x) $\int_C [2(x^2+y^2)\mathbf{i}+(x+y)^2\mathbf{j}]\,ds$ ; $C$ is the boundary of the triangle with the vertices $[1,1]$, $[2,2]$, $[1,3]$, oriented positively

**4.** Evaluate the work done by force $\mathbf{f}$ over curve $C$.

a) $\mathbf{f}=(x-y,x)$, $C$ is the boundary of the square with the vertices $[-2,-2]$, $[1,-2]$, $[1,1]$, $[-2,1]$, oriented clockwise

b) $\mathbf{f}=(x+y,2x)$, $C$ is the circle $x=a\cos t$, $y=a\sin t$, $t\in\langle 0,2\pi\rangle$

c) $\mathbf{f}=(y,2)$, $C$ is the closed curve which consists of the semi–axes and the quarter of the ellipse $x=2\cos t$, $y=\sin t$, $t\in\langle 0,2\pi\rangle$ in the first quadrant

**5.** $C_1$ is the line segment from $[0,0]$ to $[1,1]$, $C_2$ is the part of the parabola $y=x^2$ from $[0,0]$ to $[1,1]$, $I_1=\int_{C_1}(x+y)^2\,dx-(x-y)^2\,dy$, $I_2=\int_{C_2}(x+y)^2\,dx-(x-y)^2\,dy$. Applying the Green theorem, evaluate the difference $I_1-I_2$.

**6.** Using the line integral, evaluate the area of the interior of the closed curve which consists of the arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$, $t\in\langle 0,2\pi\rangle$ and the line segment connecting the points $[0,0]$ and $[2\pi a,0]$.

**7.** Using the line integral, derive the formula for the area of the interior of the ellipse $x^2/a^2+y^2/b^2=1$.

**8.** Using the line integral, evaluate the area of the interior of the closed curve whose equation in the polar coordinates is $r=a(1-\cos\varphi)$ where $a>0$ (a so called "cardioid").

**9.** Using the line integral, evaluate the area of the interior of a so called asteroid, whose equation is $x^{2/3}+y^{2/3}=a^{2/3}$ $(a>0)$. (You can use its parametric equations $x=a\cos^3 t$, $y=a\sin^3 t$, $t\in\langle 0,2\pi\rangle$.)

**10.** Evaluate the circulation of vector field $\mathbf{f}$ around the closed curve $C$. If it is possible, apply the Green theorem.

a) $\mathbf{f}(x,y)=\left(e^x\sin y-y^2,\ e^x\cos y-1\right)$, $C=C_1\cup C_2$, $C_1=\left\{[x,y]\in\mathbf{E}_2;\ x^2+y^2+2x=0,\ y\le 0\right\}$, $C_2=\left\{[x,y]\in\mathbf{E}_2;\ -1\le x\le 0,\ y=0\right\}$, $C$ is oriented positively.

b) $\mathbf{f}(x,y)=(x+y)\mathbf{i}+(y-x)\mathbf{j}$, $C=\left\{[x,y]\in\mathbf{E}_2;\ x^2/a^2+y^2/b^2=1\right\}$, $C$ is oriented negatively.

c) $\mathbf{f}(x,y)=(x^2,y^2)$, $C$ is the perimeter of a triangle with the vertices $A=[1,1]$, $B=[2,1]$, $D=[2,3]$, oriented positively.

d) $\mathbf{f}(x,y)=(y^2,-x)$, $C$ is the perimeter of a triangle with the vertices $A=[1,1]$, $B=[1,3]$, $D=[2,2]$, oriented negatively.



# V. Surface Integrals

## V.1. Simple surfaces.

We will work with two types of surfaces in $\mathbf{E}_3$: so called simple smooth surfaces and so called simple piecewise–smooth surfaces.

The idea of the definition of a simple smooth surface is the following. Imagine that you have a subset $B$ of $\mathbf{E}_2$, bounded by a closed curve $\Gamma$. You cut set $B$ from $\mathbf{E}_2$, you move it somewhere to $\mathbf{E}_3$ and you deform it elastically so that you do not disturb its smoothness. This means that you can stretch it in various directions, but you cannot break it and you cannot paste two different points together. Thus, you get a simple smooth surface in $\mathbf{E}_3$.

This can be easily expressed mathematically. The described procedure moves every point $[u,v]\in B$ to some other point $P(u,v)=(\phi(u,v),\psi(u,v),\vartheta(u,v))$ in $\mathbf{E}_3$. Thus, $P$ is a mapping of $B$ to $\mathbf{E}_3$. The requirement that the deformation of $B$ is elastic and smooth leads to the condition that $P$ (i.e. the functions $\phi$, $\psi$ and $\vartheta$) is continuous and has continuous partial derivatives in a sufficiently large subset of $B$. The requirement that two different points belonging originally to $B$ cannot be pasted together leads to the condition that mapping $P$ is one–to–one in $B$.

The functions

$$x=\phi(u,v),\qquad y=\psi(u,v),\qquad z=\vartheta(u,v)$$

are called the <u>*coordinate functions*</u> of mapping $P$.

We shall denote the partial derivatives of mapping $P$ with respect to the variables $u$, $v$ by $P_u$ and $P_v$ and we shall work with them as with vectors. Hence

$$P_u(u,v)=\left(\frac{\partial\phi(u,v)}{\partial u},\frac{\partial\psi(u,v)}{\partial u},\frac{\partial\vartheta(u,v)}{\partial u}\right)\quad\text{or shortly}\quad P_u=\left(\frac{\partial\phi}{\partial u},\frac{\partial\psi}{\partial u},\frac{\partial\vartheta}{\partial u}\right),$$

$$P_v(u,v)=\left(\frac{\partial\phi(u,v)}{\partial v},\frac{\partial\psi(u,v)}{\partial v},\frac{\partial\vartheta(u,v)}{\partial v}\right)\quad\text{or shortly}\quad P_v=\left(\frac{\partial\phi}{\partial v},\frac{\partial\psi}{\partial v},\frac{\partial\vartheta}{\partial v}\right).$$

The vector product of vectors $P_u$ and $P_v$ will be denoted by $P_u\times P_v$. Have in mind that

$$P_u\times P_v=\begin{vmatrix}\mathbf{i}, & \mathbf{j}, & \mathbf{k}\\ \frac{\partial\phi}{\partial u}, & \frac{\partial\psi}{\partial u}, & \frac{\partial\vartheta}{\partial u}\\ \frac{\partial\phi}{\partial v}, & \frac{\partial\psi}{\partial v}, & \frac{\partial\vartheta}{\partial v}\end{vmatrix}=\left(\frac{\partial\psi}{\partial u}\frac{\partial\vartheta}{\partial v}-\frac{\partial\vartheta}{\partial u}\frac{\partial\psi}{\partial v},\ \frac{\partial\vartheta}{\partial u}\frac{\partial\phi}{\partial v}-\frac{\partial\phi}{\partial u}\frac{\partial\vartheta}{\partial v},\ \frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial v}-\frac{\partial\psi}{\partial u}\frac{\partial\phi}{\partial v}\right).$$

We describe the notion of a simple smooth surface once again, this time precisely, in the following definition.

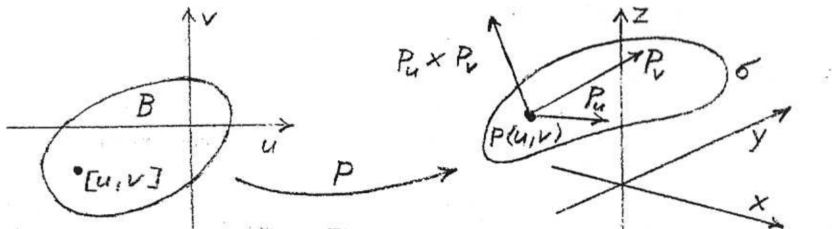


Fig. 10

**V.1.1. Simple smooth surface.** Let $\Omega \subset \mathbf{E}_2$, $P = (\phi, \psi, \vartheta)$ be a mapping of $\Omega$ to $\mathbf{E}_3$, $\Gamma$ be a closed simple piecewise–smooth curve in $\Omega$, and let $B = \Gamma \cup Int\,\Gamma$. Suppose that

a) mapping $P$ is continuous and one–to–one in $B$,

b) $P$ has continuous and bounded partial derivatives $P_u$ and $P_v$ in $B - K$ where $K = \emptyset$ or $K$ is a finite set of points on the boundary $\Gamma$ of set $B$,

c) $P_u \times P_v \neq \vec{0}$ in $B - K$.

