IV. Line Integrals

IV.1. Simple curves.

We need to specify exactly what we understand under the notion of a curve (i.e. a line) before we begin to deal with line integrals. A variety of definitions of many types of curves appear in literature. We will restrict ourselves to two types of curves: so called simple smooth curves and simple piecewise–smooth curves. They are also called simple regular or simple piecewise–regular curves.

The idea of the definition of a simple smooth curve is the following: Imagine that a mass point moves in $\mathbf{E}_2$ or in $\mathbf{E}_3$ in a time interval $\langle a,b\rangle$ and its position at time t is $P(t)$. Then P is a mapping of the interval $\langle a,b\rangle$ to $\mathbf{E}_k$ (with $k=2$ or $k=3$). Suppose that the velocity of the moving mass point is continuous, bounded and different from zero in all times $t\in(a,b)$. (This means that mapping P has a continuous, bounded and non–zero derivative at all points $t\in(a,b)$). Suppose further that the mass point cannot be at the same place at two different times, with a possible exception when the motion starts and finishes at the same point. (This means that mapping P is one–to–one in the interval $\langle a,b\rangle$ with a possible exception when $P(a)=P(b)$.) Then the path travelled by the mass point is called a simple smooth curve. The position function P is called the parametrization of the curve. You will find the precise definition of a simple smooth curve in paragraph IV.1.1.

Since $P(t)$ (for fixed t) is a point in $\mathbf{E}_k$ (where $k=2$ or $k=3$), it has two or three coordinates. Let us denote them $\phi(t)$, $\psi(t)$, respectively $\phi(t)$, $\psi(t)$, $\vartheta(t)$. Then ϕ, ψ, respectively ϕ, ψ, ϑ, are the functions of one variable t defined in the interval $\langle a,b\rangle$. They will be called the *coordinate functions* of mapping P and we will write

$$P(t)=[\phi(t),\psi(t)] \qquad \text{if } k=2,$$

$$P(t)=[\phi(t),\psi(t),\vartheta(t)] \qquad \text{if } k=3.$$

The derivative of P with respect to the parameter will be denoted by the dot, in accordance with the customs in physics. We will take the derivative for a vector and we will therefore enclose its components in parentheses:

$$\dot P(t)=(\dot\phi(t),\dot\psi(t)) \qquad \text{if } k=2,$$

$$\dot P(t)=(\dot\phi(t),\dot\psi(t),\dot\vartheta(t)) \qquad \text{if } k=3.$$

The coordinate functions of parametrization P are also often denoted by $x(t)$, $y(t)$, respectively by $x(t)$, $y(t)$, $z(t)$.

IV.1.1. Simple smooth curve. Let P be a continuous mapping of a closed bounded interval $\langle a,b\rangle$ to $\mathbf{E}_k$ (where $k=2$ or $k=3$). Suppose that

a) mapping P is one–to–one in the interval $\langle a,b\rangle$ with a possible exception when $P(a)=P(b)$,

b) P has a bounded, continuous and non–zero derivative $\dot P$ in the interval (a,b).

Then the set $C=\{X=P(t)\in\mathbf{E}_k;\ t\in\langle a,b\rangle\}$ is called a *simple smooth curve* in $\mathbf{E}_k$. Mapping P is called the *parametrization* of curve C.

The simple smooth curve C is called *closed* if $P(a)=P(b)$.

The vector $\dot P(t)$ is tangent to the simple smooth curve C at every "interior" point $P(t)$ of curve C (i.e. point $P(t)$ corresponding to $t\in(a,b)$). The vector $\dot P(t)/|\dot P(t)|$ is also tangent to C at point $P(t)$ and moreover, its length is equal to one. We can choose the *orientation* of curve C so that we put the *unit tangent vector* $\vec\tau$ to C at point $P(t)$

either $$\vec\tau=\frac{\dot P(t)}{|\dot P(t)|} \quad (\text{for all } t\in(a,b)) \qquad \text{(IV.1)}$$

or $$\vec\tau=-\frac{\dot P(t)}{|\dot P(t)|} \quad (\text{for all } t\in(a,b)). \qquad \text{(IV.2)}$$

We say that curve C is *oriented in accordance with its parametrization P* if the unit tangent vector $\vec\tau$ to C is given by formula (IV.1).

In other words, we say that simple smooth curve C is oriented in accordance with its parametrization P if the parametrization defines the motion along C in the direction corresponding to the orientation of C.

If the simple smooth curve C is oriented in accordance with the parametrization P then the point $P(a)$ is called the *initial point of C* (we denote it $i.p.\,C$) and the point $P(b)$ is called the *terminal point of C* (we denote it $t.p.\,C$).

If the orientation of C is opposite to parametrization P then the position of the initial and the terminal point of C is also opposite: $i.p.\,C=P(b)$ and $t.p.\,C=P(a)$.

A simple smooth curve C which is not closed can also be oriented so that one of the points $P(a)$, $P(b)$ is chosen to be the initial point of C and the second one to be the terminal point of C.

Every simple smooth curve has infinitely many parametrizations. This is clear if you take into account that every path can be travelled by infinitely many possible motions.

IV.1.2. Example. Every line segment in $\mathbf{E}_k$ is a simple smooth curve. For instance, the line segment AB in $\mathbf{E}_3$ with $A=[1,2,4]$ and $B=[3,-1,7]$ can be parametrized by the mapping

$$P(t)=A+t\cdot(B-A);\quad t\in\langle 0,1\rangle.$$

This means that the coordinate functions ϕ, ψ and ϑ of parametrization P are:

$$x=\phi(t)=1+2t,\quad y=\psi(t)=2-3t,\quad z=\vartheta(t)=4+3t;\quad t\in\langle 0,1\rangle.$$

The simple smooth curve C, identical with the line segment AB, is oriented in accordance with the above parametrization if $A=i.p.\,C$ and $B=t.p.\,C$.

