mathbf{E}_3$ are r, φ, w whose geometric meaning is the following: r, φ are the polar coordinates of the point $[x,y]$ in the xy–plane and $w = z$. Thus, the relation between

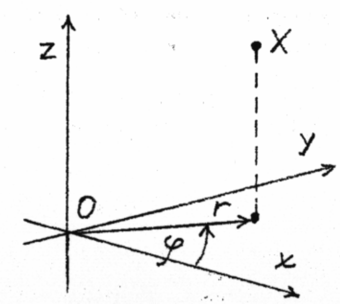

Fig. 6

the Cartesian coordinates x, y, z of point X and its cylindrical coordinates are:

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = w. \tag{III.12}$$

When transforming a volume integral to the cylindrical coordinates, we must also substitute for $dx\,dy\,dz$. It can be proved that the right substitution is

$$dx\,dy\,dz = r\,dr\,d\varphi\,dw. \tag{III.13}$$

(See Section III.9 for more details.)

It follows from the geometric sense of r, φ and w that $r \geq 0$, φ can be taken from any interval whose length is at most 2π and $w \in \mathbf{R}$. The transformation of a volume integral to the cylindrical coordinates usually simplifies the integral if the domain of integration is a cylinder (or a sector of a cylinder) or if the integrand depends on x and y mainly through the expression $x^2 + y^2$.

The transformation of the volume integral $\iiint_D f\;dx\,dy\,dz$ leads to another integral, in variables r, φ, w, on a set D'. Optimally, equations (III.12) should define a one–to–one mapping of D' onto D. Nevertheless, since the behaviour of the integrand on a set of measure zero (the three–dimensional measure m_3 because we are dealing with the volume integral) does not affect the integral, the one–to–one correspondence between the points of D' and D can be disturbed on a set of measure zero both on the side of D' and on the side of D. This is also valid for transformations to other coordinates (spherical, generalized cylindrical, generalized spherical, etc.) and so we will not deal with it in detail in this section.

III.7.7. Example. Find the volume of the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

The volume of D is identical with the three–dimensional Jordan measure of D (see paragraph III.5.5). So we denote it by $m_3(D)$. It is defined by the integral $\iiint_D dx\,dy\,dz$. Let us evaluate this integral by the transformation to the cylindrical coordinates.

Region D is a set of points $[x,y,z] \in \mathbf{E}_3$ such that $0 \leq z \leq x^2 + y^2$ and $x^2 + (y-1)^2 \leq 1$. The last inequality is equivalent to $x^2 + y^2 - 2y \leq 0$. Transforming it to the cylindrical coordinates r, φ and w, we obtain

$$r^2 - 2r\sin\varphi \leq 0,$$
$$r \leq 2\sin\varphi.$$

The inequalities $0 \leq z \leq x^2 + y^2$ are equivalent to

$$0 \leq w \leq r^2.$$

These inequalities give the limits of integration with respect to r and w. The φ–limits of integration can be found by means of the orthogonal projection of D on the xy–plane. The projection is the disk D_{xy} with the center $[0,1]$ and radius 1. The angle made by all possible straight lines passing through the origin and entering D_{xy}, measured from the positive part of the x–axis, runs from $\varphi = 0$ to $\varphi = \pi$. Hence the volume is

$$m_3(D) = \iiint_D dx\,dy\,dz = \int_0^{\pi}\int_0^{2\sin\varphi}\int_0^{r^2} r\,dw\,dr\,d\varphi = \int_0^{\pi} 4\sin^4\varphi\,d\varphi = \tfrac{3}{2}\pi.$$

III.7.8. Spherical coordinates in $\mathbf{E}_3$. The spherical coordinates of the point $X = [x,y,z]$ in $\mathbf{E}_3$ are r, φ and ϑ. They have this geometric sense: r is the distance of point X from the origin O. φ is the angle between the line segment OX' (where X' is the orthogonal projection of point X to the xy–plane) and the positive part of the x–axis (measured from the x–axis). ϑ is the angle between the line segment OX' and the line segment OX (measured from the line segment OX'. This geometric interpretation of the spherical coordinates easily leads to the following equations:

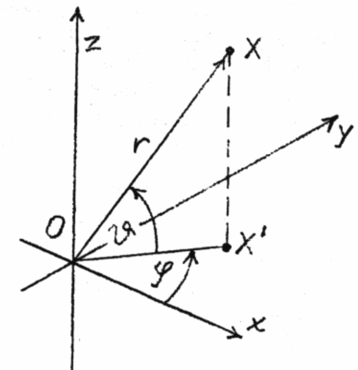

Fig. 7

$$x = r\cos\vartheta\cos\varphi, \quad y = r\cos\vartheta\sin\varphi, \quad z = r\sin\vartheta. \tag{III.14}$$

When transforming a volume integral from the Cartesian coordinates x, y, z to the spherical coordinates r, φ, ϑ, it is also necessary to transform $dx\,dy\,dz$. It can be proved that

$$dx\,dy\,dz = r^2\cos\vartheta\,dr\,d\varphi\,d\vartheta. \tag{III.15}$$

(See Section III.9 for more details.)