Then the set $\sigma = \{X = P(u,v) \in \mathbf{E}_3;\ [u,v] \in B\}$ is called the ***simple smooth surface*** in $\mathbf{E}_3$. Mapping $P$ is called the ***parametrization*** of surface $\sigma$. The set $C = \{X = P(u,v) \in \mathbf{E}_3;\ [u,v] \in \Gamma\}$ is called the ***boundary*** of surface $\sigma$.

The boundary of a simple smooth surface is a closed simple piecewise–smooth curve in $\mathbf{E}_3$. Instead of the word boundary, you may also find the denotation "contour" or "margin" in literature.

Every simple smooth surface has infinitely many parametrizations. (Compare with the analogous statement about the simple smooth curve in paragraph IV.1.1.)

Our definition of a simple smooth surface is relatively straightforward. However, this is paid for by the fact that, for instance, a "nice" surface like a sphere is not a simple smooth surface. All attempts to modify the definition of a simple smooth surface so that it will also include the sphere always lead to such great complications that they do not pay off. This is a consequence of the geometrical structure of the three-dimensional space $\mathbf{E}_3$ – it provides such a variety of possible forms of surfaces that we must be very careful in order to avoid confusion in our definitions and theorems. However, you will see that we do not exclude spheres (and other similar surfaces) from the class of surfaces that we will deal with – they can be treated as so called simple piecewise–smooth surfaces, whose definition is given in paragraphs V.1.5 and V.1.6.

**V.1.2. Orientation of a simple smooth surface. Normal vector.** Let $P$ be a parametrization of a simple smooth surface $\sigma$, defined in set $B \subset \mathbf{E}_2$, and let $X = P(u,v)$ for $[u,v] \in B - K$ (see definition V.1.1). Then the vectors $P_u(u,v)$ and $P_v(u,v)$ are tangent to $\sigma$ at point $X$ and due to condition c) from definition V.1.1, they are linearly independent. Their vector product is perpendicular to both of them,



and so it is also perpendicular to surface $\sigma$. If we divide the vector product by its length, we obtain a unit vector, perpendicular to $\sigma$ at point $X$.

We can choose the ***orientation*** of surface $\sigma$ in such a way that we put the ***normal vector*** $\mathbf{n}$ (i.e. the vector which is perpendicular to $\sigma$, its length is one and its direction shows the orientation of $\sigma$)

either
$$\mathbf{n} = \frac{P_u(u,v) \times P_v(u,v)}{|P_u(u,v) \times P_v(u,v)|} \quad \text{for all } [u,v] \in B - K \tag{V.1}$$

or
$$\mathbf{n} = -\frac{P_u(u,v) \times P_v(u,v)}{|P_u(u,v) \times P_v(u,v)|} \quad \text{for all } [u,v] \in B - K. \tag{V.2}$$

If $\mathbf{n}$ is given by formula (V.1) then we say that the simple smooth surface $\sigma$ is oriented in accordance with its parametrization $P$.

Thus, the simple smooth surface $\sigma$ is oriented by the choice of a normal vector $\mathbf{n}$ (i.e. a unit vector perpendicular to $\sigma$) at any point where the perpendicular direction to $\sigma$ is defined. The normal vector is oriented to the same side of $\sigma$ at every point where it exists. This means that it changes continuously if you move on $\sigma$.

**V.1.3. The relation between the orientation of a simple smooth surface and its boundary.** We say that boundary $C$ of a simple smooth surface $\sigma$ is ***oriented in accordance with*** $\sigma$ if your left hand shows the direction of the normal vector $\mathbf{n}$ on $\sigma$ when you move on $C$ in the sense of its orientation. (See Fig. 11.)

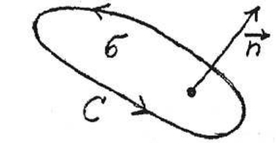


Fig. 11

It is obvious that this definition is not logically quite correct (for the same reasons as in the case of definition IV.5.3). Nevertheless, it is instructive, simple and it cannot lead to confusion.

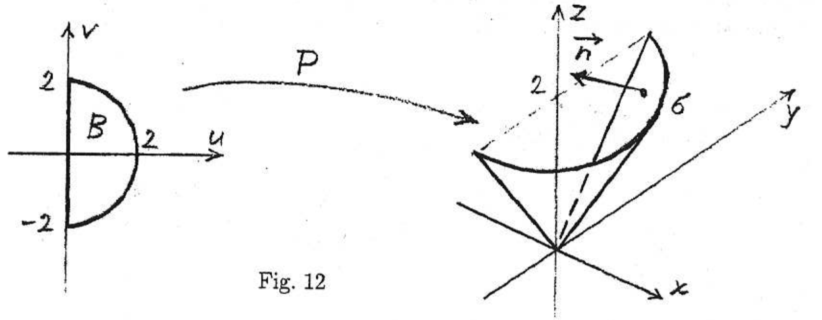


Fig. 12

**V.1.4. Example.** Surface $\sigma$ is a part of the cone $z = \sqrt{x^2 + y^2}$, corresponding to $x \geq 0$ and $x^2 + y^2 \leq 4$. It is oriented "upwards", i.e. the third component of the normal vector is positive. Show that $\sigma$ is a simple smooth surface, find its parametrization, decide whether $\sigma$ is or is not oriented in accordance with your parametrization, and define the orientation of the boundary of $\sigma$ so that it is oriented in accordance with the orientation of $\sigma$.

In order to parametrize $\sigma$, we can use the equation $z = \sqrt{x^2+y^2}$: We can put

$$P: \quad x = \phi(u,v) = u, \quad y = \psi(u,v) = v, \quad z = \vartheta(u,v) = \sqrt{u^2+v^2}$$

for $[u,v] \in B$ where set $B$ is defined by means of the conditions $u \geq 0$ and $u^2+v^2 \leq 4$:

$$B = \{[u,v] \in \mathbf{E}_2;\ u \geq 0,\ u^2+v^2 \leq 4\}.$$

It can be verified that mapping $P$ has all the properties which are required in definition V.1.1, and so $\sigma$ is a simple smooth surface and $P$ is the parametrization of $\sigma$. The partial derivatives $P_u$ and $P_v$ are

$$P_u(u,v) = \left(1,\ 0,\ \frac{u}{\sqrt{u^2+v^2}}\right), \qquad P_v(u,v) = \left(0,\ 1,\ \frac{v}{\sqrt{u^2+v^2}}\right),$$

their vector product is

$$P_u \times P_v = \left(-\frac{u}{\sqrt{u^2+v^2}},\ -\frac{v}{\sqrt{u^2+v^2}},\ 1\right)$$

and the length of this vector product is $\sqrt{2}$. Thus, the unit vector $P_u \times P_v / |P_u \times P_v|$ perpendicular to $\sigma$ equals $P_u \times P_v/\sqrt{2}$ and it is seen that the third component of this vector is positive. Hence it coincides with the given normal vector $\mathbf{n}$ and so we can say that surface $\sigma$ is oriented in accordance with our parametrization $P$.

The orientation of the boundary $C$ of $\sigma$ which corresponds to the orientation of $\sigma$ is marked in Fig. 12. For example, the unit tangent vector to $C$ at the point $X = [2,0,2]$ is $\vec{\tau} = (0,1,0)$.

**V.1.5. A simple piecewise–smooth surface consisting of two simple smooth surfaces.** Suppose that $\sigma_1$ and $\sigma_2$ are two oriented simple smooth surfaces whose boundaries $C_1$ and $C_2$ are either both oriented in accordance with $\sigma_1$ and $\sigma_2$ or they are both oriented opposite to $\sigma_1$ and $\sigma_2$. Suppose that

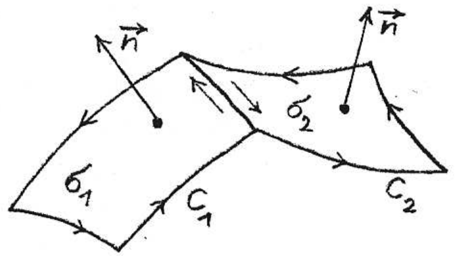

Fig. 13

a) $\sigma_1 \cap \sigma_2 = C_1 \cap C_2$ and this set forms a simple piecewise–smooth curve or more (a finite number of) such curves,



b) the orientation of $C_1$ and $C_2$ is opposite in all points of $C_1 \cap C_2$. (See Fig. 13.)

Then the union $\sigma = \sigma_1 \cup \sigma_2$ is called a *simple piecewise-smooth surface* in $\mathbf{E}_3$, consisting of two simple smooth surfaces $\sigma_1$ and $\sigma_2$.

