Mathematics I

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Part I: **Linear Algebra**

Part II: **Analytic Geometry in $\mathbb{E}_3$**

Contents

Introduction

This text contains an approximate list of definitions and theorems that students meet in the Mathematics I course in their first term of studies at the Faculty of Mechanical Engineering at the Czech Technical University, together with brief remarks, comments and examples. Some proofs and derivations of formulas are also included. These can be regarded as useful exercises leading to a better understanding of the sense and properties of notions that we deal with. The text was not written to be a completely independent textbook, especially due to brief explications and the limited number of solved and unsolved problems. It is important to emphasize that, in order to be well prepared for the examination in Mathematics I, it is necessary to solve problems individually and to think over a large number of examples. Appropriate examples and exercises can be found e.g. in the textbook [NK].

This revised edition contains Chapter V on the definite (Riemann) integral. This topic is studied at the end of the winter term in the Mathematics I course in the last years, but it is usually examined in the summer term together with double, triple and other integrals. However, it is logical to include the chapter on the Riemann integral into the textbook Mathematics I because it completes the calculus of functions of one variable.

This text is a free English version of the textbook [Ne].

Numbers of paragraphs or sections, whose contents are not actually required for the examination and which are addressed to more interested readers, are marked by the symbol $*$ on the right side above.

The author wishes to express his thanks to Mr. Robin Healey for carefully reading the text, correcting the language and especially for his readiness to discuss with the author the right sense of various formulations and to explain to him some fine points of the English language. If you still find some misprints, incorrect or not quite clear expressions or connections in the text then it is only the author who is responsible. It will be a pleasure for both Mr. Healey and the author if the text helps readers not only in their studies of mathematics, but if it also contributes to their orientation in English mathematical terminology and phraseology, and if it encourages them to go on and study further scientific literature written in English.

We suppose that readers are familiar with the notion of a *set* and that they know the set operations *union*, *intersection*, *complement* and *difference*. Let us remind the reader that an empty set is denoted by $\emptyset$. We can use this notation for describing a set: $M = \{\, x;\ V(x)\,\}$. We read this as: M is the set of all elements x such that $V(x)$ holds. ($V(x)$ is some statement that can be made about x.)

We also assume knowledge of the following notions:

- *mapping* (of a set A to a set B – we use for instance the denotation $F : A \to B$),
- *one–to–one mapping* (also called *injective mapping*),
- *mapping* of the set A *onto* the set B (also called *surjective mapping*),
- *bijective mapping* of the set A onto the set B (a mapping which is injective and surjective),
- *inverse mapping* (we denote this by F_{-1}),
- *composite mapping* (we denote this by $F * G$ or $F \circ G$),

- *domain of a mapping* (we denote this by $D(F)$),
- *range of a mapping* (we denote this by $R(F)$).

Further topics which are also assumed to be known from secondary school include some elementary notions of mathematical logic, i.e. a *statement* and operations with statements:

- *negation of a statement X* (we denote this *non* X),
- *conjunction of statements X and Y* (we denote this $X \wedge Y$ and we read it "both X and Y hold" or briefly "X and Y"),
- *alternative of statements X and Y* (we denote this $X \vee Y$ and we read it "X or Y"),
- *implication* (we denote this $X \Longrightarrow Y$ and we read it "X implies Y", "Y follows from X", "if X holds then Y also holds", "X is a sufficient condition for Y", "Y is a necessary condition for X", "Y holds provided that X holds", etc.) and
- *equivalence* (we denote this $X \Longleftrightarrow Y$ and we read it "X holds if and only if Y holds", "X is equivalent to Y", "X is a necessary and sufficient condition for Y", "Y is a necessary and sufficient condition for X", etc.).

We shall often use so called quantifiers:

- a *universal quantifier* is denoted by $\forall$ and it can be used for example in the sentence: $\forall\, x \in I \ : \ V(x)$ – we read it "for each $x \in I$ the statement $V(x)$ holds" or "each $x \in I$ has the property $V(x)$",
- an *existential quantifier* is denoted by $\exists$ and it can be used for example in this way: $\exists\, x \in I \ : \ V(x)$ – we read it "There exists $x \in I$ such that the statement $V(x)$ holds."

Quantifiers can also be used to create more complicated statements and assertions. Do not underestimate them! Their incorrect usage can entirely change the sense of various statements. You can compare e.g. these two sentences which differ only in the order of the quantifiers: 1) "To every married man there exists a woman who is his wife." 2) "There exists a married man such that every woman is his wife."

If n is a natural number (we use the denotation: $n \in \mathbb{N}$) then the set of all ordered n–tuples of real numbers is denoted by $\mathbb{R}^n$. Thus, $\mathbb{R}^2$ is the set of all ordered pairs of real numbers, $\mathbb{R}^3$ is the set of all ordered 3–tuples of real numbers, etc. An exception is made in the case $n = 1$ where instead of $\mathbb{R}^1$, we write only $\mathbb{R}$. Elements of $\mathbb{R}^n$ are written for example in this way: $[a_1, a_2]$, $[1, 3]$ (if $n = 2$), $[x_1, x_2, x_3]$ (if $n = 3$), $[x_1, x_2, \dots, x_n]$, etc. If the distance of any two elements $X = [x_1, x_2, \dots, x_n]$ and $Y = [y_1, y_2, \dots, y_n]$ from $\mathbb{R}^n$ is defined in such a way that it is equal to

$$d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$$

then $\mathbb{R}^n$ becomes the so called *n–dimensional Euclidean space*. This is denoted by $\mathbb{E}_n$. Elements of $\mathbb{E}_n$ are often called "points" and the distance of two points X and Y is also denoted by $\|X - Y\|$. $\mathbb{E}_1$ can be imagined as a straight line, $\mathbb{E}_2$ as a plane, etc.

I. Linear algebra

I.1. Vector spaces

I.1.1. The n–dimensional arithmetic space. Let us define the sum of any two elements $[x_1, x_2, \dots, x_n]$, $[y_1, y_2, \dots, y_n]$ from $\mathbb{R}^n$ by the formula

$$[x_1, x_2, \dots, x_n] + [y_1, y_2, \dots, y_n] = [x_1 + y_1, x_2 + y_2, \dots, x_n + y_n]$$

and the product of any element $[x_1, x_2, \dots, x_n]$ from $\mathbb{R}^n$ and any real number λ by the formula

$$\lambda \cdot [x_1, x_2, \dots, x_n] = [\lambda x_1, \lambda x_2, \dots, \lambda x_n].$$

The set $\mathbb{R}^n$ with these two operations is called the _n–dimensional arithmetic space_. Its elements (i.e. n–tuples of real numbers) are called _arithmetic vectors_.

I.1.2. Vectors in $\mathbb{E}_2$ and in $\mathbb{E}_3$. Oriented segments AB and CD in $\mathbb{E}_2$ are called _equivalent_ if they can be identified by parallel shifting. Each class of all oriented segments in $\mathbb{E}_2$ which are equivalent one to another is called a _vector_ in $\mathbb{E}_2$. Any segment from this class is called a _representative_ of the vector. The set of all vectors in $\mathbb{E}_2$ is denoted by $\mathbf{V}(\mathbb{E}_2)$.

Each vector in $\mathbb{E}_2$ is uniquely defined by any of its representatives. Vectors are denoted by small boldface letters (for example $\mathbf{u}$, $\mathbf{v}$, etc.). In a chosen Cartesian coordinate system, every vector can be given by means of its _coordinates_. These are an ordered pair of numbers (in round brackets) which is obtained in such a way that the representative of the vector is chosen to be the oriented segment coming out of the origin, and the coordinates of the end point of this segment are the coordinates of the vector.

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors in $\mathbb{E}_2$ and λ is a real number, then we can define the sum of $\mathbf{u}$ and $\mathbf{v}$ and the product of $\mathbf{u}$ and the number λ by the equalities:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1,\ u_2 + v_2),$$
$$\lambda \cdot \mathbf{u} = (\lambda u_1,\ \lambda u_2).$$

It can easily be verified that the operations defined above have the properties:

(a) $\mathbb{E}_2$ is closed with respect to both operations. That means: if $\mathbf{u}, \mathbf{v} \in \mathbf{V}(\mathbb{E}_2)$ and $\lambda \in \mathbb{R}$ then the sum $\mathbf{u} + \mathbf{v}$ and the product $\lambda \cdot \mathbf{u}$ also belong to $\mathbf{V}(\mathbb{E}_2)$.

(b) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbf{V}(\mathbb{E}_2)$ and α, $\beta \in \mathbb{R}$ then

(b1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$,
(b2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$,
(b3) $1 \cdot \mathbf{u} = \mathbf{u}$,
(b4) $\alpha \cdot (\beta \cdot \mathbf{u}) = (\alpha \cdot \beta) \cdot \mathbf{u}$,
(b5) $\alpha \cdot (\mathbf{u} + \mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{v}$,
(b6) $(\alpha + \beta) \cdot \mathbf{u} = \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{u}$.

(c) There exists the so called _zero vector_ $\mathbf{o} = (0, 0)$ in $\mathbf{V}(\mathbb{E}_2)$. If $\mathbf{u}$ is any vector from $\mathbf{V}(\mathbb{E}_2)$ then

$$\mathbf{u} + \mathbf{o} = \mathbf{u}.$$

(d) To every vector $\mathbf{u} \in \mathbf{V}(\mathbb{E}_2)$ there exists a vector $-\mathbf{u}$ (the so called <u>opposite vector</u> to $\mathbf{u}$) so that

$$\mathbf{u} \; + \; (-\mathbf{u}) \; = \; \mathbf{o}.$$

Due to property (a), $\mathbf{V}(\mathbb{E}_2)$ is said to be <u>closed</u> with respect to the two operations "addition of vectors" and "multiplication of vectors by real numbers".

By analogy, it is possible to define $\mathbf{V}(\mathbb{E}_3)$ (or even $\mathbf{V}(\mathbb{E}_n)$) and operations "addition of vectors" and "multiplication of vectors by real numbers" in this set. These operations in $\mathbf{V}(\mathbb{E}_3)$ (or in $\mathbf{V}(\mathbb{E}_n)$) also have properties (a) – (d).

I.1.3. Vector spaces. It can easily be verified that the operations "addition" and "multiplication by real numbers" in the n–dimensional arithmetic space $\mathbb{R}^n$ have properties (a) – (d), too. Thus, the sets $\mathbb{R}^n$, $\mathbf{V}(\mathbb{E}_2)$, $\mathbf{V}(\mathbb{E}_3)$ and $\mathbf{V}(\mathbb{E}_n)$ are in a certain sense similar. Generally, it can be observed that various nonempty sets with the operations "addition" and "multiplication by real numbers" (which satisfy conditions (a) – (d)) appear very often in mathematics and its applications. All these sets are called <u>*vector spaces*</u>.

Elements of concrete vector spaces need not always be classical vectors as is the case in $\mathbf{V}(\mathbb{E}_2)$ and in $\mathbf{V}(\mathbb{E}_3)$. As examples of further vector spaces, we can mention:

- the set of all polynomials whose degree is less than or equal to n (i.e. functions that have the form $f(x) = a_0+a_1x+\cdots+a_nx^n$, where $a_0, a_1, \ldots, a_n$ are real numbers),
- the set of all functions defined by the equation $f(x) = a_0 + a_1 \cdot \sin x + a_2 \cdot \cos x$, where a_0, a_1, a_2 are real numbers,
- the set of all sequences of real numbers, etc.

Try to suggest for yourself how it is possible to define the operations "addition" and "multiplication by real numbers" in these spaces so that the operations have properties (a) – (d).

Let us return to the vector space $\mathbf{V}(\mathbb{E}_2)$. Two vectors $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ from $\mathbf{V}(\mathbb{E}_2)$ are equal if and only if the corresponding coordinates are equal, i.e. $u_1 = v_1$, $u_2 = v_2$. This simple assertion follows immediately from properties (a) – (d) from paragraph I.1.2. Analogous assertions also hold in $\mathbf{V}(\mathbb{E}_3)$ and $\mathbb{R}^n$.

We are going to study a general vector space $\mathbf{V}$ in the next part of this chapter. If the approach seems to be too abstract for you, you can imagine that $\mathbf{V}$ coincides for instance with $\mathbf{V}(\mathbb{E}_2)$, $\mathbf{V}(\mathbb{E}_3)$ or $\mathbf{V}(\mathbb{E}_n)$. Elements of $\mathbf{V}$ will also be called vectors and they will be denoted in the same way as elements of $\mathbf{V}(\mathbb{E}_2)$ or $\mathbf{V}(\mathbb{E}_3)$, i.e. by small boldface letters. The zero vector will be again denoted by $\mathbf{o}$.

I.1.4. Theorem (uniqueness of the zero vector). *There exists only one zero vector in the vector space* $\mathbf{V}$.

P r o o f : Suppose that there are two different zero vectors $\mathbf{o}$ and $\mathbf{o}'$ in $\mathbf{V}$. Using property (c) from paragraph I.1.2, we get: $\mathbf{o} = \mathbf{o} + \mathbf{o}' = \mathbf{o}' + \mathbf{o} = \mathbf{o}'$. This is in contradiction with the assumption that the vectors $\mathbf{o}$ and $\mathbf{o}'$ are different. Hence two different zero vectors in the vector space $\mathbf{V}$ cannot exist.

I.1.5. Theorem. *The following identities hold for any vector* $\mathbf{u} \in \mathbf{V}$ *and any real number* α*:*

1) $0 \cdot \mathbf{u} = \mathbf{o}$, 2) $(-1) \cdot \mathbf{u} = -\mathbf{u}$, 3) $\alpha \cdot \mathbf{o} = \mathbf{o}$.

(We do not show the proof of this theorem here. Nevertheless, it could easily be done by means of conditions (a) – (d) from I.1.2.)

I.1.6. Linear dependence and independence of vectors. If $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is a group of vectors in the vector space $\mathbf{V}$ and $\alpha_1, \alpha_2, \dots, \alpha_n$ are real numbers then the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_n \mathbf{u}_n$$

is called the *linear combination* of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. (The linear combination of vectors from $\mathbf{V}$ is a vector that also belongs to $\mathbf{V}$. It is an easy consequence of the statements in item (a) in paragraph I.1.2.)

The group of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is called *linearly dependent* if there exist coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ such that at least one of them is different from zero and

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_n \mathbf{u}_n = \mathbf{o}.$$

A group of vectors which is not linearly dependent is called *linearly independent*.

I.1.7. Theorem. *If one of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ in the vector space $\mathbf{V}$ is equal to the zero vector then the group $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is linearly dependent.*

P r o o f : Let for instance $\mathbf{u}_1 = \mathbf{o}$. Then $1 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + \dots + 0 \cdot \mathbf{u}_n = \mathbf{o}$. So we have the linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ which is equal to the zero vector $\mathbf{o}$ and not all coefficients in this linear combination are zeros. Thus, in accordance with definition I.1.6, the group $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is linearly dependent.

I.1.8. Theorem. *The group of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ (where $n > 1$) from the vector space $\mathbf{V}$ is linearly dependent if and only if at least one vector from this group can be expressed as a linear combination of the other vectors of the group.*

P r o o f : a) Suppose that the group $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is linearly dependent. Then there exist coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ (such that at least one of them is different from zero – let it be for instance α_1) and $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_n\mathbf{u}_n = \mathbf{o}$. Since $\alpha_1 \neq 0$, we can divide the equality by α_1 and express $\mathbf{u}_1$:

$$\mathbf{u}_1 = -\frac{\alpha_2}{\alpha_1}\mathbf{u}_2 - \frac{\alpha_3}{\alpha_1}\mathbf{u}_3 - \dots - \frac{\alpha_n}{\alpha_1}\mathbf{u}_n .$$

So the vector $\mathbf{u}_1$ is a linear combination of the other vectors of the group. Similarly, if $\alpha_2 \neq 0$ then it is possible to express $\mathbf{u}_2$ in the form of a linear combination of the other vectors of the group, etc.

b) Suppose now that for example the vector $\mathbf{u}_1$ is a linear combination of the other vectors, i.e. there exist coefficients $\beta_2, \dots, \beta_n$ such that $\mathbf{u}_1 = \beta_2\mathbf{u}_2 + \dots + \beta_n\mathbf{u}_n$. If we put $\alpha_1 = -1$, $\alpha_2 = \beta_2, \dots, \alpha_n = \beta_n$ then we can see that at least one of these numbers is nonzero and $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_n\mathbf{u}_n = \mathbf{o}$. Hence the group $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is linearly dependent.