IV.1.3. Example. The part of the parabola $y=x^2+1$ between the points $[1,2]$ and $[3,10]$ (oriented from $[1,2]$ to $[3,10]$) is a simple smooth curve in $\mathbf{E}_2$. Its possible parametrization is e.g. the mapping

$$P: \quad x = \phi(t) = t, \; y = \psi(t) = t^2 + 1; \quad t \in \langle 1, 3 \rangle.$$

Since this parametrization defines the motion on the curve from its initial to its terminal point, the curve is oriented in accordance with parametrization P.

IV.1.4. Example. The arc $x^2 + y^2 = 9$, $x \geq 0$, $y \geq 0$, oriented from the point $[3, 0]$ to the point $[0, 3]$, is a simple smooth curve in $\mathbf{E}_2$. Its possible parametrization, generating the same orientation, is

$$x = \phi(t) = 3 \cos t, \quad y = \psi(t) = 3 \sin t; \quad t \in \langle 0, \pi/2 \rangle.$$

IV.1.5. Example. The circle $C: \; (x-3)^2 + y^2 = 4$ in $\mathbf{E}_2$ (oriented counter-clockwise) is a closed simple smooth curve. Its parametrization is for instance the mapping

$$x = \phi(t) = 3 + 2 \cos t, \quad y = \psi(t) = 2 \sin t; \quad t \in \langle 0, 2\pi \rangle.$$

You can easily verify that C is oriented in accordance with this parametrization.

IV.1.6. Simple piecewise–smooth curve. Let $C_1, \ldots, C_m$ be simple smooth curves in $\mathbf{E}_k$ such that

a) $t.p.\,C_1 = i.p.\,C_2$, $t.p.\,C_2 = i.p.\,C_3$, $\ldots$, $t.p.\,C_{m-1} = i.p.\,C_m$,

b) except for the points named in a) and except for the possibility when $i.p.\,C_1 = t.p.\,C_m$, any two of the curves $C_1, \ldots, C_m$ have no more common points.

Then the set $C = \cup_{i=1}^{m} C_i$ is called the *simple piecewise–smooth curve* in $\mathbf{E}_k$.

The orientation of the simple piecewise–smooth curve C is given by the orientation of its smooth parts $C_1, \ldots, C_m$. We put $i.p.\,C$ (the initial point of C) $= i.p.\,C_1$ and $t.p.\,C$ (the terminal point of C) $= t.p.\,C_m$.

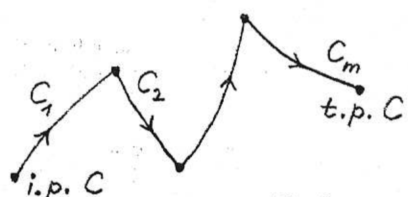

Fig. 8

The curve which differs from a simple piecewise–smooth curve C only by its orientation will be denoted by $-C$.

A simple piecewise–smooth curve C is called *closed* if $i.p.\,C = t.p.\,C$.

A simple piecewise–smooth curve in $\mathbf{E}_k$ (where $k = 2$ or $k = 3$) is a bounded set in $\mathbf{E}_k$ whose k–dimensional measure m_k equals zero.

It is obvious that the notion of a simple piecewise–smooth curve is a generalization of the notion of a simple smooth curve. Always when we will use the word "curve" in the coming sections, we will have in mind a simple piecewise–smooth curve. Similarly, a "closed curve" will mean a closed simple piecewise–smooth curve. We will give more details about the curve if they are important.

IV.2. The line integral of a scalar function.

A scalar function is a function whose values are scalars, i.e. in our case real numbers. We use the name "scalar function" only in order to distinguish it from a "vector function" which will be treated in the next sections.

IV.2.1. Physical motivation. Suppose that a spring or a wire has the form of a simple smooth curve C in $\mathbf{E}_k$ (with $k = 2$ or $k = 3$). Suppose further that the longitudinal density of the wire is ρ. ρ need not be a constant, and it is generally a function of two variables x, y (if $k = 2$) or three variables x, y, z (if $k = 3$). We wish to evaluate the mass M of the wire.

The idea of evaluation of the total mass of the wire is exactly the same as the idea of computation of the mass of a one–dimensional rod, used in paragraph II.1.1. We could explain it by means of a partition of the wire into many "short" pieces, similarly as we divided the interval $\langle a, b \rangle$ into many "short" subintervals in paragraph II.1.1. However, let us use another approach – an approach based on the idea of the partition of an interval into infinitely many "infinitely short" subintervals. This idea was explained in Section II.7 (and we promised to come back to it).

Thus, suppose that P is a parametrization of curve C which is defined in the interval $\langle a, b \rangle$. A typical "infinitely short" subinterval of $\langle a, b \rangle$ has the form $\langle t, t + dt \rangle$. Parametrization P maps this interval to the "infinitely short" part of C with the end points $P(t)$ and $P(t + dt)$. We can take this part for a line segment whose length is $ds = |P(t + dt) - P(t)| = |\dot{P}(t)|\, dt$. The mass of this segment is $dM = \rho(P(t)) \cdot ds = \rho(P(t)) \cdot |\dot{P}(t)|\, dt$. The total mass of the whole wire is

$$M = \int_a^b \rho(P(t)) \cdot |\dot{P}(t)|\, dt.$$

IV.2.2. The line integral of a scalar function on a simple smooth curve. Let C be a simple smooth curve in $\mathbf{E}_2$ or $\mathbf{E}_3$ and P be its parametrization defined in the interval $\langle a, b \rangle$. Let f be a scalar function defined on C. We say that f is *integrable* on curve C if the Riemann integral $\int_a^b f(P(t)) \cdot |\dot{P}(t)|\, dt$ exists. We denote this integral by $\int_C f \, ds$ and we call it the *line integral of a scalar function* f on the simple smooth curve C.