The orientation of $\sigma$ is given by the orientation of $\sigma_1$ and $\sigma_2$.

The boundary of $\sigma$ is the closure of the set $(C_1 \cup C_2) - (C_1 \cap C_2)$. (See Fig. 13.) It is either empty, or it is one simple piecewise–smooth curve (see Fig. 13), or it consists of more (a finite number of) simple piecewise–smooth curves.

**V.1.6. A simple piecewise–smooth surface consisting of more simple smooth surfaces.** Let $\sigma_1$ and $\sigma_2$ be the simple smooth surfaces from the previous paragraph. If we successively, respecting the same rules, connect other simple smooth surfaces $\sigma_3, \sigma_4, \dots, \sigma_m$ to the union $\sigma \cup \sigma_2$, we obtain a simple piecewise–smooth surface in $\mathbf{E}_3$ which consists of $m$ simple smooth surfaces $\sigma_1, \sigma_2, \dots, \sigma_m$. (See Fig. 14.)

The surface which differs from a simple piecewise–smooth surface $\sigma$ only by its orientation will be denoted by $-\sigma$.

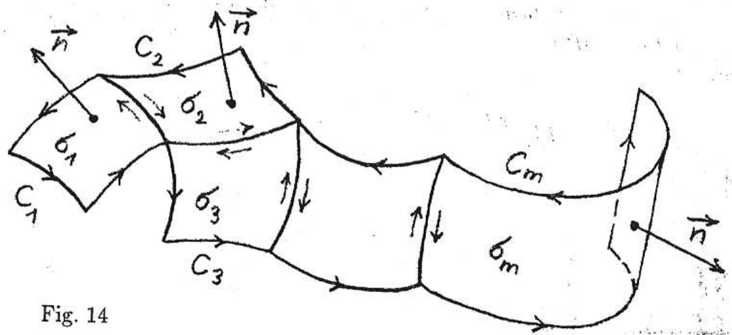

Fig. 14

**V.1.7. A closed simple piecewise– smooth surface.** A simple piecewise–smooth surface $\sigma$ whose boundary is the empty set is called *closed*.

**V.1.8. Example.** A surface which consists of two simple smooth surfaces $\sigma_1$ : $x^2+(y+1)^2 = 2;\ y \geq 0$ and $\sigma_2$ : $x^2+(y-1)^2 = 2;\ y \leq 0$ is a closed simple piecewise–smooth surface.

Other examples of closed simple piecewise–smooth surfaces are: the surface of a cube, a sphere, an ellipsoid, etc.

Similarly as in the case of curves, we can make an agreement that whenever we will use the word "surface", we will mean a simple piecewise–smooth surface. A "closed surface" will mean a closed simple piecewise–smooth surface. More details about the surfaces will be specified if they are important and necessary.

## V.2. The surface integral of a scalar function.

**V.2.1. Physical motivation.** Suppose that a desk has the form of a simple smooth surface $\sigma$ in $\mathbf{E}_3$ and its surface density (i.e. amount of mass per unit area) is $\rho$. $\rho$ is generally a function of three variables $x$, $y$, $z$. We wish to evaluate the total mass $M$ of the desk.

Suppose that $P$ is a parametrization of $\sigma$ which is defined in set $B \subset \mathbf{E}_2$. Imagine that $B$ can be decomposed to infinitely many "infinitely small" squares of the form $\langle u, u+du \rangle \times \langle v, v+dv \rangle$. $P$ maps each of these squares to the part of $\sigma$. Since the square $\langle u, u+du \rangle \times \langle v, v+dv \rangle$ is supposed to be "infinitely small", its image on $\sigma$ can be taken for an "infinitely small" parallelogram with the vertices $A_1 = P(u,v)$, $A_2 = P(u+du, v) = P(u,v) + P_u(u,v) \cdot du$, $A_3 = P(u+du, v+dv) = P(u,v) + P_u(u,v) \cdot du + P_v(u,v) \cdot dv$ and $A_4 = P(u, v+dv) = P(u,v) + P_v(u,v) \cdot dv$. (See Fig. 15.) Its area is $dp = |A_2 - A_1| \cdot |A_3 - A_1| \cdot \sin\alpha$ and this can be expressed as $dp = |(A_2 - A_1) \times (A_3 - A_1)|$. Substituting here for points $A_1$, $A_2$, $A_3$, we obtain: $dp = |P_u(u,v) \times P_v(u,v)|\, du\, dv$. The mass of the parallelogram $A_1A_2A_3A_4$ is $dM = \rho(A) \cdot dp = \rho(P(u,v))\, |P_u(u,v) \times P_v(u,v)|\, du\, dv$ and the total mass of the whole desk (surface) $\sigma$ is

$$M = \iint_B \rho(P(u,v)) \cdot |P_u(u,v) \times P_v(u,v)|\, du\, dv.$$

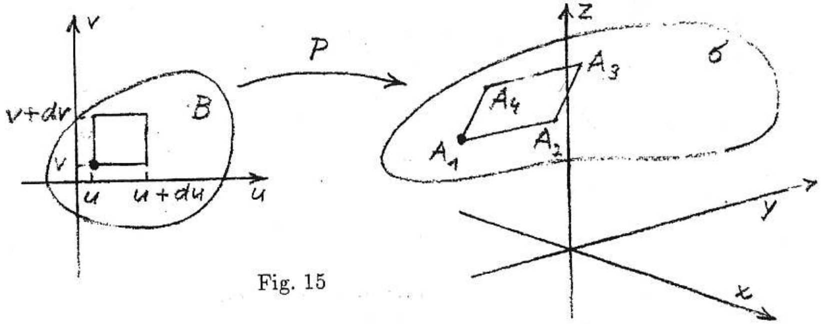


Fig. 15

**V.2.2. The surface integral of a scalar function on a simple smooth surface.** Let $\sigma$ be a simple smooth surface in $\mathbf{E}_3$ and $P$ be its parametrization defined in set $B \subset \mathbf{E}_2$. Let $f$ be a scalar function defined on $\sigma$. We say that $f$ is <u>*integrable*</u> on surface $\sigma$ if the double integral $\iint_B f(P(u,v)) \cdot |P_u(u,v) \times P_v(u,v)|\, du\, dv$ exists. We denote this integral by $\iint_\sigma f\, dp$ and we call it the <u>*surface integral of a scalar function*</u> $f$ on the simple smooth surface $\sigma$.

**V.2.3. Remark.** The integrability of function $f$ on a simple smooth surface $\sigma$ and the surface integral $\iint_\sigma f\, dp$ are defined by means of a parametrization of surface $\sigma$. However, analogously to the line integral of a scalar function (see remark IV.2.3), it can be proved that <u>neither the existence nor the value of the surface integral $\iint_\sigma f\, dp$ depends on the concrete choice of parametrization of surface $\sigma$.</u>



**V.2.4. The surface integral of a scalar function on a simple piecewise–smooth surface.** Let $\sigma$ be a simple piecewise-smooth surface in $\mathbf{E}_3$ which is a union of simple smooth surfaces $\sigma_1, \sigma_2, \ldots, \sigma_m$ (see paragraphs V.1.5 and V.1.6). Let $f$ be a scalar function defined on $\sigma$. If function $f$ is integrable on each of surfaces $\sigma_1, \sigma_2, \ldots, \sigma_m$ then we say that it is integrable on surface $\sigma$ and we put

$$\iint_\sigma f\, dp = \sum_{i=1}^{m} \iint_{\sigma_i} f\, dp. \tag{V.3}$$

The integral on the left hand side is called the <u>*surface integral of the scalar function*</u> $f$ on the simple piecewise-smooth surface $\sigma$.

The surface integral of a scalar function is also often called the <u>*surface integral of the 1st kind*</u>.

Instead of $\iint_\sigma f\, dp$, we can also write for example $\iint_\sigma f(x,y,z)\, dp$.

**V.2.5. The area of a surface.** If $\sigma$ is a simple piecewise-smooth surface, then the value of the integral $\iint_\sigma dp$ is called the <u>*area*</u> of surface $\sigma$.