I.1.9. Theorem. *The group of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ from the vector space $\mathbf{V}$ is linearly independent if and only if the vector equation*

(I.1.1) $$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_n\mathbf{u}_n = \mathbf{o}$$

(for unknowns $\alpha_1, \alpha_2, \dots, \alpha_n$) has only the zero solution $\alpha_1 = 0, \dots, \alpha_n = 0$.
(This theorem is an immediate consequence of the definition of linear dependence and linear independence of vectors from paragraph I.1.6.)

I.1.10. Example. The group of vectors (1,1,0), (0,2,3) a (3,5,3) in $\mathbb{E}_3$ is linearly dependent. Equation (I.1.1), which in our concrete case has the form

$$\alpha_1 \cdot (1,1,0) + \alpha_2 \cdot (0,2,3) + \alpha_3 \cdot 5,3,5) = (0,0,0),$$

has for instance this nonzero solution: $\alpha_1 = 3$, $\alpha_2 = 1$, $\alpha_3 = -1$. Linear dependence of the given group of vectors follows now from theorem I.1.9.

I.1.11. Theorem. *Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be a linearly dependent group of vectors from the vector space $\mathbf{V}$. Then every group of vectors from $\mathbf{V}$ which contains the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$, is also linearly dependent.*

P r o o f : The group $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is linearly dependent, hence there exist coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ such that at least one of them is different from zero and $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_n\mathbf{u}_n = \mathbf{o}$. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ be a group of vectors which contains the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. Suppose that the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are ordered so that $\mathbf{v}_1 = \mathbf{u}_1$, $\mathbf{v}_2 = \mathbf{u}_2, \dots, \mathbf{v}_n = \mathbf{u}_n$. Then $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_n\mathbf{v}_n + 0 \cdot \mathbf{v}_{n+1} + \dots + 0 \cdot \mathbf{v}_m = \mathbf{o}$. The coefficients $\alpha_1, \alpha_2, \dots, \alpha_n, 0, \dots, 0$ are surely not all equal to zero. So the group $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ is linearly dependent.

I.1.12. Dimension of a vector space. Let n be a natural number. We say that the vector space $\mathbf{V}$ is <u>n–dimensional</u> (or equivalently: its <u>*dimension*</u> is equal to n – we write $dim\,\mathbf{V} = n$) if

a) there exists a group of n vectors in $\mathbf{V}$ that is linearly independent,
b) each group of more than n vectors from $\mathbf{V}$ is linearly dependent.

I.1.13. Basis of a vector space. Let $\mathbf{V}$ be an n–dimensional vector space. Each linearly independent group of n vectors from $\mathbf{V}$ is called a <u>*basis*</u> of the space $\mathbf{V}$.

I.1.14. Remark. The dimension of vector space $\mathbf{V}$ equals the maximum number of linearly independent vectors that can be found in $\mathbf{V}$. It also equals the number of vectors in an arbitrary basis of $\mathbf{V}$.

I.1.15. Theorem. *Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be a basis of the vector space $\mathbf{V}$. Then every vector from $\mathbf{V}$ can be uniquely expressed as a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.*

P r o o f : a) <u>Existence of the expression:</u> Let $\mathbf{v}$ be an arbitrarily chosen vector from $\mathbf{V}$. The group of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}$ is linearly dependent (because it is the group of more than n vectors). Hence there exist coefficients $\alpha_1, \alpha_2, \dots, \alpha_n, \beta$ such that not all of them are equal to zero and

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \ldots + \alpha_n\mathbf{u}_n + \beta\mathbf{v} = \mathbf{o}.$$

If β were equal to zero then the above equality would imply the linear dependence of the vectors $\mathbf{u}_1, \mathbf{u}_2 \dots, \mathbf{u}_n$, which would be in contradiction with the assumptions of the theorem. So β is different from zero and it is possible to express $\mathbf{v}$:

$$\mathbf{v} \,=\, -\frac{\alpha_1}{\beta}\,\mathbf{u}_1 \,-\, \frac{\alpha_2}{\beta}\,\mathbf{u}_2 \,-\, \ldots \,-\, \frac{\alpha_n}{\beta}\,\mathbf{u}_n\,.$$

b) Uniqueness of the expression: Suppose that $\mathbf{v}$ can be also written in another way as the linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2, \ldots, \mathbf{u}_n$: $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$. If we subtract the first expression of $\mathbf{v}$ from the second one, we obtain:

$$\mathbf{o} \,=\, \big(c_1 + \frac{\alpha_1}{\beta}\big)\,\mathbf{u}_1 \,+\, \big(c_2 + \frac{\alpha_2}{\beta}\big)\,\mathbf{u}_2 \,+\, \ldots \,+\, \big(c_n + \frac{\alpha_n}{\beta}\big)\,\mathbf{u}_n\,.$$

The linear independence of the vectors $\mathbf{u}_1$, $\mathbf{u}_2, \ldots, \mathbf{u}_n$ implies: $c_1 + \alpha_1/\beta = 0$, $c_2 + \alpha_2/\beta = 0$, $\ldots$, $c_n + \alpha_n/\beta = 0$. So we get equalities $c_1 = -\alpha_1/\beta$, $c_2 = -\alpha_2/\beta$, $\ldots$, $c_n = -\alpha_n/\beta$, which show that both expressions of the vector $\mathbf{v}$ are same.

I.1.16. Remark. Theorem I.1.15. can be "reversed". We mean by this that one can also prove correctness of the inverse implication: If $\mathbf{V}$ is a vector space and $\mathbf{u}_1$, $\mathbf{u}_2, \ldots, \mathbf{u}_n$ is a group of vectors with the property that each vector from $\mathbf{V}$ can be uniquely expressed as a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2, \ldots, \mathbf{u}_n$ then these vectors form a basis of the space $\mathbf{V}$.

I.1.17. Example. The vectors $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$ form the basis of the vector space $\mathbf{V}(\mathbb{E}_3)$ because each vector $\mathbf{a} = (a_1, a_2, a_3) \in \mathbf{V}(\mathbb{E}_3)$ can be uniquely written as a linear combination of the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$: $\mathbf{a} = a_1 \cdot (1,0,0) + a_2 \cdot (0,1,0) + a_3 \cdot (0,0,1)$. Taking also into account remark I.1.14, we come to the by no means surprising assertion that the space $\mathbf{V}(\mathbb{E}_3)$ is three–dimensional.

By analogy, the arithmetic vectors $\mathbf{e}_1 = [1,0,...,0]$, $\mathbf{e}_2 = [0,1,...,0]$, $\ldots$, $\mathbf{e}_n = [0,0,...,1]$ form the basis of the space $\mathbb{R}^n$. (So this space was correctly called n–dimensional in paragraph I.1.1).

I.1.18. Remark. The basis of a vector space is not unique! For example – you can easily verify that two groups of vectors $\mathbf{i} = (1,0)$, $\mathbf{j} = (0,1)$ and $\mathbf{u} = (2,-1)$, $\mathbf{v} = (1,1)$ are both bases of the vector space $\mathbf{V}(\mathbb{E}_2)$. Moreover, every vector space (with the exception of the so called trivial vector space, which contains only one element – the zero vector) has infinitely many various bases.

If $\mathbf{V}$ is an n–dimensional vector space and $\mathbf{u}_1$, $\mathbf{u}_2, \ldots, \mathbf{u}_j$ (where $j < n$) is a linearly independent group of vectors in $\mathbf{V}$ then this group can always be filled up to the basis of $\mathbf{V}$ by adding appropriate vectors from $\mathbf{V}$.

I.1.19. Subspace of a vector space. Suppose that $\mathbf{W}$ is a subset of vector space $\mathbf{V}$. If $\mathbf{W}$ is the vector space (with the same operations "addition" and "multiplication by real numbers" as in $\mathbf{V}$) then we call $\mathbf{W}$ the *subspace* of the vector space $\mathbf{V}$.

I.1.20. How to recognize a subspace. Let $\mathbf{W}$ be a subset of a vector space $\mathbf{V}$. We wish to find out whether $\mathbf{W}$ is a subspace of $\mathbf{V}$. The operations "addition" and "multiplication by real numbers" are defined in $\mathbf{W}$ because these operations are defined in $\mathbf{V}$ and $\mathbf{W} \subset \mathbf{V}$. Thus, $\mathbf{W}$ is an individual vector space itself (and consequently, it is the subspace of $\mathbf{V}$) if it is closed with respect to these operations. This means if the sum of two arbitrary vectors from $\mathbf{W}$ remains in $\mathbf{W}$ and the λ–multiple of an arbitrary vector from $\mathbf{W}$ also remains in $\mathbf{W}$ (for any $\lambda \in \mathbb{R}$).

I.1.21. Example. The set of all arithmetic vectors, that can be written in the form

$[\alpha, \beta, 0]$ (where α, β are real numbers), is a subspace of $\mathbb{R}^3$.

The set of all arithmetic vectors, that have the form $[\alpha, \beta, \gamma]$ (where α, β, γ are real numbers and $\alpha > 0$), is not a subspace of $\mathbb{R}^3$.

I.1.22.* Remark. Try to prove these simple assertions:

a) *If* $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ *is a family of vectors from vector space* $\mathbf{V}$ *then the set of all linear combinations of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ *is a subspace of* $\mathbf{V}$. *This subspace is called the* <u>*linear hull*</u> *of the vectors* $\mathbf{u}_1, \dots, \mathbf{u}_k$.

b) *If the family of vectors* $\mathbf{u}_1, \dots, \mathbf{u}_k$ *is linearly independent then it forms a basis of its linear hull and the dimension of the hull is thus equal to* k.

I.2. Matrices and determinants

I.2.1. Matrices. A rectangular array of $m \cdot n$ real numbers written in m rows and n columns is called a <u>*matrix*</u> of the type $m \times n$ (read m by n) or shortly an $m \times n$ matrix. The numbers which are contained in the matrix are called its <u>*entries*</u> or its <u>*elements*</u>. Matrices are usually denoted by capital letters and their entries are denoted by the same small letters with two indices. The indices are related to the position of the entry. For example, a_{ij} denotes the entry in the i–th row and j–th column in matrix A.

I.2.2. Example.

$$A = \begin{pmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots, & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}, & a_{m2}, & \dots, & a_{mn} \end{pmatrix} \qquad B = \begin{pmatrix} 0, & 2, & 3, & 4, & 8, & -5 \\ 2, & 3, & 5, & -1, & 9, & 17 \\ 3, & -8, & 7, & 6, & -4, & 23 \end{pmatrix}$$

A is the $m \times n$ matrix, B is the 3×6 matrix. If the type of matrix A is known, then A can be written down in a shorter way: $A = (a_{ij})$.

I.2.3. Identity of two matrices. Two matrices are identical if they are of the same type and if they have the same entries at corresponding positions.

I.2.4. Main diagonal, upper triangular matrix, zero matrix, transposed matrix. Suppose that $A = (a_{ij})$ is an $m \times n$ matrix.

The entries $a_{11}, a_{22}, \dots$ form the so called <u>*main diagonal*</u> in matrix A.

If all entries under the main diagonal are equal to zero, then matrix A is called the <u>*upper triangular matrix*</u>.

A matrix whose all entries are equal to zero is called a <u>*zero matrix*</u>.

The $n \times m$ matrix $B = (b_{ij})$ whose entries satisfy $b_{ij} = a_{ji}$ ($i = 1, \dots, n$; $j = 1, \dots, m$) is called a <u>*transposed matrix*</u> to matrix A. It is denoted by A^T. (The 1st column of matrix A^T is identical with the 1st row of matrix A, the 2nd column of A^T is equal to the 2nd row of A, etc. In other words: the transposed matrix to matrix A can be obtained by turning A over the main diagonal.)

I.2.5. Square matrix, identity matrix. A matrix with the same number of rows as columns is said to be a <u>*square matrix*</u>.

A square matrix all of whose entries on the main diagonal equal 1 and all of whose other entries are zeros is called the *identity matrix.* It is denoted by E.

I.2.6. Addition of matrices. If the matrices $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ then their *sum* is the $m \times n$ matrix $C = (c_{ij})$ with the entries $c_{ij} = a_{ij} + b_{ij}$ $(i = 1, \dots, m;\ j = 1, \dots, n)$. We use the notation $C = A + B$.

I.2.7. Multiplication of matrices by real numbers. If $A = (a_{ij})$ is an $m \times n$ matrix and $\lambda \in \mathbb{R}$, then the *product of the number λ and matrix A* (or in other words the *λ–multiple of matrix A*) is the matrix $C = (c_{ij})$ of the same type $m \times n$ with the entries $c_{ij} = \lambda \cdot a_{ij}$ $(i = 1, \dots, n;\ j = 1, \dots, m)$. We use the notation $C = \lambda \cdot A$ or $C = \lambda A$.

I.2.8. Example.

$$\begin{pmatrix} 1, & -3, & 2 \\ 4, & -1, & 3 \end{pmatrix} + \begin{pmatrix} 1, & 2, & 2 \\ -1, & 3, & -8 \end{pmatrix} = \begin{pmatrix} 2, & -1, & 4 \\ 3, & 2, & -5 \end{pmatrix}, \qquad 2 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

I.2.9. Remark. Matrices of the same type can also be subtracted: *The difference of matrices A and B* is the matrix $C = A + (-1) \cdot B$. We write: $C = A - B$.

I.2.10. Multiplication of matrices. If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix then the *product* of the matrices A and B is the $m \times p$ matrix $C = (c_{ij})$ whose entries satisfy: $c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{in} \cdot b_{mj}$ $(i = 1, \dots, m;\ j = 1, \dots, p)$. We write: $C = A \cdot B$.

I.2.11. Remark. The definition of the multiplication of matrices seems to be artificial at first sight and so we shall analyze it once more. It can be explained by means of the following new notion:

If $[\, u_1, u_2, \dots, u_n \,]$ and $[\, v_1, v_2, \dots, v_n \,]$ are arithmetic vectors from $\mathbb{R}^n$ then the number $u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$ is called their *scalar product*.

The rows of matrix A can be identified with arithmetic vectors from $\mathbb{R}^n$ (the number of rows is m, and each of them has n entries). Similarly, we can regard the columns of matrix B as arithmetic vectors also from $\mathbb{R}^n$ (their number is p, and each of them has n entries). If you read definition I.2.10 carefully, you can observe that the entry c_{ij} in matrix C is the scalar product of the i–th row of matrix A with the j–th column of matrix B.

Matrices A and B can be multiplied (in this order) only if matrix A has the same number of columns as the number of rows of matrix B. We can easily recognize whether this is fulfilled: we write down the types of matrices A and B (for example $m \times n$, $n \times p$) and the 2nd and 3rd number must be the same. Otherwise matrices A and B cannot be multiplied (in this order).

I.2.12. Example. Verify for yourself by calculation that it holds:

$$\begin{pmatrix} 3, & 2, & 5 \\ 2, & -4, & 6 \\ 1, & 0, & 0 \end{pmatrix} \cdot \begin{pmatrix} 1, & 5 \\ 3, & -2 \\ -2, & 0 \end{pmatrix} = \begin{pmatrix} -1, & 11 \\ -22, & 18 \\ 1, & 5 \end{pmatrix}, \qquad \begin{pmatrix} 3, & 5 \\ 6, & -2 \\ -1, & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 19 \\ 14 \\ -3 \end{pmatrix}.$$

I.2.13. Rules for operations with matrices. Suppose that A, B and C are matrices and α, β are real numbers. Then each of the equalities

a) $A + B = B + A$,	b) $(A + B) + C = A + (B + C)$,
c) $\alpha \cdot (A + B) = \alpha \cdot A + \alpha \cdot B$,	d) $(\alpha + \beta) \cdot A = \alpha \cdot A + \beta \cdot A$,
e) $A \cdot (B + C) = A \cdot B + A \cdot C$,	f) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$,
g) $A \cdot E = A$,	h) $E \cdot B = B$,
i) $(A + B)^T = A^T + B^T$,	j) $(A \cdot B)^T = B^T \cdot A^T$

holds if the types of matrices are such that the operations on the left hand sides have a sense. Try to prove equalities a) – j) for yourself under this assumption.