IV.2.3. Remark. The integrability of function f on a simple smooth curve C and the line integral $\int_C f \, ds$ are defined by means of a parametrization of curve C. Since C can be parametrized in infinitely many ways, there arises the question whether the integrability of f on curve C as well as the value of the integral $\int_C f \, ds$ can depend on a concrete choice of parametrization of C. The answer is NO. It can be proved that neither the existence, nor the value of the line integral $\int_C f \, ds$ depends on the concrete choice of parametrization of curve C.

IV.2.4. The line integral of a scalar function on a simple piecewise–smooth curve. Let C be a simple piecewise–smooth curve in $\mathbf{E}_2$ or $\mathbf{E}_3$ which is a union of simple smooth curves $C_1, C_2, \ldots, C_m$ (see paragraph IV.1.3). Let f be a scalar function defined on C. If function f is integrable on each of curves $C_1, C_2, \ldots, C_m$ then we say that it is integrable on curve C and we put

$$\int_C f\ ds = \sum_{i=1}^{m} \int_{C_i} f\ ds. \tag{IV.5}$$

The integral on the left hand side is called the *line integral of the scalar function* f on the simple piecewise–smooth curve C.

The line integral of a scalar function is also often called the *line integral of the 1st kind*.

Instead of $\int_C f\ ds$, we can also write for example $\int_C f(x,y)\ ds$ (if $C \subset \mathbf{E}_2$ and f is a function of two variables) or $\int_C f(x,y,z)\ ds$ (if $C \subset \mathbf{E}_3$ and f is a function of three variables). The symbol ds at the end of the integral can also be replaced e.g. by dl, dr, etc.

IV.2.5. The length of a curve. If C is a simple piecewise–smooth curve then the value of the integral $\int_C ds$ is called the *length* of curve C.

Specially, if C is a simple smooth curve and P is its parametrization defined in the interval $\langle a,b\rangle$ then the length of curve C is

$$l(C) = \int_C ds = \int_a^b |\dot{P}(t)|\ dt. \tag{IV.6}$$

IV.2.6. Some important properties of the line integral of a scalar function. Since the line integral of a scalar function is defined by means of the Riemann integral, most of its properties are quite analogous. Let us mention only some of them:

a) **(Existence of the line integral.)** *If function f is continuous on curve C then it is integrable on C (i.e. the integral $\int_C f\ ds$ exists).*

b) **(Linearity of the line integral.)** *If functions f and g are integrable on curve C and $\alpha \in \mathbf{R}$ then*

$$\int_C (f+g)\ ds = \int_C f\ ds + \int_C g\ ds,$$

$$\int_C \alpha \cdot f\ ds = \alpha \cdot \int_C f\ ds.$$

c) *If function f is integrable on curve C and function g differs from f at most in a finite number of points then g is also integrable on C and*

$$\int_C g\ ds = \int_C f\ ds.$$

d) *If function f is integrable on curve C then it is also integrable on curve $-C$ and*

$$\int_{-C} f\ ds = \int_C f\ ds.$$

Assertion a) can be generalized in this way: *If C is a simple piecewise–smooth curve and f is continuous on each of its smooth parts then f is integrable on C.*

Assertion d) says that neither the existence, nor the value of the line integral of a scalar function depends on the orientation of the curve!

IV.2.7. Evaluation of the line integral of a scalar function. The line integral of function f on a simple smooth curve C can be evaluated by means of a parametrization of C. Thus, if P is such a parametrization, defined in the interval $\langle a,b\rangle$, and function f is integrable on C then we can use the formula

$$\int_C f\ ds = \int_a^b f(P(t)) \cdot |\dot{P}(t)|\ dt. \tag{IV.7}$$

This formula follows immediately from the definition of the line integral on a simple smooth curve – see paragraph IV.2.2.

If C is a simple piecewise–smooth curve which is a union of simple smooth curves $C_1, C_2, \ldots, C_m$ (see paragraph IV.1.6) then the line integral of function f on curve C can be computed by means of formula (IV.5).

IV.2.8. Example. C is the union of two line segments $C_1 = OP$ and $C_2 = PQ$ where $O = [0,0,0]$, $P = [1,1,0]$ and $Q = [1,1,1]$. Integrate the function $f(x,y,z) = x - 3y^2 + z$ over C.

The simplest parametrizations of C_1 and C_2 we can think of are:

$$C_1: \quad P_1(t) = O + (P-O)\,t = [t,t,0]; \quad t \in \langle 0,1\rangle,$$
$$C_2: \quad P_2(t) = P + (Q-P)\,t = [1,1,t]; \quad t \in \langle 0,1\rangle.$$

We can easily find that $\dot{P}_1(t) = (1,1,0)$, $\dot{P}_2(t) = (0,0,1)$ and so $|\dot{P}_1(t)| = \sqrt{2}$ and $|\dot{P}_2(t)| = 1$. Using formulas (IV.7) and (IV.5), we obtain

$$\int_C (x-3y^2+z)\ ds = \int_{C_1} (x-3y^2+z)\ ds + \int_{C_2} (x-3y^2+z)\ ds =$$
$$= \int_0^1 (t-3t^2)\,\sqrt{2}\ dt + \int_0^1 (t-2)\ dt = -\frac{\sqrt{2}+3}{2}.$$

IV.2.9. Example. Evaluate the line integral $\int_C (x^2+y)\ ds$ where C is the circle $x^2 + (y-5)^2 = 4$. We do not specify the orientation of C because it does not affect the line integral of a scalar function on C.