Specifically, if $\sigma$ is a simple smooth surface and $P$ is its parametrization defined in set $B \subset \mathbf{E}_2$ then the area of surface $\sigma$ is

$$p(\sigma) = \iint_\sigma dp = \iint_B |P_u(u,v) \times P_v(u,v)|\, du\, dv. \tag{V.4}$$

**V.2.6. Some important properties of the surface integral of a scalar function.** Since the surface integral of a scalar function is defined by means of the double integral, its basic properties are the same as the corresponding properties of the double integral. Let us mention only some of them:

a) **(Existence of the surface integral.)** *If function $f$ is continuous on surface $\sigma$ then it is integrable on $\sigma$ (i.e. the integral $\iint_\sigma f\, dp$ exists).*

b) **(Linearity of the surface integral.)** *If functions $f$ and $g$ are integrable on surface $\sigma$ and $\alpha \in \mathbf{R}$ then*

$$\iint_\sigma (f+g)\, dp = \iint_\sigma f\, dp + \iint_\sigma g\, dp,$$

$$\iint_\sigma \alpha \cdot f\, dp = \alpha \cdot \iint_\sigma f\, dp.$$

c) *If function $f$ is integrable on surface $\sigma$ and function $g$ differs from $f$ at most in a finite number of points or curves then $g$ is also integrable on $\sigma$ and*

$$\iint_\sigma g\, dp = \iint_\sigma f\, dp.$$

d) *If function $f$ is integrable on surface $\sigma$ then it is also integrable on surface $-\sigma$ and*

$$\iint_{-\sigma} f\, dp = \iint_\sigma f\, dp.$$

Assertion a) can be generalized in this way: *If $\sigma$ is a simple piecewise–smooth surface and $f$ is continuous on each of its smooth parts then $f$ is integrable on $\sigma$.*

Assertion d) says that neither the existence nor the value of the surface integral of a scalar function depends on the orientation of the surface.

**V.2.7. Evaluation of the surface integral of a scalar function.** The surface integral of function $f$ on a simple smooth surface $\sigma$ can be evaluated by means of a parametrization of $\sigma$. Thus, if $P$ is such a parametrization, defined in set $B \subset \mathbf{E}_2$, and function $f$ is integrable on $\sigma$ then we can use the formula

$$\iint_\sigma f\,dp = \iint_B f(P(u,v)) \cdot |P_u(u,v) \times P_v(u,v)|\,du\,dv. \tag{V.5}$$

This formula follows immediately from the definition of the surface integral on a simple smooth surface – see paragraph V.2.2.

If $\sigma$ is a simple piecewise–smooth surface which is a union of simple smooth surfaces $\sigma_1, \sigma_2, \dots, \sigma_m$ (see paragraphs V.1.5 and V.1.6) then the surface integral of function $f$ on surface $\sigma$ can be computed by means of formula (V.3).

**V.2.8. Example.** Integrate the function $f(x,y,z) = x + 2y$ over the the surface $\sigma:\ x+y+z=1,\ x \geq 0,\ y \geq 0,\ z \geq 0.$

Surface $\sigma$ can be parametrized by the mapping

$$P(u,v):\quad x=u,\quad y=v,\quad z=1-u-v;\quad [u,v] \in B$$

where $B = \{[u,v] \in \mathbf{E}_2;\ 0 \leq u \leq 1,\ 0 \leq v \leq 1-u\}$. We can find that $P_u = (1,\,0,\,-1)$, $P_v = (0,\,1,\,-1)$, $P_u \times P_v = (1,1,1)$ and $|P_u \times P_v| = \sqrt{3}$. Using formula (V.5), we get

$$\iint_\sigma (x+2y)\,dp = \iint_B (u+2v)\sqrt{3}\,du\,dv = \sqrt{3}\int_0^1\int_0^{1-u}(u+2v)\,dv\,du =$$
$$= \sqrt{3}\int_0^1 (1-u)\,du = \sqrt{3}/2.$$

**V.2.9. Example.** Integrate the function $g(x,y,z) = xyz$ over the surface of the cube cut from the first octant by the planes $x=1$, $y=1$ and $z=1$.

The cube can also be expressed as the Cartesian product $\langle 0,1\rangle \times \langle 0,1\rangle \times \langle 0,1\rangle$. Its surface has six sides. Since $xyz=0$ on the three sides that lie in the coordinate planes, the integral over the surface of the cube is equal to

$$\int_{\sigma_1} xyz\,dp + \int_{\sigma_2} xyz\,dp + \int_{\sigma_3} xyz\,dp$$

where $\sigma_1$ is the square region $x=1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1$, $\sigma_2$ is the square region $y=1,\ 0 \leq x \leq 1,\ 0 \leq z \leq 1$ and $\sigma_3$ is the square region $z=1,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1$. $\sigma_1$ can be naturally parametrized by the mapping

$$P(u,v):\quad x=1,\quad y=u,\quad z=v;\quad [u,v] \in B = \langle 0,1\rangle \times \langle 0,1\rangle.$$



It is obvious that $P_u = (0,1,0)$, $P_v = (0,0,1)$, $P_u \times P_v = (1,0,0)$ and $|P_u \times P_v| = 1$. Using formula (V.5), we obtain

$$\iint_{\sigma_1} xyz\,dp = \int_0^1\int_0^1 uv\,du\,dv = \tfrac{1}{4}.$$

Due to the symmetry, the integrals over $\sigma_2$ and $\sigma_3$ are also $\frac{1}{4}$. Hence, the integral over the surface of the cube is equal to $\frac{3}{4}$.

**V.2.10. Example.** Although the sphere $\sigma_R:\ x^2+y^2+z^2 = R^2$ is not a simple smooth surface, there exists a mapping $P$ of a closed set $B \subset \mathbf{E}_2$ onto the sphere which has all the properties of a parametrization (see paragraph V.1.1) with one exception: it is not one–to–one on the whole set $B$. (It is one–to–one in the interior of $B$, but not on the boundary of $B$.) Mapping $P$ is defined by the equations

$$x = \phi(u,v) = R\cos u\cos v,$$
$$y = \psi(u,v) = R\sin u\cos v,$$
$$z = \vartheta(u,v) = R\sin v$$

for $u \in \langle 0, 2\pi\rangle$, $v \in \langle -\pi/2, \pi/2\rangle$. (You can observe that the background of $P$ is the expression of the coordinates of the points of sphere $\sigma_R$ in the spherical coordinates.) Since $P$ fail to satisfy all the requirements on the parametrization only on the set of two–dimensional measure zero (the boundary of $B$) and it is already known that the behaviour of integrands on sets of two–dimensional measure zero does not affect double or surface integrals, $P$ can be used in the evaluation of the surface integral on sphere $\sigma_R$ in the same way as if it were a parametrization. (In fact, mappings whose properties differ from the required properties of parametrizations only on sets of two–dimensional measure zero are also often, not quite correctly, called the parametrizations.)

Thus, if for example $f(x,y,z) = x^2+y^2$ then, using formula (V.5), we obtain

$$\iint_{\sigma_R} f(x,y,z)\,dp = \iint_B R^2\cos^2 v\,|P_u(u,v) \times P_v(u,v)|\,du\,dv.$$

Vectors $P_u$, $P_v$, $P_u \times P_v$ and the number $|P_u \times P_v|$ are:

$$P_u(u,v) = \big(-R\sin u\cos v,\ R\cos u\cos v,\ 0\big),$$
$$P_v(u,v) = \big(-R\cos u\sin v,\ -R\sin u\sin v,\ R\cos v\big),$$
$$P_u(u,v) \times P_v(u,v) = \big(R^2\cos u\cos^2 v,\ R^2\sin u\cos^2 v,\ R^2\sin v\cos v\big),$$
$$|P_u(u,v) \times P_v(u,v)| = R^2\cos v.$$

Substituting this to the above integral and applying Fubini's theorem III.3.2, we get

$$\iint_B R^2\cos^2 v\,|P_u(u,v) \times P_v(u,v)|\,du\,dv = \int_0^{2\pi}\Big(\int_{-\pi/2}^{\pi/2} R^4\cos^3 v\,dv\Big)\,du = \tfrac{8}{3}\pi R^4.$$

**V.2.11. Remark.** The approach explained in example V.2.10 can also be used in connection with other simple piecewise–smooth surfaces, such as ellipsoids and conic surfaces. If for instance $\sigma$ is the conic surface $x^2+y^2=z^2$ corresponding to $z \in \langle 0,4\rangle$ then the mapping

$$P: \quad x=\phi(u,v)=u\cos v, \quad y=\psi(u,v)=u\sin v, \quad z=\vartheta(u,v)=u$$

(defined for $u \in \langle 0,2\rangle$, $v \in \langle 0,2\pi\rangle$) has similar properties as mapping $P$ from example V.2.10: It satisfies all the conditions of the parametrization (see paragraph V.1.1) with the exception that it is one–to–one only in the interior of its domain, i.e. in $(0,2\pi)\times(0,2)$ and not in $\langle 0,2\pi\rangle \times \langle 0,2\rangle$. Nevertheless, mapping $P$ can be used in the evaluation of the surface integral on $\sigma$ in the same way as if it were a parametrization of $\sigma$.