Multiplication of matrices is not commutative, i.e. it does not generally hold that $A \cdot B = B \cdot A$! If, for example, A is a 3×5 matrix and B is a 5×7 matrix then the product $A \cdot B$ is a 3×7 matrix, while the product $B \cdot A$ has no sense, it cannot be created. Moreover, even if both products $A \cdot B$ and $B \cdot A$ have a sense, there exist examples when $A \cdot B \neq B \cdot A$.

I.2.14. The rank of a matrix. The maximum number of linearly independent rows of matrix A (taken as arithmetic vectors) is called the *rank* of matrix A. We denote it $r(A)$.

I.2.15. Example. Let

$$A = \begin{pmatrix} 2, & 1, & 5 \\ 1, & 3, & 7 \\ 4, & -3, & 1 \end{pmatrix}.$$

For instance – it can be verified by means of theorem I.1.9 that the first two rows of the matrix are linearly independent. The third row is the linear combination of the first two rows (it is equal to the difference of the 1st row (multiplied by 3) and the 2nd row (multiplied by 2)). Hence all three rows form the linearly dependent group. The maximum number of linearly independent rows is two, and for this reason $r(A) = 2$.

I.2.16. Remark. It is natural to put the question whether it is possible to define the rank of a matrix by means of its columns instead of its rows (i.e. as the maximum number of linearly independent columns, if the columns are identified with arithmetic vectors). The answer is simple: YES. The "row definition" and the "column definition" assign to the matrix the same number as its rank. However, we should mention that the exact proof of this assertion is not quite simple.

If we deal with more complicated matrices than matrix A from example I.2.15, we are not able to recognize at first sight which rows form a linearly independent group and conversely, which ones are linear combinations of other rows. That is why we shall study the problem of how to specify the rank of a matrix in greater detail in several following paragraphs.

I.2.17. Theorem. *Let A be an $m \times n$ upper triangular matrix and let all entries on the main diagonal be different from zero. Then the rank of A is equal to the minimum of the numbers m, n.*

I.2.18. Example. Instead of showing a general proof of theorem I.2.17, let us study this special case: Suppose that

$$A = \begin{pmatrix} 3, & 3, & 5, & 0, & -3, & 8 \\ 0, & 1, & 7, & -6, & 4, & 3 \\ 0, & 0, & 5, & 14, & 4, & 2 \end{pmatrix}.$$

Let us verify that the rows of matrix A are linearly independent. We can write the vector equation

$$\alpha \cdot (3,3,5,0,-3,8) + \beta \cdot (0,1,7,-6,4,3) + \gamma \cdot (0,0,5,14,4,2) = (0,0,0,0,0,0).$$

If we write down the corresponding equations for all coordinates, we obtain a system of six linear algebraic equations for three unknowns: α, β and γ. We can easily find that there exists only one solution: $\alpha = \beta = \gamma = 0$. The linear independence of the rows of matrix A now follows from theorem I.1.9. Their number is three, hence $r(A) = 3$. By analogy, the matrices

$$\begin{pmatrix} 1, & 3, & -2 \\ 0, & 4, & 7 \\ 0, & 0, & 5 \\ 0, & 0, & 0 \end{pmatrix}, \quad \begin{pmatrix} -2, & 3, & 0, & 5 \\ 0, & -1, & 5, & 0 \\ 0, & 0, & 15, & 1 \\ 0, & 0, & 0, & 7 \end{pmatrix}, \quad \begin{pmatrix} 2, & 1 \\ 0, & 4 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1, & 2, & -5, & 8 \end{pmatrix}$$

have the ranks 3, 4, 2, 1, 1.

I.2.19. Elementary row and column operations. If we have to find the rank of a general matrix A which is not triangular, then we can transform the matrix to an upper triangular matrix (with non–zero entries on the main diagonal) using so called elementary row and column operations, which do not change the rank of the matrix, and afterwards we specify the rank by means of theorem I.2.17. We shall use the following elementary row operations:

a) *change of order of rows*,
b) *multiplication of some row by a nonzero number*,
c) *addition to some row of a linear combination of the other rows (specially, addition of a multiple of another row)*,
d) *omission of a row which is a linear combination of the other rows (specially, omission of a row all of whose entries are zeros or omission of a row which is a multiple of another row).*

(All the operations can also be performed with columns. We are not going to prove that these row and column operations do not change the rank of a matrix.)

The procedure of transformation of an arbitrary matrix to an upper triangular matrix (all of whose entries on the main diagonal are different from zero) by means of the elementary row and column operations is called the *Gauss algorithm*. The algorithm is explained in the next example:

I.2.20. Example. We find the rank of the matrix

$$A = \begin{pmatrix} 2, & -1, & 1, & 8, & 2 \\ 4, & -3, & 5, & 1, & 7 \\ 8, & -6, & 8, & 12, & 12 \\ 6, & -4, & 6, & 9, & 9 \end{pmatrix}.$$

Step 1. Since $a_{11} \neq 0$, we rewrite the 1st row. (If $a_{11} = 0$ then we interchange the rows or columns so that there is a nonzero entry on the position 1,1.) Now we want to get only zeros under the entry a_{11}. That is why we first multiply the 1st row by 2 and subtract it from the 2nd row, then we multiply the 1st row by 4 and subtract it from the 3rd row and finally, we multiply the 1st row by 3 and subtract it from the 4th row. We obtain the matrix:

$$\begin{pmatrix} 2, & -1, & 1, & 8, & 2 \\ 0, & -1, & 3, & -15, & 3 \\ 0, & -2, & 4, & -20, & 4 \\ 0, & -1, & 3, & -15, & 3 \end{pmatrix}.$$

Step 2. We rewrite the 1st row. Since the entry at position 2,2 is nonzero, we also rewrite the 2nd row. Now we want to get only zeros under the entry at position 2,2. Thus, we multiply the 2nd row by 2 and subtract from the 3rd row. Finally, we subtract the 2nd row from the 4th row. We get the matrix:

$$\begin{pmatrix} 2, & -1, & 1, & 8, & 2 \\ 0, & -1, & 3, & -15, & 3 \\ 0, & 0, & 2, & -36, & 0 \\ 0, & 0, & 0, & 0, & 0 \end{pmatrix}.$$

Step 3. The last row contains only zeros, hence we omit it. We obtain the upper triangular matrix:

$$\begin{pmatrix} 2, & -1, & 1, & 8, & 2 \\ 0, & -1, & 3, & -15, & 3 \\ 0, & 0, & 2, & -36, & 0 \end{pmatrix}.$$

Due to theorem I.2.17, the last matrix has the rank 3. Thus, $r(A) = 3$.

I.2.21. Determinant. Let A be a square matrix. The *determinant* of matrix A is the number which is denoted by $\det A$ and which is assigned to matrix A in accordance with these rules:

a) If $A = (a)$ is a 1×1 square matrix then $\det A = a$.
b) If $A = (a_{ij})$ is an $n \times n$ square matrix (for $n > 1$) then we choose an arbitrary row of A (let us denote this row as the i–th one) and we put

$$\det A = a_{i1} \cdot A_{i1} + a_{i2} \cdot A_{i2} + \ldots + a_{in} \cdot A_{in}, \tag{I.2.1}$$

where A_{ij} is the so called *co–factor of the entry a_{ij}*. The co–factor is equal to $(-1)^{i+j} \cdot A^*_{ij}$ where A^*_{ij} is the determinant of the $(n-1) \times (n-1)$ square matrix which arises from A by omission the i–th row and the j–th column. (A^*_{ij} is called the *minor*, which is the abbreviation for "minor determinant".)

I.2.22. Remark. The sum on the right side of (I.2.1) is called the *expansion of the determinant along the i–th row*. It can be proved that the choice of the row along which the determinant is expanded is not important because the result is always the same. The determinant can even be expanded along an arbitrary column. The *expansion of the determinant along the j–th* column is:

$$\det A \;=\; a_{1j}\cdot A_{1j} \;+\; a_{2j}\cdot A_{2j} \;+\; \ldots \;+\; a_{nj}\cdot A_{nj}.$$

It can be easily verified that the determinant of a 2×2 square matrix A is:

$$\det A \;=\; a_{11}\,a_{22} \;-\; a_{12}\,a_{21}.$$

Remember this simple formula!

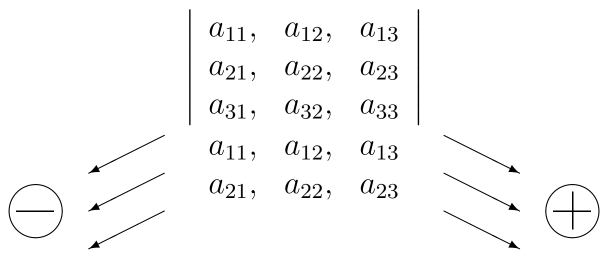
The expansion of the $n\times n$ determinant along some row or column leads to the expression of this determinant by means of n $(n-1)\times(n-1)$ determinants. Each of these determinants can be further expanded along some of its rows or columns and so the problem is transformed to a question of calculation of $(n-2)\times(n-2)$ determinants. We can proceed in this way until we come to 2×2 determinants (or even 1×1 determinants) which we already know how to compute.

I.2.23. Remark. The determinant of matrix A is often written down analogously as matrix A, only instead of round brackets we use straight vertical lines.

I.2.24. Saruss' rule. The determinant of a 3×3 matrix can also be, apart from the expansion along some row or column, computed by the so called "Saruss' rule":

$$\begin{aligned}\det A \;=\;& a_{11}\cdot a_{22}\cdot a_{33} \;+\; a_{12}\cdot a_{23}\cdot a_{31} \;+\; a_{13}\cdot a_{21}\cdot a_{32}\\ &-\, a_{13}\cdot a_{22}\cdot a_{31} \;-\; a_{11}\cdot a_{23}\cdot a_{32} \;-\; a_{12}\cdot a_{21}\cdot a_{33}.\end{aligned}$$

The correctness of this formula can be verified by comparison with the result that can be obtained by expansion along some row or column. You can easily remember the formula by means of the following scheme:



I.2.25. Problems. Verify that

a) $\begin{vmatrix}2, & 5\\ 3, & 7\end{vmatrix} = -1,$ b) $\begin{vmatrix}4, & 8, & 3\\ 5, & -1, & 0\\ 3, & 2, & -4\end{vmatrix} = 215,$ c) $\begin{vmatrix}4, & 2, & 5, & 0\\ 2, & -1, & 0, & 2\\ 3, & 6, & -8, & 2\\ 7, & 1, & 0, & 1\end{vmatrix} = -501.$

To compute the last determinant, it is advantageous to use the expansion along the 3rd column. (Why?)

I.2.26. Geometrical meaning of the determinant. a) Suppose at first that A is a 2×2 square matrix. We can consider its rows to be the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Let us fill in these two vectors to the parallelogram in plane $\mathbb{E}_2$. You can verify by an easy calculation that the area of the parallelogram is equal to $|\det A|$.

b) Assume now that A is a 3×3 square matrix whose rows are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. Let us fill in these vectors to the parallelogram in space $\mathbb{E}_3$. Then the volume of the parallelogram is equal to $|\det A|$.

c) Generally, let A be an $n \times n$ square matrix whose rows are the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Let us fill in these vectors to the n–dimensional parallelogram in $\mathbb{E}_n$. Then the n–dimensional volume of the parallelogram is $|\det A|$.

Assertions a), b) and c) remain valid if we work with the columns of matrix A instead of its rows.

I.2.27. Important properties of determinants. Knowledge of the following assertions is very useful for computing determinants. Assume that A is an $n \times n$ square matrix (where $n > 1$).

a) If all entries in some row (or column) of A are zero then $\det A = 0$. (This is seen if we expand the determinant along the zero row or column.)

b) $\det A = \det A^T$

c) Interchanging two rows (or columns) changes the sign of the determinant.

d) If two rows (or columns) are identical, the determinant is zero. (This is the consequence of item c): Interchanging two same columns changes the sign of the determinant. However, the new matrix is identical with A, so its determinant is equal to $\det A$. The equality $-\det A = \det A$ implies that $\det A = 0$.)

e) If we multiply some row (or column) of matrix A by a number λ then the determinant of the new matrix is equal to $\lambda \cdot \det A$. (This can be easily proved by expanding the determinant along the multiplied row or column.)

f) If any row (respectively column) of A is a multiple of another row (respectively column) of A, the determinant of A is zero. (This is an easy consequence of items d) and e).)

g) If any row (respectively column) of A is a linear combination of the other rows (respectively columns) of A, the determinant is zero. (We can prove this if we expand the determinant along the row (respectively column) that is the linear combination of the other rows (respectively columns) and apply the assertions from the items d) and e).)

h) If A and B are $n \times n$ square matrices then $\det(A \cdot B) = \det A \cdot \det B$.

I.2.28. Remark. The determinant of the $n \times n$ identity matrix (for arbitrary $n \in \mathbb{N}$) is equal to 1.

More generally: *The determinant of a square upper triangular matrix is equal to the product of all entries on the main diagonal.* Try to verify for yourself that this simple assertion is true.

(The expression "square upper triangular matrix" sounds rather strange, however when you read carefully the definition of a square and of an upper triangular matrix, you can see that the adjectives "square" and "upper triangular" are not in contradiction.)

The determinant of an $n \times n$ matrix was defined in paragraph I.2.21 by means of the expansion along some row (or column). Determinants of "smaller" matrices can really be calculated by means of these expansions. However, the calculation of determinants of "larger" matrices by means of the expansions along rows or columns would require an extremely large number of operations. (For instance, in the case of a 100×100 matrix, no modern computer would be able to do it in the epoch of existence of our universe.) Such determinants can be calculated by other (so called numerical) methods. For example: Applying the operations from item h) of paragraph I.2.27, the determinant can be transformed to the determinant of an upper triangular matrix. Then the assertion from the first part of this paragraph can be applied.

I.2.29. Regular and singular matrices. An $n \times n$ square matrix which has the maximum possible rank (i.e. n) is called a *regular matrix*.

A square matrix which is not regular is called *singular*.

I.2.30. Inverse matrix. Suppose that A is an $n \times n$ square matrix and E is the $n \times n$ identity matrix. An $n \times n$ matrix A^{-1} is called the *inverse matrix* to matrix A if

$$A \cdot A^{-1} = E.$$

You will see in the following paragraphs that the inverse matrix A^{-1} need not exist to each square matrix A!

I.2.31. Theorem. *Let A be a square matrix. Then the following statements are equivalent:*

a) *A is regular.*

b) $\det A \neq 0$.

c) *The inverse matrix A^{-1} exists.*

(We omit the proof of this theorem. The theorem states, among other things, when the inverse matrix does exist.)

I.2.32. Theorem. *If A and B are $n \times n$ regular matrices then matrix $A \cdot B$ is also regular. Moreover, it holds:* $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

P r o o f : Matrices A and B are regular, hence their determinants are different from zero. (See theorem I.2.31.) Assertion I.2.27 h) implies: $\det(A \cdot B) = \det A \cdot \det B$, which is different from zero. Using theorem I.2.31, we can see that matrix $A \cdot B$ is regular. The formula for $(A \cdot B)^{-1}$ follows from the equalities $(A \cdot B) \cdot (B^{-1} \cdot A^{-1}) = A \cdot (B \cdot B^{-1}) \cdot A^{-1} = A \cdot E \cdot A^{-1} = A \cdot A^{-1} = E$.