C can be parametrized e.g. by the mapping

$$x = \phi(t) = 2\ \cos t, \quad y = \psi(t) = 5 + 2\ \sin t; \quad t \in \langle 0,\ 2\pi\rangle.$$

In order to use formula (IV.6), we also need to express $|\dot{P}(t)|$:

$$|\dot{P}(t)| = \big|(-2\ \sin t,\ 2\ \cos t)\big| = 2.$$

Thus, we obtain

$$\int_C (x^2+y)\ ds = \int_0^{2\pi} (4\ \cos^2 t + 5 + 2\ \sin t) \cdot 2\ dt = 20\,\pi.$$

IV.3. Some physical applications of the line integral of a scalar function.

Suppose that a wire or a spring has the form of a curve C in $\mathbf{E}_k$ ($k = 2$ or $k = 3$). The wire need not be homogeneous and so its longitudinal density (amount of mass per unit of length) need not be constant. Let the density be given by function $\rho(x, y)$ (if $k = 2$) or $\rho(x, y, z)$ (if $k = 3$). The line integral of a scalar function enables us to define and evaluate some fundamental mechanical characteristics of curve C. Suppose that ρ is expressed in $[\mathrm{kg} \cdot \mathrm{m}^{-1}]$. Then we have:

I. $\underline{k = 2}$ Mass $M = \int_C \rho(x, y)\, ds$ [kg],

Static moment about the x–axis $M_x = \int_C y \cdot \rho(x, y)\, ds$ [kg · m],

Static moment about the y–axis $M_y = \int_C x \cdot \rho(x, y)\, ds$ [kg · m],

Center of mass $[x_m, y_m]$ $x_m = \dfrac{M_y}{M}$, $y_m = \dfrac{M_x}{M}$ [m],

Moment of inertia about the x–axis $J_x = \int_C y^2 \cdot \rho(x, y)\, ds$ [kg · m²],

Moment of inertia about the y–axis $J_y = \int_C x^2 \cdot \rho(x, y)\, ds$ [kg · m²],

Moment of inertia about the origin $J_0 = \int_C (x^2 + y^2) \cdot \rho(x, y)\, ds$ [kg · m²].

II. $\underline{k = 3}$ Mass $M = \int_C \rho(x, y, z)\, ds$ [kg],

Static moment about the xy–plane $M_{xy} = \int_C z \cdot \rho(x, y, z)\, ds$ [kg · m],

Static moment about the xz–plane $M_{xz} = \int_C y \cdot \rho(x, y, z)\, ds$ [kg · m],

Static moment about the yz–plane $M_{yz} = \int_C x \cdot \rho(x, y, z)\, ds$ [kg · m],

Center of mass $[x_m, y_m, z_m]$ $x_m = \dfrac{M_{yz}}{M}$, $y_m = \dfrac{M_{xz}}{M}$, $z_m = \dfrac{M_{xy}}{M}$ [m],

Moment of inertia about the x–axis $J_x = \int_C (y^2 + z^2) \cdot \rho(x, y, z)\, ds$ [kg · m²],

Moment of inertia about the y–axis $J_y = \int_C (x^2 + z^2) \cdot \rho(x, y, z)\, ds$ [kg · m²],

Moment of inertia about the z–axis $J_z = \int_C (x^2 + y^2) \cdot \rho(x, y, z)\, ds$ [kg · m²],

Moment of inertia about the origin $J_0 = \int_C (x^2 + y^2 + z^2) \cdot \rho(x, y, z)\, ds$ [kg · m²].

Try to suggest the formula for the moment of inertia about a general straight line in $\mathbf{E}_3$ whose parametric equations are $x = x_0 + u_1 t$, $y = y_0 + u_2 t$, $z = z_0 + u_3 t$; $t \in \mathbf{R}$!

IV.4. The line integral of a vector function.

A vector function in a domain $D \subset \mathbf{E}_3$ is a mapping which assigns to every point $[x, y, z] \in D$ a vector. We shall denote vectors and vector functions by boldface letters, like for example $\mathbf{u}$, $\mathbf{f}$, etc. However, you can also use $\vec{u}$, $\vec{f}$, etc.

If $\mathbf{f}$ is a vector function in domain D then $\mathbf{f}(x, y, z)$ has three components. We shall denote them by $U(x, y, z)$, $V(x, y, z)$ and $W(x, y, z)$. U, V and W are scalar functions in domain D. We shall write

$$\mathbf{f}(x, y, z) = (U(x, y, z),\ V(x, y, z),\ W(x, y, z)) \quad \text{or simply} \quad \mathbf{f} = (U, V, W).$$

If we denote by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ the unit vectors oriented successively in accordance with the x–axis, the y–axis and the z–axis, we can also write:

$$\mathbf{f}(x, y, z) = U(x, y, z)\,\mathbf{i} + V(x, y, z)\,\mathbf{j} + W(x, y, z)\,\mathbf{k} \quad \text{or} \quad \mathbf{f} = U\,\mathbf{i} + V\,\mathbf{j} + W\,\mathbf{k}.$$

A vector function in domain $D \subset \mathbf{E}_3$ is also often called a *vector field* in D. We shall say that the vector function (or the vector field) $\mathbf{f} = (U, V, W)$ is continuous (respectively differentiable) in D if all its components U, V and W are continuous (respectively differentiable) functions in domain D.