## V.3. Some physical applications of the surface integral of a scalar function.

Suppose that a desk has the form of surface $\sigma$ in $\mathbf{E}_3$. The desk need not be homogeneous, and so its surface density (amount of mass per unit of area) need not be constant. Let the density be given by function $\rho(x,y,z)$. The surface integral of a scalar function can be used to define and evaluate some mechanical characteristics of surface $\sigma$. Suppose that $\rho$ is expressed in $[\mathrm{kg}\cdot\mathrm{m}^{-2}]$. Then we have:

Mass $\quad M = \iint_\sigma \rho(x,y,z)\,dp \quad [\mathrm{kg}]$,

Static moment about the $xy$–plane $\quad M_{xy} = \iint_\sigma z\cdot\rho(x,y,z)\,dp \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Static moment about the $xz$–plane $\quad M_{xz} = \iint_\sigma y\cdot\rho(x,y,z)\,dp \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Static moment about the $yz$–plane $\quad M_{yz} = \iint_\sigma x\cdot\rho(x,y,z)\,dp \quad [\mathrm{kg}\cdot\mathrm{m}]$,

Center of mass $[x_m,y_m,z_m]$ $\quad x_m=\dfrac{M_{yz}}{M},\quad y_m=\dfrac{M_{xz}}{M},\quad z_m=\dfrac{M_{xy}}{M} \quad [\mathrm{m}]$,

Moment of inertia about the $x$–axis $\quad J_x = \iint_\sigma (y^2+z^2)\cdot\rho(x,y,z)\,dp \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the $y$–axis $\quad J_y = \iint_\sigma (x^2+z^2)\cdot\rho(x,y,z)\,dp \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the $z$–axis $\quad J_z = \iint_\sigma (x^2+y^2)\cdot\rho(x,y,z)\,dp \quad [\mathrm{kg}\cdot\mathrm{m}^2]$,

Moment of inertia about the origin $\quad J_0 = \iint_\sigma (x^2+y^2+z^2)\cdot\rho(x,y,z)\,dp \quad [\mathrm{kg}\cdot\mathrm{m}^2]$.

Derive the formula for the moment of inertia about a general straight line in $\mathbf{E}_3$ whose parametric equations are $x=x_0+u_1t$, $y=y_0+u_2t$, $z=z_0+u_3t$; $t\in\mathbf{R}$!



## V.4. The surface integral of a vector function.

**V.4.1. Physical motivation.** Suppose that $\sigma$ is a surface in the flow of an incompressible fluid and we wish to express the flux of the fluid through surface $\sigma$ per unit time. By "flux", we understand the volume of the fluid that flows through the surface. Suppose that the fluid moves with a steady velocity $\mathbf{v}(x,y,z)$, and $\mathbf{n}(x,y,z)$ is the normal vector to $\sigma$ at the point $[x,y,z]$. The flux of the fluid through an "infinitely small" part of surface $\sigma$ which finds itself at $[x,y,z]$ and its area is $dp$ is $\mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)\;ds$. Thus, the total flux through the whole surface $\sigma$ is $\iint_\sigma \mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)\,ds$.

The same approach can also be used if we wish to evaluate e.g. the flux of a magnetic field through a given surface.

Let us recall that the idea of an "infinitely small" part of $\sigma$ is not logically quite precise (see also Section II.7 for further details). However, if we apply the idea carefully, it can be useful especially in situations when we need to derive formulas expressing various geometrical and physical quantities.

**V.4.2. The surface integral of a vector function.** Let $\sigma$ be a simple piecewise–smooth surface in $\mathbf{E}_3$ and let $\mathbf{f}$ be a vector function (with three components) defined on $\sigma$. We say that the vector function $\mathbf{f}$ is integrable on surface $\sigma$ if the scalar function $\mathbf{f}\cdot\mathbf{n}$ is integrable on $\sigma$ (in the sense explained in paragraphs V.2.2 and V.2.4). The integral $\iint_\sigma \mathbf{f}\cdot\mathbf{n}\;dp$ is called the *surface integral of a vector function* $\mathbf{f}$ on surface $\sigma$, and it is usually denoted by $\iint_\sigma \mathbf{f}\cdot d\mathbf{p}$.

The surface integral of a vector function is also often called the *surface integral of the 2nd kind*. It defines the *flux* of vector field $\mathbf{f}$ through surface $\sigma$.

**V.4.3. Remark.** The fact that the normal vector $\mathbf{n}$ need not exist in all points of a simple piecewise–smooth surface $\sigma$ does not matter. $\mathbf{n}$ need not be defined at points where the smooth parts of $\sigma$ are connected and they form at most a finite number of lines. The surface integral of a vector function is defined by means of the surface integral of a scalar function and we already know that this integral does not depend on the behaviour of the integrand in a finite number of points or curves. (See paragraph V.2.6, part c).)

**V.4.4. Remark.** It is very important to understand various ways in which the surface integral of a vector function can be written down, and to recognize correctly what they mean.

If the vector function $\mathbf{f}$ has components $U$, $V$ and $W$ then the integrals

$$\iint_\sigma \mathbf{f}\cdot d\mathbf{p}, \quad \iint_\sigma \mathbf{f}\cdot\mathbf{n}\,dp, \quad \iint_\sigma (U,V,W)\cdot d\mathbf{p},$$

$$\iint_\sigma (U,V,W)\cdot\mathbf{n}\,dp, \quad \iint_\sigma (U\mathbf{i}+V\mathbf{j}+W\mathbf{k})\cdot d\mathbf{p}, \quad \iint_\sigma (U\mathbf{i}+V\mathbf{j}+W\mathbf{k})\cdot\mathbf{n}\,dp$$

have the same meaning.

Another denotation of the surface integral of a vector function also sometimes appears in literature. It is based on the idea of expressing vector $d\mathbf{p}$ in the form

$$d\mathbf{p} = (dy\,dz,\ dx\,dz,\ dy\,dy) = \mathbf{i}\ dy\,dz + \mathbf{j}\ dx\,dz + \mathbf{k}\ dx\,dy.$$

Substituting this to the integral $\iint_\sigma (U,V,W)\cdot d\mathbf{p}$ and computing the scalar product, we obtain

$$\iint_\sigma \mathbf{f}\cdot d\mathbf{p} = \iint_\sigma (U,V,W)\cdot d\mathbf{p} = \iint_\sigma U\ dy\,dz + V\ dx\,dz + W\ dx\,dy.$$

Nevertheless, we think that this last notation of the surface integral of a vector function can lead to confusion, so we will not use it.

**V.4.5. Remark.** The surface integral of vector function $\mathbf{f}$ is defined by means of the surface integral of the scalar function $\mathbf{f}\cdot\mathbf{n}$ and so the main properties of the surface integral of a vector function are the same as the properties of the surface integral of a scalar function. Thus, we can rewrite items a), b) and c) of paragraph V.2.6 with the function $\mathbf{f}\cdot\mathbf{n}$ instead of $f$ and we obtain valid statements for the surface integral of a vector function. (Do it for yourself!)

The main difference between the surface integral of a scalar function and the surface integral of a vector function is that the surface integral of a vector function depends on the orientation of the surface. This is the content of the following theorem:

**V.4.6. Theorem.** *If a vector function* $\mathbf{f}$ *is integrable on surface* $\sigma$ *then it is also integrable on surface* $-\sigma$ *and*

$$\iint_{-\sigma} \mathbf{f}\cdot d\mathbf{p} = -\iint_\sigma \mathbf{f}\cdot d\mathbf{p}.$$

This theorem is an immediate consequence of the definition of the surface integral of a vector function. The integral $\iint_\sigma \mathbf{f}\cdot d\mathbf{p}$ equals $\iint_\sigma \mathbf{f}\cdot\mathbf{n}\ dp$ where $\mathbf{n}$ is the normal vector to $\sigma$. $\mathbf{n}$ defines the orientation of surface $\sigma$. If we change the orientation then vector $\mathbf{n}$ changes its sign and hence also the surface integral $\iint_\sigma \mathbf{f}\cdot\mathbf{n}\ dp$ changes its sign.