It follows from the geometric sense of r, φ and ϑ that $r \geq 0$, φ can be taken from any interval whose length is at most 2π and ϑ can be taken from an interval whose length does not exceed π (usually $(-\pi/2, \pi/2)$). The transformation of a volume integral to the spherical coordinates usually simplifies the integral if the domain of integration is a ball or a sector of a ball (a ball is the interior of a sphere) or if the integrand depends on x, y and z mainly through the expression $x^2 + y^2 = z^2$.

III.7.9. Example. Find the volume of the upper region D cut from the ball $x^2 + y^2 + z^2 \leq 1$ by the cone $z^2 = \frac{1}{3}(x^2 + y^2)$.

The inequality defining the ball in the spherical coordinates r, φ and ϑ is simple: $r \leq 1$. Substituting from (III.14) to the equation of the cone, we get:

$$r^2\sin^2\vartheta = \tfrac{1}{3}r^2\cos^2\vartheta\left(\cos^2\varphi + \sin^2\varphi\right),$$
$$\sin^2\vartheta = \tfrac{1}{3}\cos^2\vartheta,$$
$$\tan\vartheta = \pm\sqrt{3}/3$$

which means that $\vartheta = \pm\pi/6$. Since D is the region above the cone, the ϑ–coordinates of its points satisfy: $\vartheta \in \langle\pi/6, \pi/2)$. Finally, all possible straight lines passing through the origin sweep over D as the angle φ runs from 0 to 2π. Thus,

$$m_3(D) = \iiint_D dx\,dy\,dz = \int_0^1 \int_0^{2\pi} \int_{\pi/6}^{\pi/2} r^2 \cos\vartheta \, d\vartheta\, d\varphi\, dr = \tfrac{1}{3}\pi.$$

III.7.10. Generalized cylindrical coordinates in $\mathbf{E}_3$. We will again denote these coordinates of the point $[x,y,z] \in \mathbf{E}_3$ by r, φ, w. They are analogous to the cylindrical coordinates, though their origin need not coincide with the origin O of the Cartesian coordinates. r, φ represent the generalized polar coordinates of the point $[x,y]$ in the xy–plane and w is a linear function of z (and vice versa). Thus, the relations between the Cartesian coordinates and the generalized cylindrical coordinates are:

$$x = x_0 + ar\cos\varphi, \quad y = y_0 + br\sin\varphi, \quad z = z_0 + cw, \qquad \text{(III.16)}$$

where $[x_0, y_0, z_0]$ is a chosen point in $\mathbf{E}_3$ (the origin of the generalized cylindrical coordinates) and a, b, c are positive parameters.

Analogously to (III.8) and (III.13), when we transform a volume integral to the generalized cylindrical coordinates, we must substitute for $dx\,dy\,dz$ in accordance with the following equation:

$$dx\,dy\,dz = abc\,r\,dr\,d\varphi\,dw. \qquad \text{(III.17)}$$

(See Section III.9 for more details.)

The transformation of a volume integral to the generalized cylindrical coordinates can simplify the integral either if the domain of integration is a part of the elliptic cylinder $(x-x_0)^2/a^2 + (y-y_0)^2/b^2 < R^2$ or if the integrand depends on x and y mainly through the expression $(x-x_0)^2/a^2 + (y-y_0)^2/b^2$.

III.7.11. Generalized spherical coordinates in $\mathbf{E}_3$. We will denote these coordinates of the point $[x,y,z] \in \mathbf{E}_3$ in the same way as the spherical coordinates, i.e. by r, φ, w. The difference between the spherical coordinates and the generalized spherical coordinates is that the generalized spherical coordinates need not have their origin at the same point as the Cartesian coordinates and they are not "isotropic". This means that r can increase with the distance from the origin with the different rate in the x–direction, y–direction and z–direction. The relations between the Cartesian coordinates and the generalized spherical coordinates are:

$$x = x_0 + ar\cos\vartheta\cos\varphi, \quad y = y_0 + br\cos\vartheta\sin\varphi, \quad z = z_0 + cr\sin\vartheta, \qquad \text{(III.18)}$$

where $[x_0, y_0, z_0]$ is a chosen point in $\mathbf{E}_3$ (the origin of the generalized spherical coordinates) and a, b, c are positive parameters.