I.2.33. Theorem. *If matrix A is regular then matrix A^{-1} is also regular. Moreover, it holds:*

$$\text{a)} \quad (A^{-1})^{-1} = A, \qquad\qquad \text{b)} \quad A \cdot A^{-1} = A^{-1} \cdot A = E.$$

P r o o f : a) If matrix A^{-1} were singular, it would be $1 = \det E = \det(A \cdot A^{-1}) = \det A \cdot \det A^{-1} = \det A \cdot 0 = 0$, which is impossible. Hence A^{-1} is regular. Further, one has: $(A^{-1})^{-1} = (A \cdot A^{-1}) \cdot (A^{-1})^{-1} = A \cdot [\, A^{-1} \cdot (A^{-1})^{-1} \,] = A \cdot E = A$.
b) The formula $A \cdot A^{-1} = E$ is already known. It remains to show that it also holds $A^{-1} \cdot A = E$. Denote $B = A^{-1}$. Obviously, one has: $A^{-1} \cdot A = B \cdot (A^{-1})^{-1} = B \cdot B^{-1} = E$.

I.2.34. Theorem (uniqueness of the inverse matrix). *If a square matrix A has an inverse matrix then the inverse matrix is unique.*

P r o o f : Suppose that both A_I^{-1} and A_{II}^{-1} are inverse matrices to matrix A. Then we have:

a) $A_I^{-1} \cdot A \cdot A_{II}^{-1} = (A_I^{-1} \cdot A) \cdot A_{II}^{-1} = E \cdot A_{II}^{-1} = A_{II}^{-1}$,

b) $A_I^{-1} \cdot A \cdot A_{II}^{-1} = A_I^{-1} \cdot (A \cdot A_{II}^{-1}) = A_I^{-1} \cdot E = A_I^{-1}$.

This means that $A_I^{-1} = A_{II}^{-1}$. Thus, the inverse matrix (if it exists) is unique.

I.2.35. Remark. A practical question is how to compute the inverse matrix A^{-1} to a given regular square matrix A. For instance, it can be proved that

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11}, & \dots, & A_{1n} \\ \vdots & & \vdots \\ A_{n1}, & \dots, & A_{nn} \end{pmatrix}^T$$

where A_{ij} are co–factors of the entries a_{ij} in matrix A (see paragraph I.2.21, item c). However, the practical application of this formula for larger n is not advantageous, due to the necessity to compute the co–factors A_{ij}, which can be very laborious. There exists a less laborious procedure, based on the same idea as the Gauss algorithm described in I.2.20. It is explained in example 159, pp. 9–10, in the textbook [NK].

I.3. Systems of linear algebraic equations

I.3.1. Basic notions. The system of equations

$$\text{(I.3.1)} \qquad \begin{array}{ccccccccc} a_{11} \cdot x_1 & + & a_{12} \cdot x_2 & + & \dots & + & a_{1n} \cdot x_n & = & b_1 \\ a_{21} \cdot x_1 & + & a_{22} \cdot x_2 & + & \dots & + & a_{2n} \cdot x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1} \cdot x_1 & + & a_{m2} \cdot x_2 & + & \dots & + & a_{mn} \cdot x_n & = & b_m \end{array}$$

(where a_{11}, a_{12}, …, a_{mn}, b_1, b_2, …, b_m are given real numbers and x_1, x_2, …, x_n are unknowns) is called the <u>*system of linear algebraic equations*</u>. The matrices

$$A = \begin{pmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots, & a_{2n} \\ & & \vdots & \\ a_{m1}, & a_{m2}, & \dots, & a_{mn} \end{pmatrix}, \qquad (A \,|\, B) = \left(\begin{array}{cccc|c} a_{11}, & a_{12}, & \dots, & a_{1n} & b_1 \\ a_{21}, & a_{22}, & \dots, & a_{2n} & b_2 \\ & & \vdots & & \vdots \\ a_{m1}, & a_{m2}, & \dots, & a_{mn} & b_m \end{array}\right)$$

are called the <u>*matrix of the system*</u> (I.3.1) and the <u>*augmented matrix of the system*</u> (I.3.1). If we further denote

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

we can write the system (I.3.1) in the much shorter way:

$$\text{(I.3.1)} \qquad A \cdot X = B.$$

A solution of the system (I.3.1) is every ordered n–tuple of real numbers x_1, x_2, … , x_n which satisfies the system. Solutions of the system can also be regarded as arithmetic vectors, i.e. as elements of $\mathbb{R}^n$. We believe that no misunderstanding can arise if the solution taken as an arithmetic vector is denoted in the same way as the solution taken as an $n \times 1$ matrix (i.e. for example by X in both cases).

If all the numbers $b_1, b_2, \ldots, b_m$ are equal to zero, then the system (I.3.1) is called *homogeneous*. In the opposite case, system (I.3.1) is called *non–homogeneous*. The homogeneous system can be shortly written as the matrix equation

$$A \cdot X = O. \tag{I.3.2}$$

where O is the $m \times 1$ zero matrix.

A non–homogeneous system need not always have a solution. (Example: $x_1 + x_2 = 1, \quad x_1 + x_2 = 2$) On the other hand, an homogeneous system always has at least one (zero) solution $x_1 = \cdots = x_n = 0$. (This solution is called the *trivial solution.*) Of course, in some cases it also has other, non–zero (= non–trivial) solutions.

Two systems of equations are called *equivalent* if they have identical sets of solutions.

Our next task is to learn to find all solutions of the system (I.3.1) and to study the structure of the set of all solutions of the system (I.3.1) (or (I.3.2)).

I.3.2. Gaussian elimination. This method can be used to solve the system (I.3.1). You will study it in detail in the exercises. However, here are the basic steps:

Step 1. Write down the augmented matrix of the system. The matrix can be transformed to an upper triangular one by means of elementary operations described in items a) – d) in paragraph I.2.19 and in example I.2.20. If possible, avoid interchanges of columns – they correspond to interchanges in the order of unknowns and they are often sources of mistakes for beginners. If you cannot avoid interchanges of columns, then you should write the unknowns above the column which contains coefficients standing at this unknown. The $(n+1)$–th column cannot be interchanged with any other one because it contains right hand sides of the equations in the system (I.3.1). Nevertheless, in what follows we suppose that interchanges of columns were not necessary.

Step 2. Write down the system of equations which corresponds to the last matrix. (It is equivalent to the original system we have to solve.) This system can be successively solved from the last to the first equation. Each of the equations can be regarded as the equation for one unknown only – that unknown with the lowest index.

2a) If the last equation has the form $c_n x_n = \gamma$ (where $c_n \neq 0$) then one can use it to express x_n, and substituting its value to the preceding equation, one can get x_{n-1}, etc.

2b) If the last equation has the form $0 = \gamma$ (where $\gamma \neq 0$) then the system has no solution.

2c) If the last equation has the form $c_k x_k + c_{k+1} x_{k+1} + \cdots + c_n x_n = \gamma$ (where $c_k \neq 0$) then $x_{k+1}, x_{k+2}, \ldots, x_n$ can be put equal to parameters which can be denoted for example by $p_1, p_2, \ldots, p_{n-k}$. We can express x_k from this equation. Substituting for $x_k, x_{k+1}, \ldots, x_n$ to the preceding equation, we can use it to get x_{k-1}, etc. The system has now infinitely many solutions. Concrete solutions can be obtained by a concrete choice of values of parameters $p_1, p_2, \ldots, p_{n-k}$.

I.3.3. Remark. Let us first study the homogeneous system (I.3.2). It is obvious that if we solve it by Gaussian elimination, case 2b) cannot appear. (The last column of the augmented matrix contains only zeros, and row operations a) – d) from paragraph I.2.19 cannot affect it. That is also why it is sufficient to work only with the matrix of the system. The augmented matrix is not necessary.) Thus, the homogeneous system is always solvable – it has the unique, trivial, solution in case 2a) or infinitely many solutions in case 2c). The following theorem gives important information on the structure of the set of all solutions of the system (I.3.2).

I.3.4. Theorem. *The set of all solutions of the homogeneous system (I.3.2) is a subspace of the n–dimensional arithmetic space $\mathbb{R}^n$ whose dimension is equal to $n-r(A)$.*

P r o o f : Let us first show that the set of all solutions of the system (I.3.2) is a vector space which is a subspace of $\mathbb{R}^n$. If X and Y are solutions of (I.3.2) then it holds: $A\cdot(X+Y) = A\cdot X + A\cdot Y = O + O = O$, i.e. $X+Y$ also is a solution of the system (I.3.2). Similarly, if X is a solution of (I.3.2) and λ is a real number then $A\cdot(\lambda\cdot X) = \lambda\cdot(A\cdot X) = \lambda\cdot O = O$, i.e. $\lambda\cdot X$ also is a solution of (I.3.2). The set of all solutions of the homogeneous system (I.3.2) is a subset of $\mathbb{R}^n$ because its every element belongs to $\mathbb{R}^n$ and it is closed with respect to the operations "addition" and "multiplication by real numbers". Therefore it is a subspace of $\mathbb{R}^n$.

The assertion on the dimension of the subspace follows from theorem I.2.17 and from the procedure described in paragraph I.3.2. (The dimension is equal to the number of parameters $p_1, p_2, \ldots, p_{n-k}$ in paragraph I.3.2 – think over this fact for yourself.)

I.3.5. Example. Let us solve the system
$$\begin{array}{rcrcrcl} x_1 &+& x_2 &-& 3x_3 &=& 0,\\ 5x_1 &-& 2x_2 &-& 8x_3 &=& 0,\\ 3x_1 &-& 2x_2 &-& x_3 &=& 0. \end{array}$$

The matrix of the system is transformed in accordance with instructions from paragraphs I.3.2 and I.2.19:
$$\begin{pmatrix} 1, & 1, & -3\\ 5, & -2, & -8\\ 3, & -4, & -2 \end{pmatrix} \sim \begin{pmatrix} 1, & 1, & -3\\ 0, & -7, & 7\\ 0, & -7, & 7 \end{pmatrix} \sim \begin{pmatrix} 1, & 1, & -3\\ 0, & -7, & 7\\ 0, & 0, & 0 \end{pmatrix} \sim \begin{pmatrix} 1, & 1, & -3\\ 0, & -7, & 7 \end{pmatrix}.$$

The system of equations corresponding to the last matrix is
$$\begin{array}{rcrcrcl} x_1 &+& x_2 &-& 3x_3 &=& 0,\\ &-& 7x_2 &+& 7x_3 &=& 0. \end{array}$$

We put $x_3 = p$ (where p is a parameter). Then the second equation yields: $x_2 = p$. Substituting for x_3 and x_2 to the first equation, we obtain: $x_1 = 2p$. The solution can be generally expressed: $[x_1, x_2, x_3] = [2p, p, p] = [2, 1, 1]\,p$. Now it is obvious that there exist infinitely many solutions and the set of all solutions forms a 1–dimensional subspace of $\mathbb{R}^3$, whose basis is the arithmetic vector $[2, 1, 1]$. This corresponds to the equality $1 = 3 - 2$ (3 is the number of unknowns, 2 is the rank of the matrix of the system) – see theorem I.3.4.

I.3.6. Remark. If the rank of matrix A is equal to the number of unknowns (i.e. $k = n$) then the set of all solutions of the homogeneous system (I.3.2) is the subspace of $\mathbb{R}^n$ of the dimension $n - n$, i.e. zero. This subspace contains only one element – the zero element – that is the n–tuple of nothing but zeros. This is exactly the case when the homogeneous system (I.3.2) has only the trivial solution.

Let us deal with the general (possibly non–homogeneous) system (I.3.1) again. The next theorem gives information about how to recognize which of the cases 2a) – 2c) (see paragraph I.3.2) occurs.

I.3.7. Frobenius' theorem. *I. The system of linear algebraic equations (I.3.1) (for n unknowns) has a solution if and only if $r(A) = r(A \,|\, B)$.*

II. If $r(A) = r(A \,|\, B) = n$ then the solution is unique.
If $r(A) = r(A \,|\, B) < n$ then the system (I.3.1) has infinitely many solutions.

I.3.8.* Remark. Let us analyze in greater detail the last case, i.e. the situation when the ranks of the matrix and the augmented matrix of the system (I.3.1) are both equal to k, where $k < n$. There is a natural question which is the structure of the set of all solutions of (I.3.1). Theorem I.3.4 states that the set of all solutions of the corresponding homogeneous system (I.3.2) forms a vector space (a subspace of $\mathbb{R}^n$) of the dimension $n - k$. If $X_1, \dots, X_{n-k}$ is a basis of this subspace then solutions of the homogeneous system (I.3.2) can generally be expressed in the form $c_1 X_1 + \dots + c_{n-k} X_{n-k}$. If some concrete, particular solution Y of the general (possibly non–homogeneous) system (I.3.1) is known, then all solutions of the system (I.3.1) can be expressed in the form:

(I.3.3)
$$X = c_1 X_1 + \dots + c_{n-k} X_{n-k} + Y.$$

This means that if $c_1, \dots, c_{n-k}$ can each run independently each of the others over the set of all real numbers, then X runs over the set of all solutions of the system (I.3.1). This is why X is often called the *general solution* of the system (I.3.1).

I.3.9. Cramer's rule. Let us now deal with the special case when system (I.3.1) is a system of n equations for n unknowns. The matrix of the system is a square matrix. Applying Frobenius' theorem, one can easily obtain this important assertion:
If matrix A of the system of equations (I.3.1) is regular then the system has a unique solution.
(Think over this fact and find for yourself reasons why this is true.) In this case, apart from by Gaussian elimination, the solution can also be obtained by means of the formulas

$$x_i = \frac{\Delta_i}{\Delta} \qquad (i = 1, \dots, n)$$

where $\Delta = \det A$ and Δ_i is the determinant of the square matrix which arises from matrix A interchanging the i–th column with the column of the right hand sides of the equations in (I.3.1).
(To derive this formula, it is possible to use the matrix form of the system: $A \cdot X = B$, the consequent form of the solution: $X = A^{-1} \cdot B$ and the expression of A^{-1} by means of the formula from paragraph I.2.35.)

I.4. Eigenvalues and eigenvectors of square matrices

I.4.1. Motivation. In order to keep a simple notation, it will be again advantageous to identify vectors with n–tuples of numbers, written in a column. Such vectors can be regarded as $n \times 1$ matrices. Thus, we shall denote vectors in the same way as matrices (i.e. by capital letters) in this chapter.

Suppose that A is an $n \times n$ square matrix. In mathematics and its applications, one often solves the question whether there exists a nonzero vector $X \in \mathbf{V}(\mathbb{E}_n)$. such that the product $A \cdot X$ is a vector, laying on the same straight line as X. The fact that $A \cdot X$ and X lay on the same straight line means that $A \cdot X = \lambda X$ for an appropriate number λ. The important problem is not only to find such vectors X, but also the corresponding numbers λ. Solution of this problem plays a big role for instance in the theory of stability of mechanical systems.

The equation $A \cdot X = \lambda X$ (for the unknown λ) can generally have complex solutions. That is why we admit the possibility that λ is a complex number and the entries (coordinates) of vector X are also complex and not only real numbers. On the other hand, the square matrix A, we work with in this section is always supposed to have only real entries.

I.4.2. Eigenvalues, eigenvectors. A complex number λ is called an *eigenvalue* of a square matrix A if there exists a nonzero vector X such that $A \cdot X = \lambda X$. Such a vector X is called an *eigenvector* of matrix A corresponding to the eigenvalue λ.

I.4.3. Remark. The eigenvector is not determined uniquely, the number of eigenvectors corresponding to the eigenvalue λ is always infinite. Clearly, if $A \cdot X = \lambda X$ (i.e. X is the eigenvector of matrix A corresponding to the eigenvalue λ) and $k \in \mathbf{C}$, $k \neq 0$, then it also holds: $A \cdot (kX) = \lambda(kX)$. This means that kX is also the eigenvector of A corresponding to the eigenvalue λ.