**V.4.7. Evaluation of the surface integral of a vector function.** The surface integral of a vector function $\mathbf{f}$ on a simple smooth surface $\sigma$ can be evaluated by means of a parametrization of $\sigma$. Let $P$ be such a parametrization, defined in set $B \subset \mathbf{E}_2$. Let $\sigma$ be oriented in accordance with parametrization $P$. Then the normal vector $\mathbf{n}$ to $\sigma$ can be expressed in all "interior points" of $\sigma$ as $\mathbf{n} = P_u \times P_v / |P_u \times P_v|$. (See paragraph V.1.2, formula (V.1).) Using the formula $\iint_\sigma \mathbf{f}\cdot d\mathbf{p} = \iint_\sigma \mathbf{f}\cdot\mathbf{n}\ dp$ and formula (V.5), we obtain

$$\iint_\sigma \mathbf{f}\cdot d\mathbf{p} = \iint_\sigma \mathbf{f}\cdot\mathbf{n}\ dp =$$

$$= \iint_B \mathbf{f}(P(u,v))\cdot \frac{P_u(u,v)\times P_v(u,v)}{|P_u(u,v)\times P_v(u,v)|}\,|P_u(u,v)\times P_v(u,v)|\ du\,dv,$$



$$\iint_\sigma \mathbf{f}\cdot d\mathbf{p} = \iint_B \mathbf{f}(P(u,v))\cdot (P_u(u,v)\times P_v(u,v))\ du\,dv. \qquad \text{(V.6)}$$

If surface $\sigma$ is not oriented in accordance with parametrization $P$ (i.e. $P$ generates the opposite orientation of $\sigma$) then formula (V.6) holds with the sign "−" in front of the integral on the right hand side.

The surface integral of a vector function on a simple piecewise–smooth surface $\sigma$ which is a union of simple smooth surfaces $\sigma_1, \sigma_2, \ldots, \sigma_m$ (see paragraph V.1.6 for details) can be computed in such a way that we first evaluate the integral on each smooth part $\sigma_1, \sigma_2, \ldots, \sigma_m$ of surface $\sigma$ (e.g. by means of the parametrization of these parts) and then we use the formula

$$\iint_\sigma \mathbf{f}\cdot d\mathbf{p} = \sum_{i=1}^{m} \iint_{\sigma_i} \mathbf{f}\cdot d\mathbf{p}.$$

Some simple piecewise–smooth surfaces, e.g. spheres, ellipsoids and parts of cones, can be described by means of a mapping whose properties differ from the required properties of parametrizations (see paragraph V.1.1) only on a set of two–dimensional measure zero. Examples of such mappings are given in paragraphs V.2.10 and V.2.11. These mappings (let us recall that they are also often, not quite correctly, called the parametrizations) can be used in formula (V.6) in the same way as parametrizations.

The other possible way of evaluating the surface integral of a vector function is to apply the Gauss–Ostrogradsky or Stokes theorem. These theorems will be explained in Section V.6.

**V.4.8. Example.** Find the flux of the vector field $\mathbf{f}(x,y,z) = yz\,\mathbf{j} + z^2\,\mathbf{k}$ through the surface $\sigma$ cut from the semicircular cylinder $y^2+z^2=4,\ z\ge 0$ by the planes $x=-1$ and $x=1$. Surface $\sigma$ is oriented by its outward normal vector.

We can parametrize surface $\sigma$ by the mapping

$$P(u,v):\quad x=u,\quad y=2\cos v,\quad z=2\sin v;\quad [u,v]\in B=\langle -1,1\rangle\times\langle 0,\pi\rangle.$$

We can find that $P_u = (1,0,0)$, $P_v = (0,-2\sin v, 2\cos v)$ and $P_u\times P_v = (0,-2\cos v,-2\sin v)$. The unit vector perpendicular to $\sigma$ for example at the point $[0,0,1]$ (which corresponds to $u=0$ and $v=\pi/2$), expressed by means of parametrization $P$ is

$$\left.\frac{P_u\times P_v}{|P_u\times P_v|}\right|_{[u,v]=[0,\pi/2]} = (0,0,-1).$$

Since surface $\sigma$ is oriented outward, the above vector is equal to $-\mathbf{n}$ (where $\mathbf{n}$ is the normal vector to $\sigma$ at the point $[0,0,1]$). Hence, parametrization $P$ generates the opposite orientation of surface $\sigma$. This means that if we use formula (V.6), we must write the "−" sign in front of the integral of the right hand side:

$$\iint_\sigma (yz\,\mathbf{j}+z^2\,\mathbf{k})\cdot d\mathbf{p} = -\iint_B [4\sin v\cos v\,\mathbf{j} + 4\sin^2 v\,\mathbf{k}]\cdot (P_u\times P_v)\ du\,dv =$$

$$= -\int_{-1}^{1}\int_{0}^{\pi}\big(0,\ 4\sin v\cos v,\ 4\sin^2 v\big)\cdot\big(0,\ -2\cos v,\ -2\sin v\big)\,dv\,du =$$

$$= -\int_{-1}^{1}\int_{0}^{\pi}\big[-8\sin v\cos^2 v - 8\sin^3 v\big]\,dv\,du = 32.$$

## V.5. Operators div and curl.

**V.5.1. Divergence of a vector field.** Let $\mathbf{f} = (U, V.W)$ be a differentiable vector field in domain $D \subset \mathbf{E}_3$. The *divergence* of $\mathbf{f}$ is a scalar field in $D$ which is denoted by div $\mathbf{f}$ and it is defined by the equation

$$\text{div } \mathbf{f} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}.$$

**V.5.2. Curling of a vector field.** Let $\mathbf{f} = (U, V.W)$ be a differentiable vector field in domain $D \subset \mathbf{E}_3$. The *curling* of $\mathbf{f}$ is a vector field in $D$ which is denoted by curl $\mathbf{f}$ and it is defined by the equation

$$\text{curl } \mathbf{f} = \begin{vmatrix} \mathbf{i}, & \mathbf{j}, & \mathbf{k} \\ \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z} \\ U, & V, & W \end{vmatrix} = \Big(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z},\ \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x},\ \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\Big).$$

Instead of the curling of a vector field, denoted by curl $\mathbf{f}$, we often speak about the *rotation* of a vector field, and we denote it by rot $\mathbf{f}$.

**V.5.3. The operator nabla.** We denote by $\nabla$ and refer to as the operator nabla the vector whose components are operators of partial differentiation with respect to $x$, $y$ and $z$. Thus

$$\nabla = \Big(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\Big).$$

(We use the word "operator" because it prescribes performance of some operations – in our case performance of partial derivatives with respect to $x$, $y$ and $z$.)

The operator nabla is often used in the denotation of various scalar or vector fields. You already know that the gradient of a scalar field $\phi$ is a vector field whose components are the partial derivatives of $\phi$ with respect to $x$, $y$ and $z$. This can be expressed by means of the operator $\nabla$ in this way:

$$\text{grad } \phi = \nabla\phi = \Big(\frac{\partial\phi}{\partial x},\ \frac{\partial\phi}{\partial y},\ \frac{\partial\phi}{\partial z}\Big).$$

On the other hand, divergence of a vector field $\mathbf{f} = (U, V, W)$, which is a scalar field, can be written down by means of the scalar product of $\nabla$ and $\mathbf{f}$:

$$\text{div } \mathbf{f} = \nabla\cdot\mathbf{f} = \nabla\cdot(U, V, W) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}.$$



Finally, curling of a vector field $\mathbf{f} = (U, V, W)$ can be written down as the vector product of $\nabla$ with $\mathbf{f}$:

$$\text{curl } \mathbf{f} = \nabla\times\mathbf{f} = \begin{vmatrix} \mathbf{i}, & \mathbf{j}, & \mathbf{k} \\ \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z} \\ U, & V, & W \end{vmatrix} = \Big(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z},\ \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x},\ \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\Big).$$

**V.5.4. Remark.** Operators grad, div and curl play an important role in the theory of all possible types of fields (flow fields of various fluids, gravity field, electrostatic field, electromagnetic field, etc.). Their mathematical properties and relations are therefore very interesting not only from the point of view of applied mathematics itself, but also from the point of view of many other disciplines. More detailed study of these properties would go beyond the scope of this text. Nevertheless, let us mention two formulas whose validity follows immediately from the definitions of grad, div and curl and it can be easily verified:

If $\phi$ is a twice–differentiable scalar field in a domain $D \subset \mathbf{E}_3$ and $\mathbf{f}$ is a twice–differentiable vector field in $D$ then

$$\text{curl grad } \phi = \vec{0} = (0, 0, 0), \tag{V.7}$$

$$\text{div curl } \mathbf{f} = 0. \tag{V.8}$$

You already know the geometrical meaning of the gradient of a scalar function $\phi$ from Chapter I – grad $\phi$ is the vector which shows the direction of the greatest growth of function $\phi$. The physical sense of the other two operators, div and curl, will be explained in paragraphs V.6.5 and V.6.8.