Analogously to (III.8) and (III.15), when we transform a volume integral to the generalized cylindrical coordinates, we must substitute for $dx\,dy\,dz$ in accordance with the following equation:

$$dx\,dy\,dz = abc\,r^2\cos\vartheta\,dr\,d\varphi\,dw. \qquad \text{(III.19)}$$

(See Section III.9 for more details.)

The transformation of a volume integral to the generalized spherical coordinates usually simplifies the integral either if the domain of integration is the ellipsoid $(x-x_0)^2/a^2 + (y-y_0)^2/b^2 + (z-z_0)^2/c^2 < R^2$ (or its sector) or if the integrand depends on x, y and z mainly through the expression $(x-x_0)^2/a^2 + (y-y_0)^2/b^2 + (z-z_0)/c^2$.

III.8. Some physical applications of the volume integral.

Suppose that a three–dimensional body has the form of a measurable set D in $\mathbf{E}_3$. The body need not be homogeneous and so its density (amount of mass per unit of volume) need not be constant. Let the density be given by function $\rho(x,y,z)$. The volume integral enables us to define and evaluate some fundamental mechanical characteristics of the body. Suppose that ρ is expressed in $[\mathrm{kg}\cdot\mathrm{m}^{-3}]$. Then we have:

Mass $\quad M = \iiint_D \rho(x,y,z)\,dx\,dy\,dz \quad [\mathrm{kg}]$,

Static moment about the xy–plane $\quad M_{xy} = \iiint_D z\cdot\rho(x,y,z)\,dx\,dy\,dz \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Static moment about the xz–plane $\quad M_{xz} = \iiint_D y\cdot\rho(x,y,z)\,dx\,dy\,dz \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Static moment about the yz–plane $\quad M_{yz} = \iiint_D x\cdot\rho(x,y,z)\,dx\,dy\,dz \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Center of mass $[x_m, y_m, z_m]$ $\quad x_m = \dfrac{M_{yz}}{M}, \quad y_m = \dfrac{M_{xz}}{M}, \quad z_m = \dfrac{M_{xy}}{M} \quad [\mathrm{m}]$,

Moment of inertia about the x–axis $\quad J_x = \iiint_D (y^2+z^2)\,\rho(x,y,z)\,dx\,dy\,dz \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the y–axis $\quad J_y = \iiint_D (x^2+z^2)\,\rho(x,y,z)\,dx\,dy\,dz \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the z–axis $\quad J_z = \iiint_D (x^2+y^2)\,\rho(x,y,z)\,dx\,dy\,dz \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the origin $\quad J_0 = \iint_D (x^2+y^2+z^2)\,\rho(x,y,z)\,dx\,dy\,dz \quad [\mathrm{kg}\cdot\mathrm{m}^2]$.

Try to suggest the formula for the moment of inertia about a general straight line in $\mathbf{E}_3$ whose parametric equations are $x = x_0 + u_1 t$, $y = y_0 + u_2 t$, $z = z_0 + u_3 t$; $t \in \mathbf{R}$!

III.9. A remark to the method of substitution in the double and volume integral.

The idea of the method of substitution is the same in the double and in the volume integral. This is why we shall treat it together for both types of integrals in this section. Thus,

- $\mathbf{E}_k$ will mean either $\mathbf{E}_2$ (if $k=2$) or $\mathbf{E}_3$ (if $k=3$),
- the integral $\int$ will mean either $\iint$ (if $k=2$) or $\iiint$ (if $k=3$),
- point $X \in \mathbf{E}_k$ will denote either $[x_1, x_2]$ (if $k=2$) or $[x_1, x_2, x_3]$, (if $k=3$) and

– dX will denote either $dx_1\,dx_2$ (if $k=2$) or $dx_1\,dx_2\,dx_3$ (if $k=3$).

We already know some widely used substitutions – they are given by equations (III.4), (III.7), (III.12), (III.14), (III.16) and (III.18) and they are called the transformation to the polar (or generalized polar) coordinates, transformation to the cylindrical coordinates, etc. We will explain the method of substitution on a general level in this section.