I.4.4. How to find the eigenvalues. The equation $A \cdot X = \lambda X$ can be written in the equivalent form $A \cdot X - \lambda E \cdot X = O$ or $(A - \lambda E) \cdot X = O$. ($E$ is the $n \times n$ identity matrix and O is the zero vector, i.e. it is the $n \times 1$ zero matrix). The vector equation $(A - \lambda E) \cdot X = O$ can be regarded as the homogeneous system of linear algebraic equations for the unknown components of the vector X. This system has a nonzero solution if and only if the rank of the matrix of the system, i.e. the matrix $A - \lambda E$, is less than n. (See theorem I.3.4 and remark I.3.6.) The inequality $r(A - \lambda E) < n$ means that the matrix $A - \lambda E$ is singular (see paragraph I.2.29), which is true if and only if

$$\det(A - \lambda E) = 0. \tag{I.4.1}$$

(See theorem I.2.31.) Hence a nonzero vector X satisfying the equation $A \cdot X = \lambda X$ exists if and only if (I.4.1) holds. Equation (I.4.1) (for the unknown λ) is called the *characteristic equation* of matrix A. Solving the characteristic equation, we obtain all eigenvalues of matrix A.

I.4.5. Example. Find eigenvalues of the matrix $A = \begin{pmatrix} 3, & 4 \\ 5, & 5 \end{pmatrix}$.

Solution: The characteristic equation of matrix A is

$$\det(A-\lambda E) = \begin{vmatrix} 3-\lambda, & 4 \\ 5, & 5-\lambda \end{vmatrix} = (3-\lambda)\cdot(5-\lambda) - 4\cdot 5 = \lambda^2 - 8\lambda - 5 = 0.$$

Its solution is $\lambda_1 = 4+\sqrt{21}$ a $\lambda_2 = 4-\sqrt{21}$.

I.4.6. How to find the eigenvectors. Eigenvectors corresponding to the eigenvalue λ can be obtained by solving the homogeneous system of linear algebraic equations $(A-\lambda E)\cdot X = O$ (for unknown components $x_1,\ldots,x_n$ of the vector X).

I.4.7. Example. Find eigenvectors of the matrix A from example I.4.5 which correspond to the eigenvalue $\lambda_1 = 4+\sqrt{21}$.

Solution: Substituting the value $4+\sqrt{21}$ for λ in the vector equation $(A-\lambda E)\cdot X = O$ and expressing this equation in coordinates, we obtain the system of linear algebraic equations

$$\begin{aligned} (-1-\sqrt{21})\,x_1 \;&+\; 4\,x_2 &= 0, \\ 5\,x_1 \;&+\; (1-\sqrt{21})\,x_2 &= 0. \end{aligned}$$

This system has infinitely many solutions: $x_1 = (1-\sqrt{21})p$, $x_2 = -5p$ (where $p \in \mathbf{C}$). Every vector X with these coordinates x_1, x_2 (where $p \neq 0$ because X cannot be the zero vector) is the eigenvector.

I.4.8. Remark. If λ is a real eigenvalue of the matrix A then the system of equations $(A-\lambda E)\cdot X = O$ (for unknown components of vector X) has a matrix all of whose entries are real numbers. Therefore it is possible to find a nonzero solution X of this system with all components also being real numbers. This means that to real eigenvalues there exist real eigenvectors.

I.4.9. Remark. The eigenvalues and the eigenvectors of square matrices have a series of interesting properties. Their detailed explication, proofs and examples of some applications would go beyond the scope of this text. Nevertheless, we still present several simple assertions in this remark.

a) *The eigenvectors, corresponding to different eigenvalues, are linearly independent.*

   (Let us show that this statement is true, for simplicity, only for the family of two eigenvectors X_1 and X_2, corresponding to the eigenvalues λ_1 and λ_2: if the vectors X_1 and X_2 are linearly dependent then there exists $k \in \mathbb{R}$, $k \neq 0$, such that $X_2 = kX_1$. Then $A\cdot X_2 = A\cdot kX_1 = kA\cdot X_1 = k\lambda_1 X_1 = \lambda_1 X_2$. We observe that X_2 is an eigenvector, corresponding to the eigenvalue λ_1. This means that $\lambda_2 = \lambda_1$. Thus, on the other hand, if $\lambda_1 \neq \lambda_2$ then the eigenvectors X_1 and X_2 cannot be linearly dependent.)

b) 0 *is the eigenvalue of matrix A if and only if A is singular.*

   (A has the eigenvalue 0 if and only if there existrs a nonzero vector X, such that $A\cdot X = 0\cdot X = O$. This vectorial equation can be regarded as a system of n linear algebraic equations with the quare matrix A. Such a system has a nonzero solution if and only if matrix A is singular – see remark I.3.6.)

Next simple statements are listed without proofs:

c) *If λ is an eigenvalue of matrix A and X is the corresponding eigenvector then $\overline{\lambda}$ is also an eigenvalue of A and $\overline{X}$ is the corresponding eigenvector.*

d) *If λ is an eigenvalue of matrix A and X is the corresponding eigenvector then λ^2 is the eigenvalue of matrix A^2 and X is the corresponding eigenvector.*

e) If the inverse matrix A^{-1} exists then λ is an eigenvalue of matrix A if and only if $1/\lambda$ is the eigenvalue of matrix A^{-1}. The corresponding eigenvectors are the same.

f)* *If A is a symmetric square matrix then all its eigenvalues are real. The eigenvectors, corresponding to different eigenvalues, are perpendicular.*

I.4.10. Example. A is a 3×3 square matrix. Decide whether A can have the given eigenvalues, eventually also the given eigenvectors:

a) $2,\ 3$ b) $3,\ 2+i,\ -3-2i$ c) $5+i,\ 5-i,\ 7$

d) $4,\ 3-i,\ 3+i,\ 5$ e) $7,\ 5,\ 1,\ \begin{pmatrix}-1\\2\\3\end{pmatrix},\ \begin{pmatrix}2\\3\\1\end{pmatrix},\ \begin{pmatrix}4\\6\\2\end{pmatrix}$

Solution: a) Matrix A can have the eigenvalues 2 and 3. However, these need not be all eigenvalues. The characteristic equation of A is a cubic equation and it can have three different roots.

b) If $2+i$ is an eigenvalue of matrix A then $2-i$ must also be an eigenvalue of A. (See remark I.4.10, item c).) By analogy, since $-3-2i$ is an eigenvalue, the complex conjugate $-3+2i$ is an eigenvalue, too. However, matrix A cannot have five eigenvalues 3, $2 \pm i$ and $-3 \pm 2i$ because it is a 3×3 matrix and as such, it can have at most three different eigenvalues.

c) $5+i,\ 5-i\ \ 7$ can be eigenvalues of matrix A.

d) The given numbers are four, hence the answer is negative.

e) The given numbers can be the eigenvalues of matrix A. The numbers are all different and so the eigenvectors should be linearly independent. (See remark I.4.10, item a).) This is not fulfilled because the third vector is a multiple of the second one. Thus, the answer is negative.

I.4.11. Problems. Find eigenvalues and eigenvectors of the following matrices.

a) $\begin{pmatrix}2, & 1\\1, & 2\end{pmatrix}$; b) $\begin{pmatrix}3, & 4\\5, & 2\end{pmatrix}$; c) $\begin{pmatrix}0, & a\\-a, & 0\end{pmatrix}$ $(a \neq 0)$; d) $\begin{pmatrix}5, & 6, & -3\\-1, & 0, & 1\\1, & 2, & -1\end{pmatrix}$.

Results: a) $\lambda_1 = 3,\ X_1 = \begin{pmatrix}p\\p\end{pmatrix},\ \lambda_2 = 1,\ X_2 = \begin{pmatrix}p\\-p\end{pmatrix}$; b) $\lambda_1 = 7,\ X_1 = \begin{pmatrix}p\\p\end{pmatrix}$, $\lambda_2 = -2,\ X_2 = \begin{pmatrix}-4p\\5p\end{pmatrix}$; c) $\lambda_{1,2} = \pm a\,i,\ X_{1,2} = \begin{pmatrix}p\\\pm p\,i\end{pmatrix}$;

d) $\lambda = 2,\ X = \begin{pmatrix}-2p+q\\p\\q\end{pmatrix}$.

(The parameters p, q can be any complex numbers such that the corresponding eigenvector is non–zero.)

I.5. Survey of equivalent properties of a square matrix

We do not explain any new notion or a new method in this chapter. On the other hand, you already know everything which follows. Nevertheless, if you are studying linear algebra for the first time, it is possible that your knowledge is still not enough classified and various assertions and theorems are not mutually interconnected. In order to show clearly the connections, we give a survey of equivalent statements one can make about a square matrix A in this chapter. (This means that for a given square matrix A either all statements are true or all statements are false.)

We assume that A is an $n \times n$ square matrix.
Then the following statements are equivalent:

1. *Matrix A is regular.*

2. $\det A \neq 0$.
(See theorem I.2.31.)

3. *An inverse matrix A^{-1} exists.*
(See theorem I.2.31.)

4. *The rank of matrix A is n.*
(See the definition of a regular matrix in paragraph I.2.29.)

5. *The rows of matrix A are linearly independent.*
(See the definition of the rank of a matrix in paragraph I.2.14 and statement 4.)

6. *The columns of matrix A are linearly independent.*
(See remark I.2.16 and statement 4.)

7. *The homogeneous system of linear algebraic equations $A \cdot X = O$ has a unique (i.e. zero) solution.*
(See remark I.3.6.)

8. *The general (i.e. homogeneous or non–homogeneous) system of linear algebraic equations $A \cdot X = B$ has a unique solution.*
(See the Frobenius theorem I.3.7.)

9. 0 *is not an eigenvalue of matrix A.*
(See remark I.4.9, item b).)

Naturally, the negations of all these statements are also equivalent. Formulate and write down all the negations yourself.

II. Analytic geometry in $\mathbb{E}_3$

II.1. Some basic notions

II.1.1. Cartesian coordinates in $\mathbb{E}_3$. To locate points and other objects in space $\mathbb{E}_3$, we use three mutually perpendicular coordinate axes, intersecting at one point. We denote them x, y, z or x_1, x_2, x_3. Their orientation is chosen so that they make a *right handed* system. This means that when you hold your right hand so that the fingers curl from the positive part of the x–axis toward the positive part of the y–axis, your thumb points along the positive part of the z–axis. The intersection of all three axes is called the *origin* of the coordinate system.

To each point in $\mathbb{E}_3$, we can uniquely assign its *Cartesian coordinates* – they are successively the distances of orthogonal projections of the considered point onto the axes x, y, z from the origin, taken with the "+" sign if the projection lays on the positive part of the axis and with the "−" sign in the opposite case.

By analogy, we can uniquely assign the Cartesian coordinates to each free vector in $\mathbb{E}_3$ – we choose the concrete position of the vector so that its initial point is at the origin of the coordinate system and we regard the cartesian coordinates of its end point as the cartesian coordinates of the vector. (See also paragraph I.1.2.)

To distinguish between points and free vectors in $\mathbb{E}_3$, we write the Cartesian coordinates of points in $\mathbb{E}_3$ in brackets (for example $[1,2,3]$) and the Cartesian coordinates of vectors in $\mathbb{E}_3$ in parentheses (for instance $(-1,2,5)$).

II.1.2. The length (the magnitude) of a vector in $\mathbb{E}_3$. If $\mathbf{u} = (u_1, u_2, u_3)$ is a vector in $\mathbb{E}_3$ then the number

$$\|\mathbf{u}\| \;=\; \sqrt{u_1^2 + u_2^2 + u_3^2}$$

is called its *length* (or its *magnitude*).

II.1.3. The scalar product of vectors in $\mathbb{E}_3$. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{E}_3$ then the number

$$\mathbf{u} \cdot \mathbf{v} \;=\; u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3.$$

is said to be their *scalar product* (or their *dot product*). (Compare with the scalar product of arithmetic vectors, defined in paragraph I.2.11.)

II.1.4. Theorem. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are non–zero vectors in* $\mathbb{E}_3$ *and* φ *is the angle between the vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *then*

$$\mathbf{u} \cdot \mathbf{v} \;=\; \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \cos\varphi.$$

Proof: Let us choose an arbitrary point A in $\mathbb{E}_3$ and put $B = A + \mathbf{u}$ and $C = A + \mathbf{v}$. Obviously, the vector $B - C$ can also be written as $\mathbf{u} - \mathbf{v}$. Applying the cosine theorem to the triangle ABC, we obtain: $\|B-C\|^2 = \|B-A\|^2 + \|C-A\|^2 - 2\,\|B-A\|\,\|C-A\| \cdot \cos\varphi$, or $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| \cdot \cos\varphi$. This yields: $(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 = u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| \cos\varphi$. The desired formula is an easy consequence of this equality.

II.1.5. Remark. Thus, the non–zero vectors $\mathbf{u}$, $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

II.1.6. Another form of vectors in $\mathbb{E}_3$. Denote by $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$. If $\mathbf{u} = (u_1, u_2, u_3)$ is a vector in $\mathbb{E}_3$ then it can also be written down in this form: $\mathbf{u} = u_1 \cdot \mathbf{i} + u_2 \cdot \mathbf{j} + u_3 \cdot \mathbf{k}$.

II.1.7. The vector product of vectors in $\mathbb{E}_3$. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{E}_3$. The vector

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i}, & \mathbf{j}, & \mathbf{k} \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2) \cdot \mathbf{i} + (u_3 v_1 - u_1 v_3) \cdot \mathbf{j} + (u_1 v_2 - u_2 v_1) \cdot \mathbf{k}$$

is called the *vector product* (or the *cross product*) of $\mathbf{u}$ and $\mathbf{v}$ (in this order).

II.1.8. Theorem. *If $\mathbf{u}$ and $\mathbf{v}$ are non–zero vectors in $\mathbb{E}_3$ and φ is the angle between them then*

a) the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both the vectors $\mathbf{u}$ and $\mathbf{v}$;
b) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \cdot \sin \varphi$;

We omit the proof of this theorem. However, the assertion a) can easily be verified computing the scalar products $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$. All of them are equal to zero. The vector product $\mathbf{u} \times \mathbf{v}$ is oriented in accordance with the so called right–hand rule: $\mathbf{u} \times \mathbf{v}$ *points the way your right thumb points when your fingers curl through the angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *from* $\mathbf{u}$ *to* $\mathbf{v}$.

II.1.9. The sum of a point and a vector. If $A = [\, a_1, a_2, a_3 \,]$ is a point in $\mathbb{E}_3$ and $\mathbf{u} = (u_1, u_2, u_3)$ is a vector in $\mathbb{E}_3$, then the *sum of point A and vector* $\mathbf{u}$ is equal to the point $B = [a_1 + u_1, a_2 + u_2, a_3 + u_3 \,]$ in $\mathbb{E}_3$. We write: $B = A + \mathbf{u}$.

On the other hand, the *difference of two points* $B = [\, b_1, b_2, b_3 \,]$ and $A = [\, a_1, a_2, a_3 \,]$ in $\mathbb{E}_3$ (in this order) is defined to be equal to the vector $\mathbf{u} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$. We write: $\mathbf{u} = B - A$.

II.1.10. The distance between two points. Remember that if $A = [\, a_1, a_2, a_3 \,]$ and $B = [\, b_1, b_2, b_3 \,]$ are two points from $\mathbb{E}_3$ then their distance $\|B - A\|$ is

$$\|B - A\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.$$

(It is the length of the vector $B - A$.)

II.1.11. The distance between a point and a set; the distance between two sets. If $A = [\, a_1, a_2, a_3 \,]$ is a point in $\mathbb{E}_3$ and $M \subset \mathbb{E}_3$, then the *distance between point A and set M* is defined as follows:

$$d(A, M) = \inf \{ \, \|A - X\|; \; X \in M \, \}.$$

(See p. 46 for the definition of “inf”.) By analogy, the *distance between sets M and N* in $\mathbb{E}_3$ is defined by the formula

$$d(M, N) = \inf \{ \, \|X - Y\|; \; X \in M, \; Y \in N \, \}.$$

II.2. Straight lines in $\mathbb{E}_3$

II.2.1. A straight line in $\mathbb{E}_3$. If $A = [\,a_1, a_2, a_3\,]$ is a point in $\mathbb{E}_3$ and $\mathbf{u} = (u_1, u_2, u_3)$ is a non–zero vector in $\mathbb{E}_3$, then the set $p = \{\, X \in \mathbb{E}_3;\ \exists t \in \mathbb{R} :\ X = A + t \cdot \mathbf{u} \,\}$ is called a *straight line* in $\mathbb{E}_3$. The equation

(II.2.1) $$X = A + t \cdot \mathbf{u}; \qquad t \in \mathbb{R}$$

is called the *parametric equation* for straight line p. This equation is also often written down in a coordinate form:

(II.2.2) $$x_1 = a_1 + t \cdot u_1, \quad x_2 = a_2 + t \cdot u_2, \quad x_3 = a_3 + t \cdot u_3; \qquad t \in \mathbb{R}.$$

The straight line p is said to be *parallel* to $\mathbf{u}$ and on the other hand, $\mathbf{u}$ is called the *directional vector* of p.