The denotation and the used terminology in the case of two–dimensional vector functions is analogous, the only difference being that we have one variable and one component less.

IV.4.1. Physical motivation. Suppose that a body moves along a curve C due to the action of a force $\mathbf{f}$. We wish to evaluate the work A done by force $\mathbf{f}$ along curve C. The force is generally the function of three variables x, y and z. Let us apply the idea of an "infinitely small" positive number explained in Section II.7 and let us imagine that curve C can be decomposed to infinitely many "infinitely short" parts. A position of a typical part is $[x, y, z]$, its length is ds and the unit tangent vector to C at point $[x, y, z]$ is $\vec{\tau}(x, y, z)$. The work dA of force $\mathbf{f}$ done by its action on the considered "infinitely short" part is $dA = \mathbf{f}(x, y, z) \cdot \vec{\tau}(x, y, z)\, ds$. Hence the total work of force f along the whole curve C is

$$A = \int_C \mathbf{f}(x, y, z) \cdot \vec{\tau}(x, y, z)\, ds.$$

Since the product $\mathbf{f}(x, y, z) \cdot \vec{\tau}(x, y, z)$ is a scalar, the integral is the integral of a scalar function which is already known.

IV.4.2. The line integral of a vector function. Let C be a simple piecewise–smooth curve in $\mathbf{E}_k$ (where $k = 2$ or $k = 3$) and let $\mathbf{f}$ be a vector function (with k components) defined on C. We say that the vector function $\mathbf{f}$ is integrable on curve C if the scalar function $\mathbf{f}\cdot\vec{\tau}$ is integrable on C (in the sense explained in paragraphs IV.2.2 and IV.2.4). The integral $\int_C \mathbf{f}\cdot\vec{\tau}\, ds$ is called the *line integral of a vector function* $\mathbf{f}$ on curve C and it is usually denoted by $\int_C \mathbf{f}\cdot d\mathbf{s}$.

The line integral of a vector function is also often called the *line integral of the 2nd kind*.

IV.4.3. Remark. The fact that the unit tangent vector $\vec{\tau}$ need not exist in all points of a simple piecewise–smooth curve C does not matter. The points where $\vec{\tau}$ need not be defined are the points where the smooth parts of C are connected and the number of these points is at most finite. The line integral of a vector function is defined by means of the line integral of a scalar function and we already know that this integral does not depend on the behaviour of the integrand at points whose number is finite. (See paragraph IV.2.6, part c).)

IV.4.4. Remark. The line integral of a vector function can be denoted and written down in various ways. It is very important to understand them and to recognize correctly what they mean. We will explain one of the other possible ways of writing the line integral of a vector function in this paragraph.

Suppose that a vector function $\mathbf{f}$ has components U, V and W. Thus, $\mathbf{f} = (U, V, W) = U\cdot\mathbf{i} + V\cdot\mathbf{j} + W\cdot\mathbf{k}$. Comparing the two integrals $\int_C \mathbf{f}\cdot\vec{\tau}\, ds$ and $\int_C \mathbf{f}\cdot d\mathbf{s}$ which mean the same, we obtain the formal equality $\vec{\tau}\, ds = d\mathbf{s}$. The term $d\mathbf{s}$ is considered as an "infinitely short" tangent vector to curve C and its components are often denoted by dx, dy and dz. Thus, we have

$$\vec{\tau}\, ds = d\mathbf{s} = (dx, dy, dz) = \mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz.$$

The scalar product $\mathbf{f}\cdot d\mathbf{s}$ can now be expressed

$$\mathbf{f}\cdot d\mathbf{s} = (U, V, W)\cdot(dx, dy, dz) = U\, dx + V\, dy + W\, dz$$

and the line integral of the vector function $\mathbf{f}$ can be written as

$$\int_C \mathbf{f}\cdot\vec{\tau}\, ds = \int_C \mathbf{f}\cdot d\mathbf{s} = \int_C (U\, dx + V\, dy + W\, dz). \tag{IV.8}$$

It is clear that if C is a curve in $\mathbf{E}_2$ and $\mathbf{f}$ is a two–dimensional vector function with the components U and V then

$$\int_C \mathbf{f}\cdot\vec{\tau}\, ds = \int_C \mathbf{f}\cdot d\mathbf{s} = \int_C (U\, dx + V\, dy). \tag{IV.9}$$

IV.4.5. Example. Using the notation explained in the previous paragraph, you can observe that the integral $\int_C (2x^2 + 3y)\, dz$ is in fact the line integral of a vector function $(0,\, 0,\, 2x^2 + 3y)$:

$$\int_C (2x^2 + 3y)\, dz = \int_C 0\cdot dx + 0\cdot dy + (2x^2 + 3y)\, dz = \int_C (0,\, 0,\, 2x^2 + 3y)\cdot d\mathbf{s}.$$

IV.4.6. Remark. The line integral of vector function $\mathbf{f}$ is defined by means of the line integral of the scalar function $\mathbf{f}\cdot\vec{\tau}$ and so the main properties of the line integral of a vector function are the same as the properties of the line integral of a scalar function. Thus, we can simply rewrite items a), b) and c) of paragraph IV.2.6 with the function $\mathbf{f}\cdot\vec{\tau}$ and we obtain valid assertions for the line integral of a vector function. (Do it for yourself!)