## V.6. The Gauss–Ostrogradsky theorem and the Stokes theorem.

We already know the "two-dimensional" Jordan theorem – see paragraph IV.5.1. The next paragraph contains a "three–dimensional" version of the same theorem. It says again something that is very clear at first sight. However, you would be surprised that it is quite complicated to prove. (We do not show the proof in this text.)

Note that if no other details are given then "surface" refers to a simple piecewise–smooth surface and "closed surface" means a closed simple piecewise–smooth surface. (See paragraph V.1.8.)

**V.6.1. Jordan's theorem.** *Let $\sigma$ be a closed surface in $\mathbf{E}_3$. Then there exist two disjoint domains $G_1$ and $G_2$ in $\mathbf{E}_3$ such that $\sigma$ is their common boundary and*

*a)* $\mathbf{E}_3 = G_1 \cup \sigma \cup G_2$,

*b) one of the domains $G_1$, $G_2$ is bounded and the second is unbounded.*

**V.6.2. Interior and exterior of a closed surface in $\mathbf{E}_3$.** Let $\sigma$ be a closed surface in $\mathbf{E}_3$ and $G_1$, $G_2$ be the domains whose existence is given by Jordan's theorem. That domain of $G_1$, $G_2$ which is bounded is called the *interior* of surface $\sigma$ and it is denoted by $Int\,\sigma$. The second domain, which is unbounded, is called the *exterior* of $\sigma$ and it is denoted by $Ext\,\sigma$.

We say that the closed surface $\sigma$ is oriented to its exterior (respectively to its interior) if its normal vector (at all points of $\sigma$ where it exists) is oriented to the exterior of $\sigma$ (respectively to the interior of $\sigma$).

**V.6.3. The Gauss–Ostrogradsky theorem.** *Suppose that*

a) *vector function* **f** *has continuous partial derivatives in domain* $D \subset \mathbf{E}_3$,

b) $\sigma$ *is a closed surface in* $D$, *oriented to its exterior and such that* $Int\,\sigma \subset D$.

Then
$$\iint_\sigma \mathbf{f} \cdot d\mathbf{p} = \iiint_{Int\,\sigma} \operatorname{div} \mathbf{f} \; dx\,dy\,dz. \qquad \text{(V.9)}$$

**V.6.4. Example.** Calculate the flux of the field $\mathbf{f}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$ outward through the surface $\sigma$ of the cube $\langle 0,1\rangle \times \langle 0,1\rangle \times \langle 0,1\rangle$.

All the components of **f** are continuously differentiable in the whole space $\mathbf{E}_3$ and the considered surface is a closed surface in $\mathbf{E}_3$, oriented outward. Thus, the Gauss–Ostrogradsky theorem yields

$$\iint_\sigma \mathbf{f} \cdot d\mathbf{p} = \iiint_{Int\,\sigma} \operatorname{div} \mathbf{f} \; dx\,dy\,dz =$$
$$= \int_0^1 \int_0^1 \int_0^1 \left( \frac{\partial(xy)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(xz)}{\partial z} \right) dx\,dy\,dz = \int_0^1 \int_0^1 \int_0^1 (y+z+x)\,dx\,dy\,dz = \tfrac{3}{2}.$$

**V.6.5. Physical sense of divergence.** Suppose that the vector function **f** has continuous partial derivatives in domain $D \subset \mathbf{E}_3$ and $A \in D$. Denote by $\sigma_r$ the sphere with center $A$ and radius $r$, oriented to its exterior. Then

$$\operatorname{div} \mathbf{f}(A) = \lim_{r\to 0+} \frac{\operatorname{div} \mathbf{f}(A)}{\frac{4}{3}\pi r^3} \iiint_{Int\,\sigma_r} dx\,dy\,dz =$$
$$= \lim_{r\to 0+} \frac{1}{\frac{4}{3}\pi r^3} \iiint_{Int\,\sigma_r} \operatorname{div} \mathbf{f}(x,y,z)\,dx\,dy\,dz = \lim_{r\to 0+} \frac{1}{\frac{4}{3}\pi r^3} \iint_{\sigma_r} \mathbf{f}(x,y,z) \cdot d\mathbf{p}.$$

If the vector field **f** has a source at point $A$ then $\iint_{\sigma_r} \mathbf{f} \cdot d\mathbf{p}$ is positive for $r > 0$ sufficiently small and the limit of this integral divided by the volume of $Int\,\sigma_r$ for $r \to 0+$ gives the intensity of the source. Thus, $\operatorname{div} \mathbf{f}(A)$ expresses the intensity of the source of **f** at point $A$.

For example, if **v** is the velocity of a moving incompressible fluid then $\operatorname{div} \mathbf{v} = 0$ in all points of the flow field. This follows from the fact that the conservation of mass, together with the incompressibility of the fluid, guarantees that the fluid cannot arise or disappear at any point $A$ and so the velocity field has no sources (positive or negative). (The equation $\operatorname{div} \mathbf{v} = 0$ is the very well known *equation of continuity* for incompressible fluids – you will hear more about it later, in mechanics of fluids.)



**V.6.6. Stokes' theorem.** *Suppose that*

a) *vector function* **f** *has continuous partial derivatives in domain* $D \subset \mathbf{E}_3$,

b) $\sigma$ *is a surface in* $D$ *whose boundary* $C$ *is oriented in accordance with* $\sigma$.

Then
$$\oint_C \mathbf{f} \cdot d\mathbf{s} = \iint_\sigma \operatorname{curl} \mathbf{f} \cdot d\mathbf{p}. \qquad \text{(V.10)}$$

**V.6.7. Example.** Evaluate the line integral $\int_C \mathbf{f} \cdot d\mathbf{s}$ where $\mathbf{f}(x,y,z) = xz\,\mathbf{i} + xy\,\mathbf{j} + 3xz\,\mathbf{k}$ and $C$ is the boundary of surface $\sigma$ which is the portion of the plane $2x + y + z = 2$ in the first octant. $C$ is oriented counter–clockwise as viewed from above.

A vector perpendicular to $\sigma$ is given by the coefficients from the equation of $\sigma$: $(2,1,1)$. If you sketch a figure, you can observe that surface $\sigma$ is oriented in accordance with its boundary $C$ if its normal vector differs from the vector $(2,1,1)$ only by the length: $\mathbf{n} = (2,1,1)/\sqrt{6}$.

The components of vector field **f** are continuously differentiable in $\mathbf{E}_3$ and $\operatorname{curl} \mathbf{f} = (0,\, x - 3z,\, y)$. Thus, the Stokes theorem yields

$$\int_C \mathbf{f} \cdot d\mathbf{s} = \iint_\sigma \operatorname{curl} \mathbf{f} \cdot d\mathbf{p} = \iint_\sigma (0,\, x-3z,\, y) \cdot d\mathbf{p}.$$

Surface $\sigma$ can be parametrized by the mapping

$$P(u,v): \quad x = u, \quad y = v, \quad z = 2 - 2u - v; \quad [u,v] \in B$$

where $B = \{[u,v] \in \mathbf{E}_2;\ 0 \le u \le 1,\ 0 \le v \le 2 - 2u\}$. We can easily find that $P_u = (1,0,-2)$, $P_v = (0,1,-1)$ and $P_u \times P_v = (2,1,1)$. Since the orientation of the last vector is the same as the orientation of the normal vector **n**, surface $\sigma$ is oriented in accordance with parametrization $P$. Using formula (V.6), we obtain

$$\iint_\sigma (0,\, x-3z,\, y) \cdot d\mathbf{p} =$$
$$= \iint_B (0,\, u - 6 + 6u + 3v,\, v) \cdot (2,1,1)\,du\,dv = \int_0^1 \int_0^{2-2u} [7u + 4v - 6]\,dv\,du = -1.$$

**V.6.8. Physical sense of curling.** Suppose that the vector function **f** has continuous partial derivatives in the domain $D \subset \mathbf{E}_3$, $A \in D$ and **a** is a vector whose length is one. Denote by $\sigma_r$ the disk with center $A$, radius $r$ and normal vector **a**. Denote further by $C_r$ the circle which is the boundary of the disk $\sigma_r$ and is oriented in accordance with $\sigma_r$. Then we have

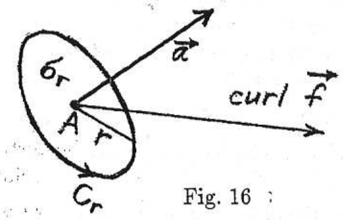

Fig. 16

$$\operatorname{curl} \mathbf{f}(A) \cdot \mathbf{a} = \lim_{r\to 0+} \frac{\operatorname{curl} \mathbf{f}(A) \cdot \mathbf{a}}{\pi r^2} \iint_{\sigma_r} dp = \lim_{r\to 0+} \frac{1}{\pi r^2} \iint_{\sigma_r} \operatorname{curl} \mathbf{f}(x,y,z) \cdot \mathbf{a}\,dp =$$

$$= \lim_{r \to 0+} \frac{1}{\pi r^2} \iint_{\sigma_r} \text{curl } \mathbf{f}(x,y,z) \cdot d\mathbf{p} = \lim_{r \to 0+} \frac{1}{\pi r^2} \oint_{C_r} \mathbf{f}(x,y,z) \cdot d\mathbf{s}.$$

Thus, curl $\mathbf{f}(A)$ is the vector whose scalar product with any unit vector $\mathbf{a}$ expresses the intensity of circulation of $\mathbf{f}$ around circles perpendicular to $\mathbf{a}$, oriented in accordance with $\mathbf{a}$.