If parameter t in (II.2.1) is taken not from the whole set $\mathbb{R}$, but for example only from the interval $[\,-1, 5\,]$, then (II.2.1) is the parametrization of a line segment. Its end points are $A - \mathbf{u}$ and $A + 5\mathbf{u}$. Similarly, if t is taken for instance from the interval $[1, = \infty)$ then (II.2.1) parametrizes a half–line, etc.

II.2.2. The distance from a point to a straight line. Let $p:\ X = A + t \cdot \mathbf{u};\ t \in \mathbb{R}$ be a straight line in $\mathbb{E}_3$ and M be a point in $\mathbb{E}_3$ which does not lie on straight line p. We show two methods for evaluating the distance $d(M, p)$.

1st method: Denote by P the nearest point to M on straight line p. (It can be proved that such a point exists and the vector $M - P$ is perpendicular to p.) Thus, for some (still unknown) value of parameter t, it holds: $P = A + t \cdot \mathbf{u}$. The value of t can be evaluated from the equation $(M - P) \cdot \mathbf{u} = 0$, i.e. $(M - P) \cdot \mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\, t = 0$. Substituting now for t back to the formula $P = A + t \cdot \mathbf{u}$, we obtain the concrete position of P. The distance $d(M, p)$ is equal to $|P - M|$.

2nd method: AMP is the right triangle. Hence

$$d(M, p) = \|M - P\| = \|M - A\|\ \sin\varphi.$$

It follows from theorem II.1.8 that

$$\|(M - A) \times \mathbf{u}\| = \|M - A\|\ \|\mathbf{u}\|\ \sin\varphi.$$

Expressing $\sin\varphi$ from this equality and substituting it to the preceding equality, we obtain:

(II.2.3) $$d(M, p) = \frac{\|(M - A) \times \mathbf{u}\|}{\|\mathbf{u}\|}.$$

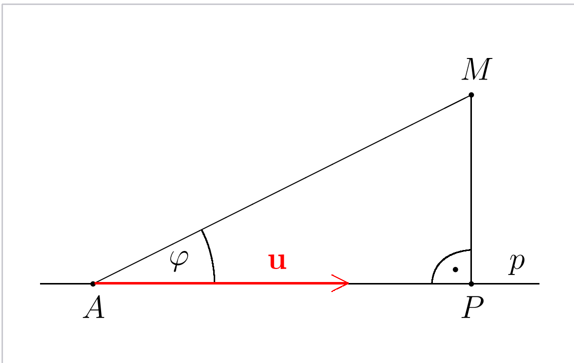


Fig. 1

II.2.3. Two straight lines in space. Let p and q be two straight lines in $\mathbb{E}_3$ whose parametric equations are

$$p:\quad X = A + t \cdot \mathbf{u}; \quad t \in \mathbb{R}, \qquad\qquad q:\quad Y = B + s \cdot \mathbf{v}; \quad s \in \mathbb{R}.$$

We say that p and q are

a) <u>*identical*</u> (if they have infinitely many common points),
b) <u>*intersecting*</u> (if they have just one common point),
c) <u>*parallel*</u>, (if they have no common points and vectors $\mathbf{u}$, $\mathbf{v}$ are linearly dependent),
d) <u>*skew*</u>, (if they have no common points and vectors $\mathbf{u}$, $\mathbf{v}$ are linearly independent).

Let us now deal with the question how to distinguish between these cases in a particular situation. We look for common points of straight lines p and q. This task can be reformulated: We seek for such values of parameters t, s that using them in the parametric equations for p and q, we get the same point, i.e. $X = Y$. This yields: $A + t \cdot \mathbf{u} = B + s \cdot \mathbf{v}$. Rewriting this into coordinates, we obtain the system of three linear equations for two unknowns t, s:

$$a_1 + t \cdot u_1 = b_1 + s \cdot v_1\,, \qquad a_2 + t \cdot u_2 = b_2 + s \cdot v_2\,, \qquad a_3 + t \cdot u_3 = b_3 + s \cdot v_3\,.$$

This is equivalent to

$$\begin{array}{rcrcl} u_1 \cdot t & - & v_1 \cdot s & = & b_1 - a_1\,, \\ u_2 \cdot t & - & v_2 \cdot s & = & b_2 - a_2\,, \\ u_3 \cdot t & - & v_3 \cdot s & = & b_3 - a_3\,. \end{array}$$

If the number of solutions is infinite then the number of common points of straight lines p, q is also infinite and so the straight lines are identical.

If the above system has a unique solution, straight lines p, q have only one common point and so they are intersecting.

If the above system of equations has no solution then straight lines p, q have no common point. Thus, they are either parallel or skew. This depends on directional vectors $\mathbf{u}$, $\mathbf{v}$. If they are linearly dependent, straight lines p, q are parallel, otherwise they are skew.

II.2.4. A straight line given by two points. If A and B are two different points in $\mathbb{E}_3$, then the straight line passing through these points can be described for example by the following parametric equation:

$$X = A + t \cdot (B - A); \quad t \in \mathbb{R}. \tag{II.2.4}$$

II.2.5. Remark. Suppose that points A, B, C, D, E are different points, all lying on the same straight line p. Put for example $\mathbf{u} = B - A$, $\mathbf{v} = 2 \cdot (B - D)$. Then the equations

$$\begin{array}{rcll} X & = & A + t \cdot (B - A); & t \in \mathbb{R}, \\ Y & = & B + s \cdot (C - A); & s \in \mathbb{R}, \\ Z & = & D + r \cdot \mathbf{u}; & r \in \mathbb{R}, \\ X & = & E + \alpha \cdot \mathbf{v}; & \alpha \in \mathbb{R} \end{array}$$

are all parametric equations for line p. (Why?)

II.2.6. A secant of two lines. A straight line which is intersecting the two lines p and q is called the <u>*secant line*</u> of p and q.

II.2.7. Example. Lines p, q are given by their parametric equations

$$p: \quad X = A + t \cdot \mathbf{u}; \quad t \in \mathbb{R}, \qquad\qquad q: \quad X = B + s \cdot \mathbf{v}; \quad t \in \mathbb{R},$$

where $A = [3,0,-1]$, $\mathbf{u} = (2,1,1)$, $B = [1,1,1]$, $\mathbf{v} = (1,3,-2)$. Find a straight line r which intersects both lines p and q and which passes through the point $M = [-7,-6,5]$.

Solution: Denote by P (respectively by Q) the point of intersection of line r with line p (respectively with line q). Then there exist such values of parameters t and s that

$$P = A + t\cdot\mathbf{u}, \qquad Q = A + s\cdot\mathbf{v}. \tag{II.2.5}$$

All of points P, Q, M lie on line r and $M \neq Q$, because point M does not lie on the line q. (This can easily be verified: If $M \in p$ then the vectors $M - B$ and $\mathbf{v}$ would be linearly dependent. However, this is not true, because $(M-B) = (-8,-7,4)$, $\mathbf{v} = (1,3,-2)$ and it is seen that none of these vectors is equal to a multiple of the second vector.) Thus, there exists a number $\alpha \in \mathbb{R}$ such that

$$P - M = \alpha\cdot(Q - M).$$

Substituting here the above expression of points P and Q, we obtain the equation:

$$A + t\cdot\mathbf{u} - M = \alpha\cdot(B + s\cdot\mathbf{v} - M).$$

Writing this in coordinates and using here known coordinates of points A, B, M and vectors $\mathbf{u}$, $\mathbf{v}$, we obtain a system of three linear algebraic equations for the unknowns t, αs, α:

$$2t - \alpha s - 8\alpha = -10, \qquad t - 3\alpha s - 7\alpha = -6, \qquad t + 2\alpha s + 4\alpha = 6.$$

This system has the unique solution $t = 2$, $\alpha s = -2$, $\alpha = 2$. Hence, we also get: $s = -1$. When used in (II.2.5), this gives: $P = [7,2,1]$, $Q = [0,-2,3]$. A parametric equation for line r can now be written down for example by means of (II.2.4), i.e. as the equation of a straight line passing through two known points P, Q:

$$X = P + \tau\cdot(Q - P); \quad \tau \in \mathbb{R},$$
$$X = [7,2,1] + \tau\cdot(-7,-4,2); \quad \tau \in \mathbb{R}.$$

The above equation can also be written in coordinates:

$$x_1 = 7 - 7\tau, \qquad x_2 = 2 - 4\tau, \qquad x_3 = 1 + 2\tau; \qquad \tau \in \mathbb{R}.$$

II.2.8. The distance between two straight lines. Suppose that the straight lines
$p:\ X = A + t\cdot\mathbf{u};\ t \in \mathbb{R}$ and
$q:\ Y = B + s\cdot\mathbf{v};\ s \in \mathbb{R}$ are parallel or skew.

To find the distance $d(p,q)$, we first find a straight line r which intersects lines p, q and is perpendicular to both of them. Denote by P, Q points of intersection of r with lines p, q. Then $P = A + t\cdot\mathbf{u}$, $Q = B + s\cdot\mathbf{v}$. for some values of parameters t, s. Line r is parallel to the vector $Q - P$. The vector $Q - P$ is orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$. So $(Q-P)\cdot\mathbf{u} = 0$, $(Q-P)\cdot\mathbf{v} = 0$. Substituting here from the equations for points P and Q, we obtain (after an easy rearrangement):

$$(\mathbf{v}\cdot\mathbf{u})\,s - (\mathbf{u}\cdot\mathbf{u})\,t = -(B-A)\cdot\mathbf{u}, \qquad (\mathbf{v}\cdot\mathbf{v})\,s - (\mathbf{u}\cdot\mathbf{v})\,t = -(B-A)\cdot\mathbf{v}.$$

We solve this system of two linear algebraic equations for unknowns t, s and use the values of t, s in the formulas for P and Q. Finally, the distance between lines p and q can be calculated as the distance between points P and Q: $d(p,q) = \|P - Q\|$.

II.2.9. The angle between two straight lines. Suppose that p, q are straight lines in $\mathbb{E}_3$ whose parametrizations are: $p: \ X = A + t \cdot \mathbf{u}; \ t \in \mathbb{R}$ and $q: \ X = B + s \cdot \mathbf{v}; \ s \in \mathbb{R}$. Denote by φ the angle between vectors $\mathbf{u}$ and $\mathbf{v}$. Its cosine is given by formula (II.1.6). The angle φ can take on the values from 0 to π (sketch a picture). The *angle between straight lines* p, q is called the angle $\vartheta \in [0, \pi/2]$, which is equal to φ if $\varphi \in [0, \pi/2]$ and it is equal to $\pi - \varphi$ if $\varphi \in (\pi/2, \pi]$. (In other words, the angle between two lines is defined to be the acute or right angle between their directional vectors.) Using (II.1.6), we obtain the formula:

$$\cos \vartheta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}. \tag{II.2.6}$$

II.2.10. Problems. a) Find the angle between the straight lines AB and CD if $A = [-1, 2, 0]$, $B = [-2, 0, 2]$, $C = [4, -4, 5]$, $D = [2, 4, 3]$.
b) Find the distance from the point $M = [1, -6, 8]$ to the straight line AB, if $A = [2, 1, 3]$, $B = [3, -1, 6]$.
c) Find the distance between the straight lines $p: \ X = [6, 4, 3] + t \cdot (1, 1, 1); \ t \in \mathbb{R}$ and $q: \ X = [7, 0, -18] + s \cdot (2, -1, 4); \ s \in \mathbb{R}$.
R e s u l t s : a) $\cos \vartheta = \sqrt{2}/2$, $\vartheta = \pi/4$, b) 4.36 c) 11.83

II.3. Planes in $\mathbb{E}_3$

II.3.1. A plane in $\mathbb{E}_3$. If $A = [A_1, a_2, a_3]$ is a point in $\mathbb{E}_3$ and $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ are two linearly independent vectors in $\mathbb{E}_3$, then the set $\sigma = \{ X \in \mathbb{E}_3; \ \exists \alpha \in \mathbb{R}, \ \exists \beta \in \mathbb{R} : \ X = A + \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{v} \}$ is called the *plane* in $\mathbb{E}_3$. The equation

$$X = A + \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{v}; \qquad \alpha, \beta \in \mathbb{R} \tag{II.3.1}$$

is called the *parametric equation* for plane σ. This equation is often used in the coordinate form:

$$\begin{aligned} x_1 &= a_1 + \alpha \cdot u_1 + \beta \cdot v_1 \\ x_2 &= a_2 + \alpha \cdot u_2 + \beta \cdot v_2 \\ x_3 &= a_3 + \alpha \cdot u_3 + \beta \cdot v_3 \, ; \qquad \alpha, \beta \in \mathbb{R}. \end{aligned}$$

II.3.2. A plane given by three points. Let A, B, C be points in $\mathbb{E}_3$ which do not lie on a line. A plane passing through all these points (it will also be called plane ABC) can be described by the parametric equation

$$X = A + \alpha \cdot (B - A) + \beta \cdot (C - A); \qquad \alpha, \beta \in \mathbb{R}. \tag{II.3.2}$$

II.3.3. Remark. Each plane has infinitely many parametric equations. (Compare with lines – see remark II.2.5).

II.3.4. A normal vector. Let σ be a plane, given by the parametric equation (II.3.1). Every non–zero vector which is perpendicular to plane σ (i.e. perpendicular to vectors $\mathbf{u}$, $\mathbf{v}$) is called the *normal vector* to plane σ.

It follows from theorem II.1.11 that for example the vector $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ is the normal vector to plane σ. All other normal vectors to plane σ are non–zero multiples of $\mathbf{n}$, i.e. they have the form $c \cdot \mathbf{n}$, where $c \in \mathbb{R}$, $c \neq 0$.

II.3.5. Another analytic description of a plane. If σ is a plane whose parametric equation is equation (II.3.1) and if its normal vector is $\mathbf{n} = (n_1, n_2, n_3)$, then any point $X = [\, x_1, x_2, x_3 \,]$ in $\mathbb{E}_3$ belongs to plane σ if and only if $(X - A) \cdot \mathbf{n} = 0$. Expressing the scalar product on the left hand side, we get:

$$(x_1 - a_1) \cdot n_1 + (x_2 - a_2) \cdot n_2 + (x_3 - a_3) \cdot n_3 \;=\; 0,$$

$$n_1\, x_1 + n_2\, x_2 + n_3 + q \;=\; 0, \tag{II.3.3}$$

where $q = -a_1 n_1 - a_2 n_2 - a_3 n_3$. Equation (II.3.3) is called the <u>*equation for plane σ*</u>.

Equation (II.3.3) can be obtained from the parametric equation for plane σ (written in coordinates) by excluding parameters α and β. Conversely, if plane σ is given by equation (II.3.3) then its parametrization can be obtained for example in such a way that we find three different points A, B, C of σ (i.e. points whose coordinates satisfy equation (II.3.3)) which do not lie on a line, and we then use (II.3.2).

II.3.6. Example. Let plane σ have the equation $5\, x_1 - 3\, x_2 + 7\, x_3 - 12 = 0$. Then the vector $\mathbf{n} = (5, -3, 7)$ is the normal vector to this plane.

II.3.7. The distance from a point to a plane. Assume that plane σ is given by the parametric equation (II.3.1) and M is a point in $\mathbb{E}_3$. We derive a formula for the distance $d(M, \sigma)$.