The main and very important difference between the line integral of a scalar function and the line integral of a vector function is that <u>the line integral of a vector function depends on the orientation of the curve</u>. More precisely:

IV.4.7. Theorem. *If a vector function* $\mathbf{f}$ *is integrable on curve* C *then it is also integrable on curve* $-C$ *and*

$$\int_{-C} \mathbf{f}\cdot d\mathbf{s} = -\int_C \mathbf{f}\cdot d\mathbf{s}.$$

This theorem follows immediately from the definition of the line integral of a vector function. The integral $\int_C \mathbf{f}\cdot d\mathbf{s}$ is equal to the integral $\int_C \mathbf{f}\cdot\vec{\tau}\, ds$ where $\vec{\tau}$ is the unit tangent vector. It shows the direction given by the orientation of the curve. If we change the orientation then the unit tangent vector changes its sign and this leads to the change of sign of the integral.

IV.4.8. Evaluation of the line integral of a vector function. The line integral of a vector function $\mathbf{f}$ on a simple smooth curve C can be evaluated by means of a parametrization of C. Suppose that P is such a parametrization, defined in the interval $\langle a, b\rangle$. Suppose further that curve C is oriented in accordance with parametrization P. Then the unit tangent vector in every "interior" point of curve C can be expressed as $\vec{\tau} = \dot{P}(t)/|\dot{P}(t)|$. Now using the definition of the line integral of the vector function $\mathbf{f}$ and formula (IV.7), we obtain

$$\int_C \mathbf{f}\cdot d\mathbf{s} = \int_C \mathbf{f}\cdot\vec{\tau}\, ds = \int_a^b \mathbf{f}(P(t))\cdot\frac{\dot{P}(t)}{|\dot{P}(t)|}\, |\dot{P}(t)|\, dt,$$

$$\int_C \mathbf{f}\cdot d\mathbf{s} = \int_a^b \mathbf{f}(P(t))\cdot\dot{P}(t)\, dt. \tag{IV.10}$$

Substituting here $\mathbf{f} = (U, V, W)$, $P(t) = [\phi(t),\, \psi(t),\, \vartheta(t)]$ and $\dot{P}(t) = (\dot{\phi}(t),\, \dot{\psi}(t),\, \dot{\vartheta}(t))$, we get:

$$\int_C \mathbf{f}\cdot d\mathbf{s} = \int_a^b \left[U\, \dot{\phi}(t) + V\, \dot{\psi}(t) + W\, \dot{\vartheta}(t)\right] dt \tag{IV.11}$$

where $U = U(\phi(t),\, \psi(t),\, \vartheta(t))$, $V = V(\phi(t),\, \psi(t),\, \vartheta(t))$ and $W = W(\phi(t),\, \psi(t),\, \vartheta(t))$. Formula (IV.11) can also be formally obtained from (IV.8) if we use the substitution $x = \phi(t)$, $y = \psi(t)$, $z = \vartheta(t)$ and $dx = \dot{\phi}(t)\, dt$, $dy = \dot{\psi}(t)\, dt$, $dz = \dot{\vartheta}(t)\, dt$.

If curve C is not oriented in accordance with parametrization P (i.e. P generates the opposite orientation of C) then formula (IV.11) holds with the sign "$-$" in front of the integral on the right hand side.

The line integral of a vector function on a simple piecewise–smooth curve C which is a union of the simple smooth curves $C_1, C_2, \dots, C_m$ (see paragraph IV.1.6 for details) can be evaluated in such a way that we first compute the integral on each smooth part $C_1, C_2, \dots, C_m$ of curve C (e.g. by means of the parametrization of these parts) and then we use the fact that the integral on C is equal to the sum of the integrals on $C_1, C_2, \dots, C_m$.

Finally, the line integral of a vector function can also be sometimes evaluated by means of the Green theorem, the Stokes theorem or formula (VI.1). (You will find the details in paragraphs IV.5.5., V.6.6 and VI.1.5.)

IV.4.9. Example. Find the work done by the force $\mathbf{f}(x,y,z) = (y-x^2)\,\mathbf{i} + (z-y^2)\,\mathbf{j} + (x-z^2)\,\mathbf{k}$ over the curve $C:\ P(t) = [t,\, t^2,\, t^3];\quad t \in \langle 0,1\rangle$ from $[0,0,0]$ to $[1,1,1]$.

Curve C is defined by means of its parametrization P. Since $[0,0,0] = i.p.\,C = P(0)$ and $[1,1,1] = t.p.\,C = P(1)$, C is oriented in accordance with parametrization P. We can easily find that $\dot{P}(t) = (1,\, 2t,\, 3t^2)$. Using formula (IV.10), we obtain

$$\int_C \mathbf{f}\cdot d\mathbf{s} = \int_C (y-x^2,\, z-y^2,\, x-z^2)\cdot d\mathbf{s} = \int_0^1 (0,\, t^3-t^4,\, t-t^6)\cdot(1,\, 2t,\, 3t^2)\, dt =$$

$$= \int_0^1 \left[2t^4 - 2t^5 + 3t^3 - 3t^8\right] dt = \left[\tfrac{2}{5}t^5 - \tfrac{2}{6}t^6 + \tfrac{3}{4}t^4 - \tfrac{3}{9}t^9\right]_0^1 = \tfrac{29}{60}.$$

IV.4.10. Circulation of a vector field around a closed curve. Let C be a closed curve in $\mathbf{E}_2$ or in $\mathbf{E}_3$ and let $\mathbf{f}$ be a vector field (= a vector function) defined on C. The line integral $\int_C \mathbf{f}\cdot d\mathbf{s}$ is called the <u>*circulation*</u> of $\mathbf{f}$ around curve C. In order to stress the fact that C is a closed curve, the integral is also often denoted as $\oint_C \mathbf{f}\cdot d\mathbf{s}$.