Suppose that $D \subset \mathbf{E}_k$ ($k=2$ or $k=3$) and we have to evaluate the integral $\int_D f(X)\,dX$. If every point $X \in D$ can be expressed as $X = \mathcal{F}(Y)$ where points Y are taken from some other set $D' \subset \mathbf{E}_k$ then the integral can be transformed to the integral in the variables Y on the domain of integration D'. However, it is clear that this can be done only if both the integrals on D and D' exist and mapping $\mathcal{F}$ has certain properties. We will discuss them in the following.

III.9.1. Regular mapping and its Jacobian. Let D and D' be domains in $\mathbf{E}_k$. Suppose that $\mathcal{F}$ is a mapping of D' to D. Equation $X = \mathcal{F}(Y)$ means

$$x_1 = \phi_1(y_1,y_2),\quad x_2 = \phi_2(y_1,y_2) \qquad \text{for } k=2,$$
$$x_1 = \phi_1(y_1,y_2,y_3),\quad x_2 = \phi_2(y_1,y_2,y_3),\quad x_3 = \phi_3(y_1,y_2,y_3) \qquad \text{for } k=3.$$

The determinant

$$J_{\mathcal{F}}(Y) = \left|\frac{\partial \phi_i}{\partial y_j}(Y)\right|_{i,j=1,2} \quad (\text{for } k=2) \quad \text{or} \quad J_{\mathcal{F}}(Y) = \left|\frac{\partial \phi_i}{\partial y_j}(Y)\right|_{i,j=1,2,3} \quad (\text{for } k=3)$$

is called the *Jacobian* or the *Jacobi determinant* of mapping $\mathcal{F}$.

Mapping $\mathcal{F}$ is called *regular* if functions ϕ_i ($i=1,2$ or $i=1,2,3$) have continuous partial derivatives in set D' and $J_{\mathcal{F}}(Y) \neq 0$ in all points $Y \in D'$.

III.9.2. A one–to–one mapping. You already know the notion of a one–to–one mapping. Mapping $\mathcal{F}$ is called *one–to–one* if

$$Y,\ Z \in D',\ Y \neq Z \implies \mathcal{F}(Y) \neq \mathcal{F}(Z).$$

III.9.3. Example. Verify that the mapping given by equations (III.4) is a one–to–one regular mapping of the open rectangle $D' = \{[r,\varphi] \in \mathbf{E}_2;\ r \in (0,2),\ \varphi \in (0,2\pi)\}$ onto the domain $D = \{[x_1,x_2] \in \mathbf{E}_2;\ x_1^2+x_2^2 < 4\} - \{[x_1,x_2] \in \mathbf{E}_2;\ x_1 \in \langle 0,1),\ x_2 = 0\}$ (D is an open disk (with its center at the origin and radius 2) minus the line segment connecting the points $O=[0,0]$ and $P=[2,0]$. Sketch a figure!

The one–to–one correspondence between the points $[x_1,x_2] \in D$ and $[r,\varphi] \in D'$ is obvious. To verify the regularity of mapping (III.4), let us evaluate the Jacobian of this mapping. Equations (III.4) can also be written as

$$x_1 = \phi_1(r,\varphi) = r\cos\varphi, \quad x_2 = \phi_2(r,\varphi) = r\sin\varphi.$$

The Jacobian is:

$$J(r,\varphi) = \begin{vmatrix} \dfrac{\partial\phi_1}{\partial r}, & \dfrac{\partial\phi_1}{\partial\varphi} \\ \dfrac{\partial\phi_2}{\partial r}, & \dfrac{\partial\phi_2}{\partial\varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi, & -r\sin\varphi \\ \sin\varphi, & r\cos\varphi \end{vmatrix} = r.$$

Now it is seen that functions ϕ_1 and ϕ_2 have continuous partial derivatives in domain D' and $r \neq 0$ in D'. Thus, mapping (III.4) is regular in D'.

III.9.4. Theorem. *Let $X = \mathcal{F}(Y)$ be a one–to–one regular mapping of domain $D' \in \mathbf{E}_k$ onto domain $D \in \mathbf{E}_k$. Then*

$$\int_D f(X)\,dX = \int_{D'} f(\mathcal{F}(Y)) \cdot |J_{\mathcal{F}}(Y)|\,dY, \tag{III.20}$$

if both integrals exist.