From the right–angled triangle APM, we obtain: $d(M, \sigma) = \|M - P\| = \|M - A\| \cdot \cos\varphi$. $\cos\varphi$ can be expressed by means of the scalar product of the vectors $\mathbf{n}$ and $(M - A)$:

$$\cos\varphi \;=\; \frac{\|\mathbf{n} \cdot (M - A)\|}{\|\mathbf{n}\| \cdot \|M - A\|}$$

($\mathbf{n}$ is an arbitrary normal vector to plane σ.) Using this in the formula for $d(M, \sigma)$, we get

$$d(M, \sigma) \;=\; \frac{|\mathbf{n} \cdot (M - A)|}{\|\mathbf{n}\|}. \tag{II.3.4}$$

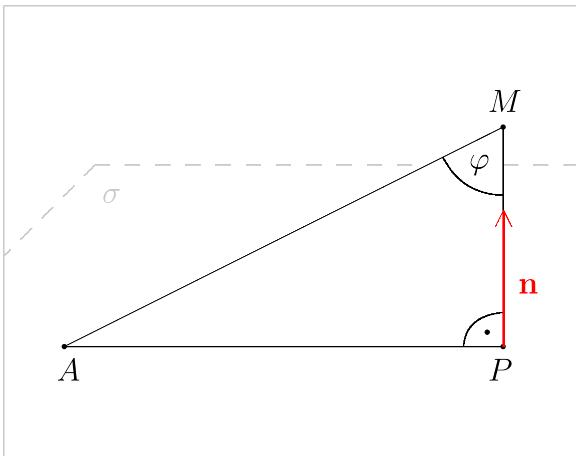


Fig. 2

Suppose now that $M = [\, m_1, m_2, m_3 \,]$. Then $\mathbf{n} \cdot (M - A) = n_1\, m_1 + n_2\, m_2 + n_3\, m_3 + q$ (where $q = -n_1\, a_1 - n_2\, a_2 - n_3\, a_3$). Substituting this to (II.3.4) and expressing $|\mathbf{n}|$ as $\sqrt{n_1^2 + n_2^2 + n_3^2}$, we obtain:

$$d(M, \sigma) \;=\; \frac{|n_1\, m_1 + n_2\, m_2 + n_3\, m_3 + q|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}. \tag{II.3.5}$$

This formula is useful especially if plane σ is given by equation (II.3.3).

II.3.8. Remark. Assume that p is a straight line, given by parametric equations (II.2.2). Excluding parameter t from the three equations (II.2.2), we obtain a system of two equations which can be written in the form

$$n'_1 x_1 + n'_2 x_2 + n'_3 x_3 + q' = 0, \qquad n''_1 x_1 + n''_2 x_2 + n''_3 x_3 + q'' = 0. \tag{II.3.6}$$

Straight lines are often described by such systems of two equations. Each of them is an equation for a plane, and the straight line is the intersection of the two planes.

Conversely, if a straight line is given by two equations of type (II.3.6) then its parametrization can be obtained for example as follows: We solve equations (II.3.6) as two equations for unknowns x_1, x_2, x_3. If the planes described by the equations in (II.3.6) are not parallel (or even identical) then the general solution contains one arbitrary parameter (see remark I.3.8). Writing down the general solution of (II.3.6), we get the parametric equation for the line.

II.3.9. Example. Find a parametrization of the line given by two equations

$$5x_1 + 7x_2 - 4x_3 + 1 = 0, \qquad 2x_1 + 4x_2 - 4x_3 - 2 = 0.$$

Applying for example Gauss' method of elimination, we express the solution of this system: $x_1 = -2t - 3$, $x_2 = 2t + 2$, $x_3 = t$ $(t \in \mathbb{R})$. These three equations represent parametric equations for the line. Obviously, the parametric representation can also be written in a vector form, as one equation: $X = A + t \cdot \mathbf{u}$, where $A = [-3, 2, 0]$ and $\mathbf{u} = (-2, 2, 1)$.

II.3.10. A position of a straight line according to a plane. Let us deal with a straight line p and a plane σ in space $\mathbb{E}_3$. p and σ can find themselves in three different positions:

a) Line p is a subset of plane σ.
b) Line p intersects plane σ at one point.
c) Line p and plane σ are disjoint, they have no common points. In this case, we say that line p is *parallel* to plane σ.

How to distinguish between the possibilities a), b), c) in a particular case? Let us analyze for example the situation when line p is given by two equations (II.3.6) and plane σ is described by one equation (II.3.3). Equations (II.3.6) and (II.3.3) together form a system of three linear equations for three unknowns x_1, x_2, x_3. We solve this system. There are three possibilities (see Frobenius' theorem): 1) The system has infinitely many solutions. 2) The system has a unique solution. 3) The system has no solution. These possibilities successively correspond to the cases a), b) and c) mentioned above. In case 2) the solution of the system gives coordinates of the point of intersection of line p with plane σ.

We will examine the position of a line according to a plane once again in example II.3.12, the line will be given parametrically this time.

II.3.11. The angle between a straight line and a plane. Let p be a straight line and σ be a plane in space $\mathbb{E}_3$. If line p is not perpendicular to σ then the *angle between line p and plane σ* is defined to be equal to the angle between line p and its orthogonal projection to plane σ. If line p is perpendicular to plane σ then the angle between them is defined to be equal to $\pi/2$.

How can the angle between line p and plane σ be evaluated in a particular case? First we determine a normal vector $\mathbf{n}$ to plane σ (see paragraph II.3.4 or possibly also II.3.5, II.3.6).
Let q be an arbitrary line parallel to vector $\mathbf{n}$ (see Fig. 3). Denote by ϑ the angle between lines p and q. $\cos\vartheta$ can be calculated from (II.2.6). The angle between line p and plane σ is equal to $\pi/2 - \vartheta$.

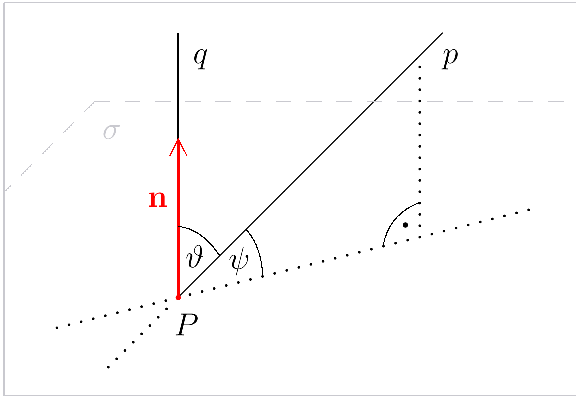


Fig. 3

II.3.12. Example. Line p is given parametrically: $x_1 = 1 + t$, $x_2 = 2 + t$, $x_3 = 3$. Plane σ is given by the equation $2x_1 + 4x_2 + 4x_3 + 2 = 0$. What is the position of p according to σ and what is the angle between p and σ?

We substitute for x_1, x_2, x_3 from the parametric equations for p to the equation for *sigma*. We obtain the equation for the unknown t: $2 + 2t + 8 + 4t + 12 + 2 = 0$. This equation has a unique solution $t = -4$. As the solution exists and is unique, line p and plane σ have a unique common point. The coordinates of this point can be obtained by the substitution of $t = -4$ to the parametric equations for line p: $x_1 = -3$, $x_2 = -1$, $x_3 = 3$.

For example, the vector $\mathbf{n} = (2, 4, 4)$ is the normal vector to plane σ. A directional vector of line p can be chosen for example to be equal to the vector $\mathbf{a} = (1, 1, 0)$. The angle between line p and a line with the directional vector $\mathbf{n}$ is the angle ϑ whose cosine can be calculated by formula (II.2.6). It is equal to $\sqrt{2}/2$. Since ϑ is the acute angle, the information on its cosine implies: $\vartheta = \pi/4$. Hence the angle between line p and plane σ is the angle $\psi = \pi/2 - \vartheta = \pi/2 - \pi/4 = \pi/4$.

II.3.13. A position of two planes in space. Two planes in space $\mathbb{E}_3$ can either be *identical* or *intersecting in a line* or *parallel*. If two particular planes are given and one wishes to recognize which of these cases obtains, then the approach one can use depends on the way the planes are given. We show two possible methods in examples II.3.15 and II.3.16.

II.3.14. The angle between two planes. The *angle between two planes σ and η* is defined to be equal to the angle between two arbitrary lines, the first of which is perpendicular to plane σ, and the second being perpendicular to plane η.

II.3.15. Example. Analyze the position of plane σ according to plane η and evaluate the angle between them. σ: $x_1 - x_2 + x_3 - 1 = 0$ and η: $2x_1 + x_2 - x_3 - 1 = 0$.

Coordinates of common points of the two planes must satisfy both equations. The equations represent the system of two linear equations for the unknowns x_1, x_2, x_3. We can easily check that the system has the solution $x_1 = \frac{2}{3}$, $x_2 = -\frac{1}{3} + t$, $x_3 = t$; $t \in \mathbb{R}$. These equations for x_1, x_2, x_3 can be regarded as parametric equations of the line which is the intersection of planes σ and η.

The vector $\mathbf{n}' = (1,-1,1)$ is the normal vector to plane σ and $\mathbf{n}'' = (2,1,-1)$ is the normal vector to plane η. Applying formula (II.2.6), we can now find out that the angle between planes σ and η is equal to $\pi/2$.

II.3.16. Problems. 1) Find the distance from point M to plane σ, if

a) $M = [7,0,-1]$, σ: $3x_1 + x_2 - 2x_3 + 5 = 0$,

b) $M = [2,3,-1]$, σ: $X = [1,0,-1] + \alpha \cdot (2,0,1) + \beta \cdot (0,2,1)$; $\alpha, \beta \in \mathbb{R}$.

2) What is the position of plane σ in relation to plane η and what is the angle between σ and η if

a) σ: $3x_1 - 6x_2 + 6x_3 + 9 = 0$ and η: $x_1 - 2x_2 + 2x_3 - 3 = 0$,

b) σ: $2x_1 - x_2 + x_3 - 1 = 0$ and η: $x_1 + x_2 + 2x_3 + 1 = 0$?

3) Find the angle between the line AB (where $A = [2,-1,2]$ and $B = [1,-1,1]$) and the plane σ: $x_1 - x_2 - 5 = 0$.

4) Find coordinates of an orthogonal projection of the point $A = [7,0,-1]$ onto the plane σ: $3x_1 + x_2 - 2x_3 + 5 = 0$.

5) In which position is line p in relation to plane σ, if p : $5x_1 + 7x_2 - 4x_3 + 15 = 0$, $12x_1 - 2x_2 + 8x_3 + 3 = 0$ and σ : $5x_1 + 7x_2 - 4x_3 - 8 = 0$?

6) In which position is the plane ABC in relation to the plane DEF, if $A = [1,1,2]$, $B = [1,1,4]$, $C = [1,2,1]$, $D = [2,0,-1]$, $E = [2,1,1]$, $F = [4,-1,3]$? Also find the angle between the planes ABC and DEF.

R e s u l t s : 1a) $2\sqrt{14}$ 1b) $4/\sqrt{30}$

2a) The planes are parallel and their distance is 2.

2b) The planes intersect in the straight line $X = [0,-1,0] + t\,(-1,-1,1)$, the angle between them is $\pi/3 = 60°$.

3) $\pi/6 = 30°$ 4) $[1,-2,3]$

5) Straight line p and plane σ are parallel, their distance is $23/\sqrt{90}$.

6) The planes intersect in the straight line $X = [1,2,0] + t(0,1,2)$, the angle between them is 36.67°.

II.4.* Quadric surfaces in $\mathbb{E}_3$

II.4.1. Quadric surfaces in $\mathbb{E}_3$. From secondary school, you are familiar with conic sections in $\mathbb{E}_2$ and with their equations. These equations are quadratic. By analogy, quadratic equations describe so called *quadric surfaces* in $\mathbb{E}_3$. A general equation for a quadric surface is

(II.4.1) $$a_{11}\,x_1^2 + a_{12}\,x_1\,x_2 + a_{13}\,x_1\,x_3 + a_{22}\,x_2^2 + a_{23}\,x_2\,x_3 + a_{33}\,x_3^2 + \\ + b_1\,x_1 + b_2\,x_2 + b_3\,x_3 + c = 0.$$

(The statement "quadric surface σ is described by equation (II.4.1)" or "(II.4.1) is the equation for quadric surface σ" means that quadric surface σ is the set of all points in $\mathbb{E}_3$ whose coordinates satisfy equation (II.4.1).)

It can be shown that there exists a Cartesian coordinate system x_1', x_2', x_3' in $\mathbb{E}_3$ which is only turned on in relation to the coordinate system x_1, x_2, x_3 and the quadric surface given by (II.4.1) can be described in the new system by an equation which does not contain the mixed products $x'_1 x'_2,\ x'_1 x'_3,\ x'_2 x'_3$.

To avoid complications, we shall further deal only with quadric surfaces whose equations even in the coordinate system x_1, x_2, x_3 do not contain the mixed products $x_1 x_2,\ x_1 x_3,\ x_2 x_3$. This approach corresponds to situations you have been used to from analytic geometry in plane from secondary school. For example, you have mostly described an ellipse in the x, y–plane by an equation which contained the terms x^2 and y^2, but it did not contain the mixed product xy. This means that you have (mostly) restricted yourself to ellipses whose axes are parallel to the x and y axes.

We present only a survey of the names of quadric surfaces which correspond to the most important special cases of equation (II.4.1) in the following. Instead of x_1, x_2, x_3, we use the denotation $x,\ y,\ z$ in the rest of this chapter.

II.4.2. Circular quadric surfaces (quadric surfaces of revolution). A surface which arises by the rotation of a conic section about its axis is called a *circular quadric surface* or a *quadric surface of revolution*. Particularly, the rotation of

- an ellipse about its axis yields .... *a circular ellipsoid,*
- a parabola about its axis yields .... *a circular paraboloid,*
- a hyperbola about its non–focal axis yields *a one sheet–circular hyperboloid,*
- a hyperbola about its focal axis yields .... *a two sheet–circular hyperboloid,*
- two skew lines about the axis of symmetry (lying in their plane) yields .... *a circular conic surface,*
- two parallel lines about the axis (lying in the middle between them) yields ... *a circular cylindrical surface.*

The first four surfaces are the so called *regular quadric surfaces*, the last two are the so called *singular quadric surfaces*.

II.4.3. How to recognize a circular quadric surface. If the equation of a quadric surface depends on x and y only through $x^2 + y^2$ (i.e. x and y appear in the equation only as a part of the expression $x^2 + y^2$) then the quadric surface is circular and its axis of revolution is the z–axis.

This holds due to the following reason: $x^2 + y^2$ is the second power of the distance of the point $[x, y, z]$ from the z–axis. Thus, x and y appear in the equation only through the distance of $[x, y, z]$ from the z–axis. The distance is the same on each circle which lies in a plane perpendicular to the z–axis and such that the z–axis passes through its center. This means that either all points of the circle satisfy the equation (and all points of the circle belong to the quadric surface) or no point of the circle satisfy the equation (and therefore no point of the circle belongs to the quadric surface).

Analogous assertions can also be formulated in the cases when the equation of some quadric surface depends on x, z only through x^2+z^2 or if it depends on y, z only through $y^2 + z^2$.

II.4.4. Quadric surfaces in the basic and in a shifted position. Conic sections in a plane can be in a *basic position* (for example the ellipse $x^2/a^2 + y^2/b^2 = 1$), or in a *shifted position* (as the ellipse $(x-\alpha)^2/a^2 + (y-\beta)^2/b^2 = 1$). By analogy, quadric surfaces in $\mathbb{E}_3$ can also be in a basic or in a shifted position. We restrict ourselves to quadric surfaces in the basic position in this text. Equations for the same surfaces in the shifted position (shifted by the vector (α, β, γ)) can be obtained from the equations for surfaces in the basic position by replacing x^2 by $(x-\alpha)^2$, y^2 by $(y-\beta)^2$ and z^2 by $(z-\gamma)^2$.