IV.5. Green's theorem.

This section deals with the line integral of a vector function on a closed curve in $\mathbf{E}_2$. The vector function is also supposed to be two–dimensional (i.e. to have two components).

The next theorem says something that is very obvious at first sight. We do not give the proof of the theorem. Nevertheless, if you were to see the proof, you would be surprised that it is not easy.

Bear in mind the convention that if nothing else is specified then "curve" denotes a simple piecewise–smooth curve and "closed curve" denotes a closed simple piecewise–smooth curve. (See paragraph IV.1.6.)

IV.5.1. Jordan's theorem. *Let C be a closed curve in $\mathbf{E}_2$. Then there exist two disjoint domains G_1 and G_2 in $\mathbf{E}_2$ such that C is their common boundary and*

a) $\mathbf{E}_2 = G_1 \cup C \cup G_2$,

b) one of the domains G_1, G_2 is bounded and the second one is unbounded.

IV.5.2. Interior and exterior of a closed curve in $\mathbf{E}_2$. Let C be a closed curve in $\mathbf{E}_2$ and G_1, G_2 be the domains whose existence is given by Jordan's theorem. That domain of G_1, G_2 which is bounded is called the <u>*interior*</u> of curve C and it is denoted by $Int\, C$. The second domain, which is unbounded, is called the <u>*exterior*</u> of C and it is denoted by $Ext\, C$.

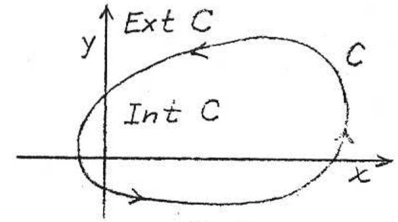

Fig. 9

IV.5.3. Positive and negative orientation of a closed curve in $\mathbf{E}_2$. Let C be a closed curve in $\mathbf{E}_2$. We say that C is oriented <u>*positively*</u> if, when moving on C in accordance with its orientation, we have its interior on our left–hand side. (See Fig. 9.) In the opposite case, i.e. in the case when the interior of C is on our right–hand side when moving along C in accordance with its orientation, we say that C is oriented <u>*negatively*</u>.

IV.5.4. Remark. Definition IV.5.3 is very simple and you can easily imagine what it says, because you know where you have your left hand and your right hand. However, you can also observe that this definition is not correct from a purely logical point of view. Why not? – It is clear: Mathematical notions must be defined precisely and must not depend on our knowledge of where we have our left hand and our right hand. In other words: How would you explain the above definition to an intelligent being (for example an extra–terrestrial) who does not have two hands and is not used to distinguishing between "left" and "right"?

Since the logically correct definition of the positive (respectively negative) orientation of a closed curve in $\mathbf{E}_2$ is not so easy and the above (not quite correct) definition is satisfactory for our purposes, we do not show the correct definition in this text.

IV.5.5. Green's theorem. *Suppose that*

a) a vector function $\mathbf{f} = (U, V)$ has continuous partial derivatives in domain $D \subset \mathbf{E}_2$,

b) C is a positively oriented closed curve in D such that $Int\ C \subset D$.

Then
$$\oint_C \mathbf{f}\cdot d\mathbf{s} = \iint_{Int\ C}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) dx\, dy. \qquad \text{(IV.11)}$$

IV.5.6. Remark. Using the formal equality $\mathbf{f}\cdot d\mathbf{s} = U\,dx + V\,dy$ as in (IV.8), we can write formula (IV.11) in the form

$$\oint_C U\,dx + V\,dy = \iint_{Int\ C}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) dx\, dy. \qquad \text{(IV.12)}$$

If all the assumptions of Green's theorem are satisfied with the exception that curve C is oriented negatively then formulas (IV.11) and (IV.12) hold with the sign "–" in front of the integrals on the right hand sides.

IV.5.7. Example. Evaluate the circulation of the vector field $\mathbf{f} = (-x^2y, xy^2)$ around the positively oriented circle $x^2 + y^2 = \alpha^2$ (where $\alpha > 0$).

If we denote the components of $\mathbf{f}$ by U and V then we get:

$$\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = \frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(-x^2y) = x^2 + y^2.$$

It can easily be verified that all the assumptions of Green's theorem are satisfied and so we obtain:

$$\oint_C -x^2y\,dx + xy^2\,dy = \iint_{x^2+y^2\leq\alpha^2} (x^2+y^2)\,dx\,dy =^{1)} \int_0^{2\pi}\Big(\int_0^{\alpha} r^3\,dr\Big)\,d\varphi = \frac{\pi\alpha^4}{2}.$$

[1]) We have transformed the double integral to the polar coordinates.

IV.5.8. Remark. If the components U and V of vector function $\mathbf{f}$ are such that $\partial V/\partial x - \partial U/\partial y = 1$ then Green's theorem can be used to evaluate the area of $Int\,C$. For example, if we choose $U = -\frac{1}{2}y$, $V = \frac{1}{2}x$ and C is a closed curve in $\mathbf{E}_2$ then

$$\tfrac{1}{2}\oint_C -y\,dx + x\,dy = \tfrac{1}{2}\iint_{Int\,C}\Big(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\Big)\,dx\,dy = \iint_{Int\,C} dx\,dy = m_2(Int\,C).$$

IV.6. Exercises.