III.9.5. Remark. We already know that adding (or subtracting) a set of measure zero to the domain of integration does not influence the existence or the value of the integral. This enables us to generalize Theorem III.9.4:

If the assumptions of Theorem III.9.4 are satisfied and A, respectively A', are sets in $\mathbf{E}_k$ which differ from D, respectively D', only in sets of measure zero (i.e. $m_k((A-D)\cup(D-A)) = m_k((A'-D')\cup(D'-A')) = 0$) then

$$\int_A f(X)\,dX = \int_{A'} f(\mathcal{F}(Y)) \cdot |J_{\mathcal{F}}(Y)|\,dY, \tag{III.21}$$

if both integrals exist.

It is seen from equations (III.20) and (III.21) that dX in the integral on the left–hand side changes to $|J_{\mathcal{F}}(Y)|\,dY$ in the integral on the right–hand side. We have already shown that if $k=2$ and $Y=[r,\varphi]$ represents the polar coordinates then the Jacobian is equal to r (see example III.9.3). Thus, the equation

$$dX = |J_{\mathcal{F}}(Y)|\,dY \tag{III.22}$$

implies, as a special case, equation (III.5). Computing the Jacobians of mappings (III.7), (III.12), (III.14), (III.16) and (III.18), we can see that general equation (III.22) also implies special equations (III.8), (III.13), (III.15), (III.17) and (III.19).

III.10. Exercises.

1. Do the following integrals exist?

$$\iint_D \frac{dx\,dy}{x+y+1};\quad D = \langle 0,1\rangle \times \langle 0,1\rangle$$

$$\iint_D \frac{\sin(x^2+y^2)}{x^2+y^2}\,dx\,dy;\quad D = \{[x,y] \in \mathbf{E}_2;\ x^2+y^2 \le 9\}$$

$$\iint \frac{x}{x^2+y^2}\,dx\,dy;\quad D = \{[x,y] \in \mathbf{E}_2;\ x^2+y^2 \le 9\}$$

$$\iint_D \frac{dx\,dy}{(1-xy)^2};\quad D \text{ is the square } PQRS \text{ where } P=[1,2],\ Q=[3,2],\ R=[3,4],\ S=[1,4]$$

$$\iiint_D \sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}-\frac{z^2}{16}}\,dx\,dy\,dz;\quad D = \left\{[x,y,z] \in \mathbf{E}_3;\ \frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16} < 1\right\}$$

$\iiint_D \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz\,;\quad D=\{[x,y,z]\in \mathbf{E}_3;\ x^2+y^2+z^2\le 0\}$

2. Find the area of the region R in the xy–plane enclosed by the curves

a) $y=2x+4,\ \ y=4-x^2$
b) $xy=1,\ \ y=x,\ \ x=4$
c) $y=x^2,\ \ 4y=x^2,\ \ y=4$
d) $y^2=4+x,\ \ x+3y=0$
e) $y=\ln x,\ \ x-y=1,\ \ y=-1$
f) $x^2+y^2=2x,\ \ x^2+y^2=4x,\ \ y=x,\ \ y=0$

3. Find the area of the region in xy–plane bounded on the right by the parabola $y=x^2$, on the left by the line $x+y=2$, and above by the line $y=4$.

4. Find the volume of set R in $\mathbf{E}_3$ if

a) R is the region under the paraboloid $z=x^2+y^2$, above the triangle enclosed by the lines $y=x,\ x=0,\ x+y=2$, in the xy–plane
b) R is the region under the parabolic cylinder $z=x^2$, above the domain enclosed by the parabola $y=6-x^2$ and the line $y=x$, in the xy–plane
c) R is the region in the half–space $z\ge 0$ bounded by the surfaces $x^2+y^2-z^2=0$, $z=6-x^2-y^2$
d) R is the region in the half–space $z\ge 0$ bounded by the surfaces $az=x^2+y^2$, $x^2+y^2+x^2=2a^2$, $(a>0)$
e) R is the region bounded by the surfaces $y^2=4a^2-3ax$, $y^2=ax$, $z=h$, $z=-h$, $(a>0,\ h>0)$
f) R is the region bounded by the surfaces $x^2+y^2=z$, $x^2+y^2=2z$, $z=0$