## V.7. Exercises.

**1.** $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2+z=4,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$, $\sigma$ is oriented so that its normal vector $\mathbf{n}$ at every point of $\sigma$ satisfies $\mathbf{n} \cdot \mathbf{i} \geq 0$.

a) Verify that the mapping $P(u,v) = [2\cos u,\ 2\sin u,\ v]$ for $[u,v] \in B = \langle -\pi/2, \pi/2 \rangle \times \langle 1,4 \rangle$ is a parametrization of $\sigma$ (i.e. that is has all the properties named in paragraph V.1.1). Decide whether $\sigma$ is oriented in accordance with this parametrization.

b) Show that the mapping $Q(u,v) = [\sqrt{4-u^2},\ u,\ v]$ for $[u,v] \in B = \langle -2,2 \rangle \times \langle 1,4 \rangle$ is not a parametrization of $\sigma$.

**2.** $\sigma$ is the half-sphere $\{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2+z^2=a^2,\ z \geq 0\}$ $(a>0)$, oriented by the normal vector $\mathbf{n} = (n_1,n_2,n_3)$ such that $n_3 \geq 0$. Set $B$ is $B = \{[u,v] \in \mathbf{E}_2;\ u^2+v^2 \leq a^2\}$.

a) Show that the mapping

$$P(u,v) = \left[\frac{2a^2u}{a^2+u^2+v^2},\ \frac{2a^2v}{a^2+u^2+v^2},\ \frac{2a^3}{a^2+u^2+v^2} - a\right]; \quad [u,v] \in B$$

is a parametrization of surface $\sigma$ (i.e. it has all the properties named in paragraph V.1.1). Decide whether $\sigma$ is oriented in accordance with this parametrization.

b) Show that the mapping $P(u,v) = [u,\ v,\ \sqrt{a^2-u^2-v^2}]$, $[u,v] \in B$, is not a parametrization of surface $\sigma$.

**3.** $\sigma$ is a simple smooth surface, oriented by the normal vector $\mathbf{n}$. Find its parametrization, show that it has all the properties named in paragraph V.1.1. and determine whether $\sigma$ is oriented in accordance with the chosen parametrization.

a) $\sigma$ is the triangle with the vertices $A=[1,-1,2]$, $B=[2,1,3]$, $C=[-1,2,4]$, $\mathbf{n} \cdot \mathbf{j} < 0$

b) $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2=4,\ x \geq 0,\ 0 \leq z \leq 4\}$, $\mathbf{n} = (1,0,0)$ at the point $P=[2,0,2]$

c) $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2=z,\ y \geq 0,\ z \leq 1\}$, $P=[0,0,0]$, $\mathbf{n}=(0,0,-1)$

d) $\sigma$ is the parallelogram with the vertices $A=[1,1,1]$, $B=[1,4,4]$, $C=[0,5,6]$, $D=[0,2,3]$, $\mathbf{n} \times \mathbf{k} > 0$ at every point of $\sigma$

e) $\sigma$ is the disk in the plane $x=2$ with its center at the point $[2,-1,3]$ and radius $r=4$, $\mathbf{n}=(-1,0,0)$ at every point of $\sigma$

f) $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2+z^2=4;\ z \geq \sqrt{2}\}$, $P=[0,0,2]$, $\mathbf{n}=(0,0,1)$

g) $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ xy-z=0,\ x^2+y^2 \leq a^2\}$ $(a>0)$, $\mathbf{n} \cdot \mathbf{k} > 0$



**4.** Verify that the set $\sigma = \{X \in \mathbf{E}_3;\ X = P(u,v),\ [u,v] \in B\}$ is a simple smooth surface in $\mathbf{E}_3$ and $P$ is its parametrization.

a) $P(u,v) = [u,\ 4u^2+9v^2,\ v]$, $B = \{[u.v] \in \mathbf{E}_2;\ u^9/9+v^2/4 \leq 1\}$

b) $P(u,v) = [u,\ v,\ 4-u-v]$, $B = \{[u,v] \in \mathbf{E}_2;\ u \geq 0,\ v \geq 0,\ u+v \leq 4\}$

c) $P(u,v) = [3\cos u \cos v,\ 3\sin u \cos v,\ 3\sin v]$, $B = \langle 0,\pi \rangle \times \langle 0,\pi/4 \rangle$

**5.** Verify that $\sigma$ is a simple piecewise–smooth surface. Find parametrizations of the simple smooth parts of $\sigma$.

a) $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ z = 4-\sqrt{x^2+y^2},\ z \geq 0\}$

b) $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ z = 4-\sqrt{x^2+y^2},\ 0 \leq z \leq 2\}$

c) $\sigma_1 \cup \sigma_2$ where $\sigma_1 = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2 \leq 16,\ z=0\}$, $\sigma_2 = \{[x,y,z] \in \mathbf{E}_3;\ z = 4-\sqrt{x^2+y^2},\ z \geq 0\}$

d) $\sigma$ is the boundary of $D = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2+z^2 \leq 4,\ 4-x^2-y^2 \leq 4z\}$

**6.** Decide about the existence of the integral $\iint_\sigma f\, dp$.

a) $f(x,y,z) = (xy \ln|x|)/z$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ (x-2)^2+y^2+z^2=1,\ z \geq 0\}$

b) $f(x,y,z) = (xy \ln|x|)/z$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ z = 1+x^2+y^2,\ z \leq 2\}$

c) $f(x,y,z) = (x^2+y^2+z^2-1)^{-1}$, $\sigma$ is the sphere with its center at the point $S=[0,0,3]$ and radius $r=a$

**7.** Evaluate the area of the surfaces from examples 4c, 5b, 5c, 3g

**8.** Evaluate the surface integrals

a) $\displaystyle\iint_\sigma xyz\, dp$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ y^2+9z^2=9,\ 1 \leq x \leq 3,\ y \geq 0,\ z \geq 0\}$

b) $\displaystyle\iint_\sigma xz\, dp$, $\sigma$ is the triangle with the vertices $A=[1,0,0]$, $[0,1,0]$ and $C=[0,0,1]$

c) $\displaystyle\iint_\sigma x(^2+y^2)\, dp$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2+z^2=a^2\}$, $(a>0)$

d) $\displaystyle\iint_\sigma (xy+yz+xz)\, dp$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ y=\sqrt{x^2+z^2},\ x^2+z^2 \leq 2x\}$

e) $\displaystyle\iint_\sigma (x+y+z)\, dp$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2+z^2=a^2,\ x \leq 0\}$ $(a>0)$

f) $\displaystyle\iint_\sigma \frac{dp}{x^2+y^2+z^2}$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2+y^2=9,\ 0 \leq z \leq 3\}$

g) $\displaystyle\iint_\sigma (x^2+y^2)\, dp$, $\sigma$ is the surface from example 5d

**9.** Find the center of mass of surface $\sigma$ if mass is distributed on $\sigma$ with the density $\rho$.

a) $\rho(x,y,z) = x$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x=\sqrt{y^2+z^2},\ y \geq 0,\ 0 \leq x \leq 2\}$

b) $\rho(x,y,z) = xyz$, $\sigma = \{[x,y,z] \in \mathbf{E}_3;\ x^2+z^2=4,\ x \geq 0,\ z \geq 0,\ 0 \leq y \leq 3\}$