1. Decide about the existence of the integral $\int_C ds/(x^2+y^2)$ where C is the circle with its center at point S and radius 1.

a) $S = [0,0]$ b) $S = [1,0]$ $S = [0,-2]$

2. Evaluate the length of curve C which is defined by its parametrization.

a) $P(t) = [3t,\ 3t^2,\ 2t^3]$, $t \in \langle 0,1\rangle$

b) $P(t) = [a\cos t,\ a\sin t,\ bt]$, $t \in \langle 0,2\pi\rangle$ $(a>0,\ b>0)$

3. Evaluate the following integrals. (Which of them are are the line integrals of a scalar function and which of them are the line integrals of a vector function?)

a) $\displaystyle\int_C \frac{ds}{x-y}$; C is the part of the straight line $y = \frac{1}{2}x - 2$ between the points $[0,-2]$ and $[4,0]$

b) $\displaystyle\int_C y\,ds$; C is the part of the parabola $y^2 = 2px$ between the points $[0,0]$ and $[2p,2p]$ $(p>0)$

c) $\displaystyle\int_C xy\,ds$; C is the part of the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the second quadrant

d) $\displaystyle\int_C \sqrt{2y}\,ds$; C is the part of the cycloid $x = a(t-\sin t)$, $y = a(1-\cos t)$ corresponding to $t \in \langle 0,2\pi\rangle$

e) $\displaystyle\int_C (x-y)\,ds$; C is the circle $x^2+y^2 = 2x$

f) $\displaystyle\int_C \frac{z^2}{x^2+y^2}\,ds$; C: $x = a\cos t$, $y = a\sin t$, $z = at$, $t \in \langle 0,2\pi\rangle$ $(a>0)$

g) $\displaystyle\int_C xyz\,ds$; C is the intersection of the surfaces $x^2+y^2+z^2 = R^2$ and $x^2+y^2 = R^2/4$ in the first octant $(R>0)$

h) $\displaystyle\int_C (x+y)\,ds$; C is the quarter of the circle $x^2+y^2+z^2 = R^2$, $y = x$ in the first octant

i) $\displaystyle\int_C x\,dy$; C is the triangle with its sides on the coordinate $x-$ and y–axes and on the line $x/2 + y/3 = 1$, oriented positively

j) $\displaystyle\int_C (x+y)\,dx$; C is the line segment from $[a,0]$ to $[0,b]$

k) $\displaystyle\int_C (x^2-y^2)\,dx$; C is the part of the parabola $y = x^2$ from $[0,0]$ to $[2,4]$

l) $\displaystyle\int_C [-x\cos y\,dx + y\sin x\,dy]$; C is the line segment from $[0,0]$ to $[\pi,\ 2\pi]$

m) $\displaystyle\int_C (y,-x)\cdot d\mathbf{s}$; C is the ellipse $x^2/a^2 + y^2/b^2 = 1$, oriented positively

n) $\displaystyle\int_C \frac{y^2\,dx - x^2\,dy}{x^2+y^2}$; C is the part of the circle $x^2+y^2 = a^2$ $(a>0)$ in the first and in the second quadrant, oriented from $[a,0]$ to $[-a,0]$

o) $\displaystyle\int_C (2a-y,\ -a+y)\cdot d\mathbf{s}$; C: $x = a(t-\sin t)$, $y = a(1-\cos t)$, $t \in \langle 0,2\pi\rangle$; C is oriented from $[2\pi a,0]$ to $[0,0]$

p) $\displaystyle\int_C [y^2\,dx + z^2\,dy + x^2\,dz]$; C is the intersection of the sphere $x^2+y^2+z^2 = R^2$ with the cylindrical surface $x^2+y^2 = Rx$ $(R>0)$, $z \geq 0$; C is oriented positively as viewed from the origin $O = [0,0,0]$

q) $\displaystyle\int_C [2xy\,\mathbf{i} - x^2\mathbf{j}]\cdot d\mathbf{s}$; C is the union of the line segments leading from $[0,0]$ to $[2,0]$ and from $[2,0]$ to $[2,1]$

r) $\displaystyle\int_C [yz\,\mathbf{i} + z\sqrt{R^2-y^2}\,\mathbf{j} + xy\,\mathbf{k}]\cdot d\mathbf{s}$; C: $x = R\cos t$, $y = R\sin t$, $z = at/(2\pi)$ $(a>0)$, C is oriented from its intersection with the plane $z = 0$ to its intersection with the plane $z = a$

s) $\displaystyle\int_C (1-x^2)y\,dx + x(1+y^2)\,dy$; C is the boundary of the square $\langle 0,2\rangle \times \langle 0,2\rangle$ oriented positively

t) $\displaystyle\int_C (e^{xy} + 2x\cos y)\,dx + (e^{xy} - x^2\sin y)\,dy$; C is the circle $x^2+y^2 = 8$ oriented positively

u) $\displaystyle\int_C (xy+x+y)\,dx + (xy+x-y)\,dy$; C is the ellipse $9x^2 + 36x + 4y^2 = 0$ oriented negatively

v) $\displaystyle\int_C (x+y)\,dx - 2x\,dy$; C is the boundary of the triangle with its sides on the lines $x = 0$, $y = 0$, $x+y = 5$, oriented negatively

w) $\displaystyle\int_C (dx/y - dy/x)$; C is the boundary of the triangle with the vertices $[1,1]$, $[2,1]$, $[2,2]$, oriented positively