5. Evaluate the following integrals.

a) $\iint_D (1+x)\,dx\,dy\,;$ D is the region in $\mathbf{E}_2$ enclosed by the lines $y=x^2-4$, $y=-3x$
b) $\iint_D \dfrac{dx\,dy}{(x+y)^2}\,;\quad D=\langle 3,4\rangle\times\langle 1,2\rangle$
c) $\iint_D xy\,dx\,dy\,;$ D is the region in $\mathbf{E}_2$ enclosed by the line $y=x-4$ and by the parabola $y^2=2x$
d) $\iiint_V (x+y+z)\,dx\,dy\,dz\,;\quad V=\langle 0,1\rangle\times\langle 0,2\rangle\times\langle 0,3\rangle$
e) $\iiint_V x\,dx\,dy\,dz\,;$ V is the region in $\mathbf{E}_3$ bounded by the surfaces $x=0,\ y=0,\ z=0,\ y=2,\ x+z=1$
f) $\iiint_V xy^2z^3\,dx\,dy\,dz\,;$ V is the region in $\mathbf{E}_3$ bounded by the surfaces $z=xy$, $y=x,\ x=1,\ z=0$
g) $\iiint_V x^3y^2z\,dx\,dy\,dz\,;\quad V=\{[x,y,z]\in\mathbf{E}_3;\ 0\le x\le 1,\ 0\le y\le x,\ 0\le z\le xy\}$

h) $\iiint_V y\cos(x+z)\,dx\,dy\,dz\,;$ V is the region in $\mathbf{E}_3$ bounded by the surfaces $y=\sqrt{x},\ y=0,\ z=0,\ x+z=\pi/2$
i) $\iint_D \sqrt{1-x^2-y^2}\,dx\,dy\,;\quad D=\{[x,y]\in\mathbf{E}_2;\ x^2+y^2\le 1\}$
j) $\iint_D \dfrac{dx\,dy}{(x^2+y^2)^2}\,;$ D is the region in the xy–plane enclosed by the lines $y=x$, $y=2x$ and by the circles $x^2+y^2=4x,\ x^2+y^2=8x$
k) $\iint_D y\,dx\,dy\,;$ D is the upper half of the disk $(x-a)^2+y^2\le a^2\ \ (a>0)$
l) $\iint_D x\,dx\,dy\,;$ D is the sector of the disk $x^2+y^2\le a^2$ consisting of the points $[x,y]$ such that $x\ge 0$ and $-x\le\sqrt{3}\,y\le 1$
m) $\iiint_V \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz\,;$ V is the ball $x^2+y^2+z^2\le a^2,\ (a>0)$
n) $\iiint_D (x+y+z)^2\,dx\,dy\,dz\,;$ D is the region in the half–space $z\ge 0$ bounded by the paraboloid $z=\frac{1}{2}(x^2+y^2)$ and by the sphere $x^2+y^2+z^2=3$
o) $\iiint_V z\,dx\,dy\,dz\,;$ V is the region in $\mathbf{E}_3$ bounded by the surfaces $z=\sqrt{x^2+y^2}$ and $z=1$
p) $\iiint_D (x^2+y^2)\,dx\,dy\,dz\,;$ D is the region in $\mathbf{E}_3$ bounded by the surfaces $x^2+y^2=2z$ and $z=2$
q) $\iiint_V \left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)dx\,dy\,dz\,;$ V is the interior of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1,\ (a>0,\ b>0,\ c>0)$

6. Find the center of mass of the homogeneous regions in $\mathbf{E}_2$ bounded by the curves

a) $y=\sin x,\ y=0;\ x\in\langle 0,\pi\rangle$,
b) $x^2+y^2=a^2,\ y=0;\ (y\ge 0,\ a>0)$,
c) $y^2=ax,\ x=0,\ y=a;\ (y>0,\ a>0)$,
d) $y^2=4x+4,\ y^2=-2x+4$.

7. Evaluate the moment of inertia with respect to the x–axis of the homogeneous region in $\mathbf{E}_2$, bounded by the lines $y=x/2,\ y=a,\ x=a\ \ (a>0)$. (The density is $\rho=1$.)

8. Evaluate the mass of the body in $\mathbf{E}_3$ bounded by the surfaces

a) $x=0,\ x=a,\ y=0,\ y=b,\ z=0,\ z=c\ (a>0,\ b>0,\ c>0)$ if the density is $\rho(x,y,z)=x+y+z$,
b) $2x+z=2a,\ x+z=a,\ y^2=ax,\ y=0$ (for $y>0$) if $a>0$ and the density is $\rho(x,y,z)=y$,
c) $x^2+y^2+z^2=a^2,\ x^2+y^2+z^2=4a^2\ \ (a>0)$ if the density is $\rho(x,y,z)=2/\sqrt{x^2+y^2+z^2}$.