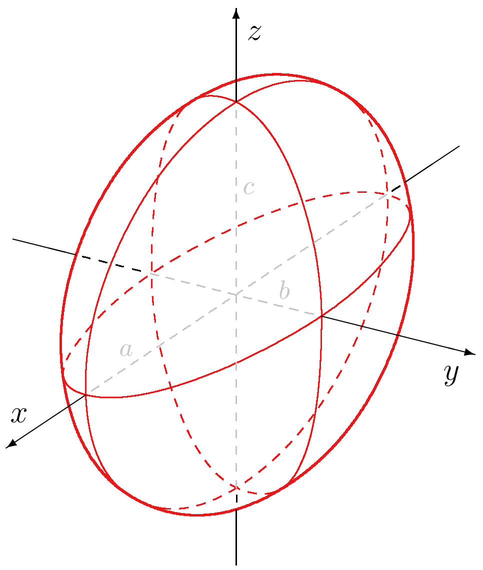


Fig. 4

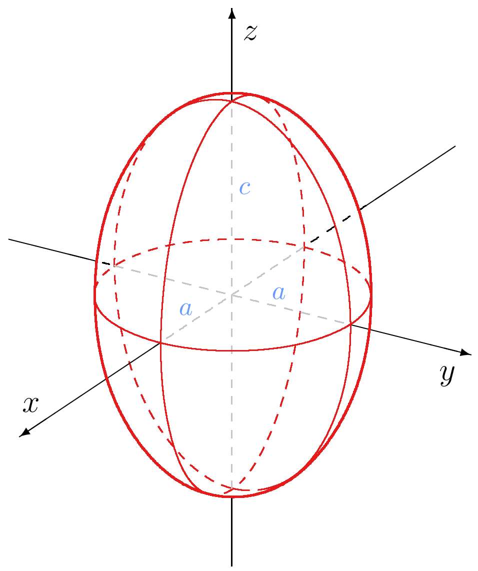


Fig. 5

II.4.5. Ellipsoid. Let a, b, c be positive numbers. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{II.4.2}$$

defines the so called *ellipsoid* with semi–axes a, b, c. The center of the ellipsoid is the point $[0,0,0]$. (See Fig. 4.)

If any two of the semi–axes are equal, the ellipsoid is circular (= the ellipsoid of revolution). (See Fig. 5.) If all three semi–axes are equal, the ellipsoid coincides with a sphere.

II.4.6. One sheet – hyperboloid. Let a, b, c be positive numbers. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{II.4.3}$$

is the equation for the so called *one sheet – hyperboloid* (= a *hyperboloid of one sheet*). (See Fig. 6.)

If $a = b$ then it is a circular one sheet – hyperboloid (= a one sheet – hyperboloid of revolution) which arises e.g. by revolution of the hyperbola $x^2/a^2 - z^2/c^2 = 1$ in the x, z–plane about the z–axis.

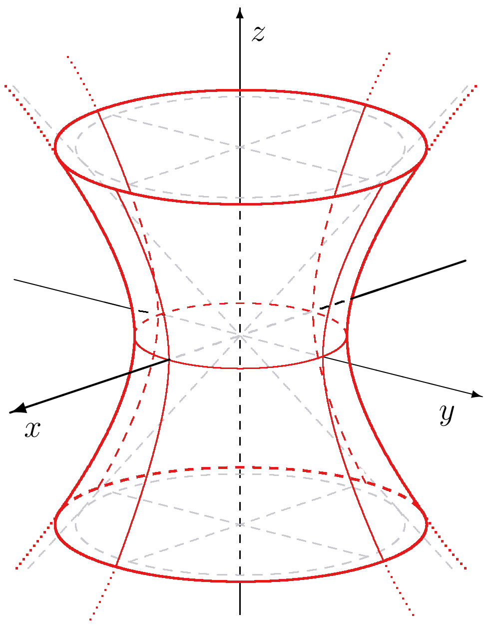


Fig. 6

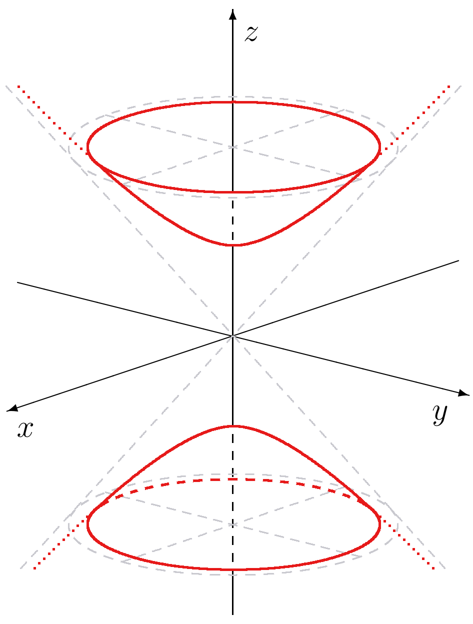


Fig. 7

II.4.7. Two sheet – hyperboloid. Let a, b, c be positive numbers. The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{II.4.4}$$

defines the so called *two sheet – hyperboloid* (= a *hyperboloid of two sheets*). (See Fig. 7.)

If $b = c$ then the hyperboloid is circular and its axis of revolution is the x–axis. It arises e.g. by revolution of the hyperbola $x^2/a^2 - z^2/c^2 = 1$ in the x, z–plane about the z–axis.

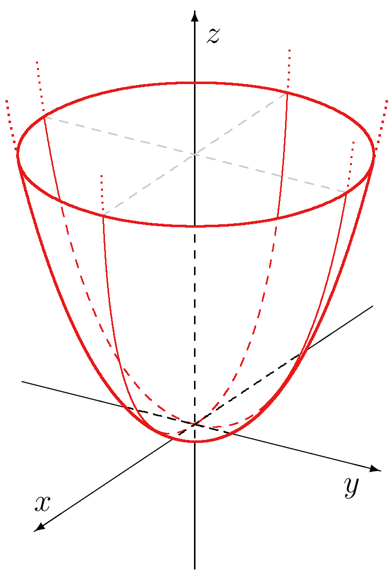


Fig. 8

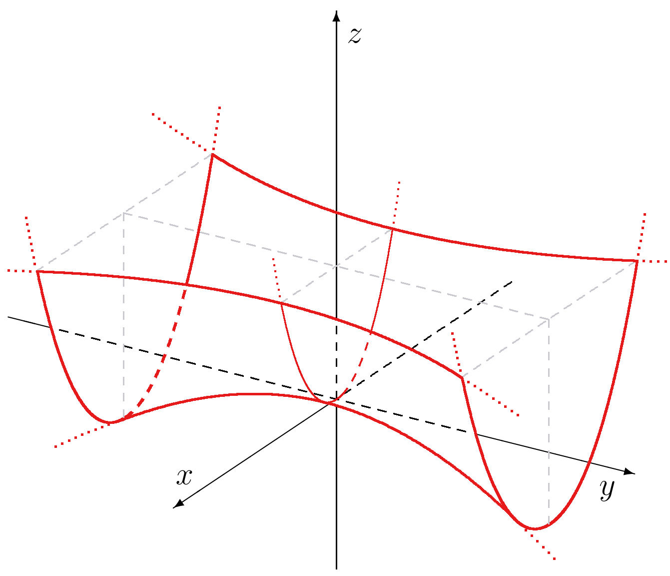


Fig. 9

II.4.8. Paraboloid. Let a, b be positive numbers. The equation

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{II.4.5}$$

is the equation for the so called *elliptic paraboloid*. Its axis is the z–axis. (See Fig. 8.)

If $a = b$ then the paraboloid is circular (in other words: it is a paraboloid of revolution). It arises e.g. by revolution of the parabola $z = x^2/a^2$ in the x, z–plane about the z–axis.

The equation

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \tag{II.4.6}$$

defines the so called *hyperbolic paraboloid*. (See Fig. 9.) This paraboloid intersects every plane of the type $z = c$ (where $c > 0$) in a hyperbola.

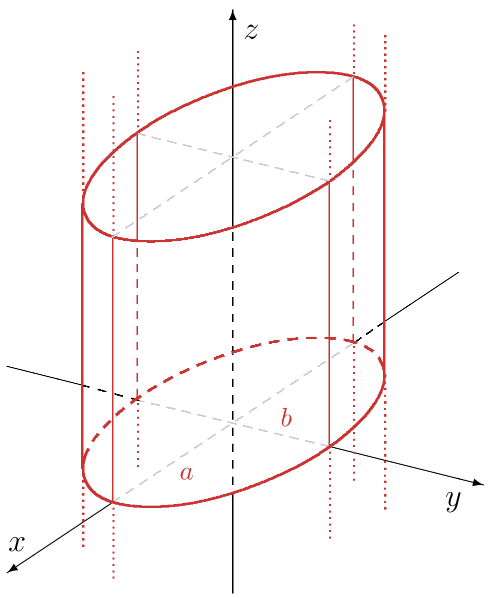


Fig. 10

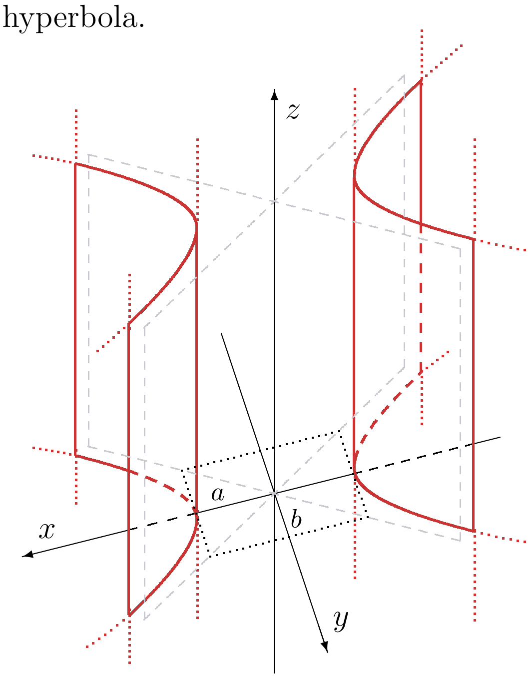


Fig. 11

II.4.9. Cylindrical surface, cylinder. Let a, b be positive numbers. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{II.4.7}$$

defines the so called *elliptic cylindrical surface* (= an *elliptic cylinder*). (See Fig. 10.)

If $a = b$ then it is a circular cylindrical surface (= a cylindrical surface of revolution). This arises e.g. by rotation of the parallel lines $x = a, y = 0$ and $x = -a, y = 0$ about the z–axis.

The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{II.4.8}$$

defines the so called *hyperbolic cylindrical surface* (= a *hyperbolic cylinder*). (See Fig. 11.)

The equation

(II.4.9) $$y = k\,x^2$$

defines a <u>*parabolic cylindrical surface*</u> (= a <u>*parabolic cylinder*</u>). (See Fig. 12.)

II.4.10. Conic surface, cone. Let a, b, c be positive numbers. The equation

(II.4.8) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

describes the so called <u>*elliptic conic surface*</u> (= an <u>*elliptic cone*</u>).

If $b = c$ then the cone is circular. This arises for example by rotation of the straight lines $x/a = z/c,\ y = 0$ and $x/a = z/c,\ y = 0$ about the z–axis.

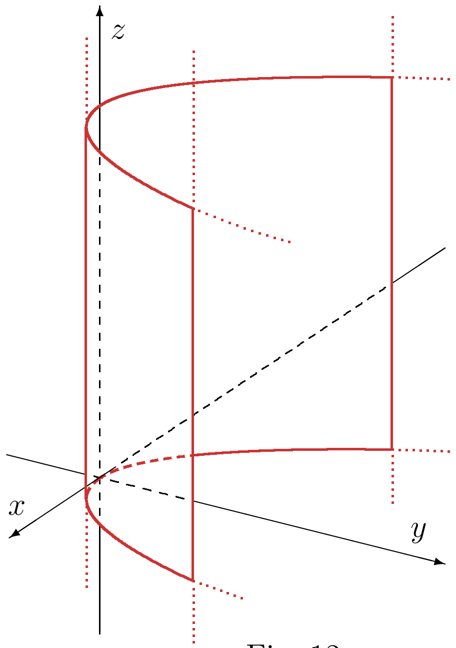


Fig. 12

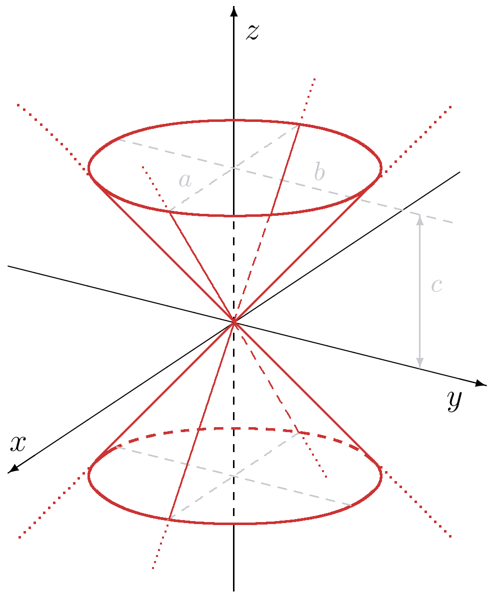


Fig. 13

You will again meet with quadric surfaces in the Mathematics II course, especially in the chapter on surface integrals.

II.4.11. Problems. The given equations define various quadric surfaces. Recognize their types, give them appropriate names and specify their axes, possibly also semi–axes.

a) $x^2 + y^2/4 + 9z^2 = 1$

b) $25\,(x-1)^2 + y^2 + z^2 = 25$

c) $x^2/4 + y^2/4 - z^2/9 = 1$

d) $x^2 - (y-2)^2/4 - (z-4)^2/4 = 1$

e) $z = x^2 + 4y^2$

f) $y = 1 - 5x^2 - 5(z-2)^2$

g) $y^2 - x^2 = z$

h) $(x+2)^2 - 5z^2 = y$

i) $x^2 + z^2 = 4$

j) $x^2/4 + (y+4)^2 = 1$

k) $4x^2 + 9z^2 = 9y^2$

l) $(y-1)^2 + z^2 = x^2$

m) $x^2 - 4z^2 = 1$

n) $(y+2)^2/4 - x^2/16 = 1$

o) $4x^2 - 9z^2 = 9y^2$

p) $y = x^2 + 2$

q) $x^2 + 4y^2 = 1$

r) $(x-4)^2/4 + y^2/16 + z^2 = 1$

R e s u l t s :

a) Ellipsoid with center $[0,0]$ and semi–axes 1, 2, $\frac{1}{3}$.

b) Circular ellipsoid with center at the point $[1,0,0]$ and semi–axes 1, 5, 5. Axis of revolution is the x–axis.

c) Circular one–sheet hyperboloid with center at the origin. Axis of revolution is the z–axis.

d) Circular two–sheet hyperboloid with center $[0,2,4]$. Axis of revolution is the straight line $y=2$, $z=4$, parallel with the x–axis.

e) Elliptic paraboloid with center at the origin and axis z, open in the positive direction of the z–axis.

f) Circular paraboloid with vertex $[0,1,2]$, open in the negative direction of the y–axis. Axis of revolution is the straight line $x=0$, $z=2$, parallel with the y–axis.

g) Hyperbolic paraboloid with vertex at the origin and axis z.

h) Hyperbolic paraboloid with vertex $[-2,0,0]$ and axis y.

i) Circular cylindrical surface. Axis of revolution is the y–axis and radius equals 2.

j) Elliptic cylindrical surface. Its axis is the straight line $x=0$, $y=4$, parallel with the z–axis.

k) Elliptic conic surface with vertex at the origin and axis y.

l) Circular conic surface with vertex $[0,1,0]$. Axis of revolution is the straight line $y=1$, $z=0$, parallel with the x–axis.

m) Hyperbolic cylindrical surface, parallel with the y–axis, symmetric according to this axis.

n) Hyperbolic cylindrical surface, parallel with the z–axis, symmetric according to the straight line $x=0$, $y=-2$.

o) Circular conic surface with vertex at the origin. Axis of revolution is the x–axis.

p) Parabolic cylindrical surface, parallel with the z–axis. It meets the x, y–plane in the parabola with vertex $[0,2]$ and axis y. (Its equation in the $x,,y$–plane is identical with the equation of the whole cylindrical surface: $y=x^2+2$.)

q) Elliptic cylindrical surface. Its axis is the z–axis.

r) Ellipsoid with center at the point $[4,0,0]$ and semi–axes 2, 4, 1.