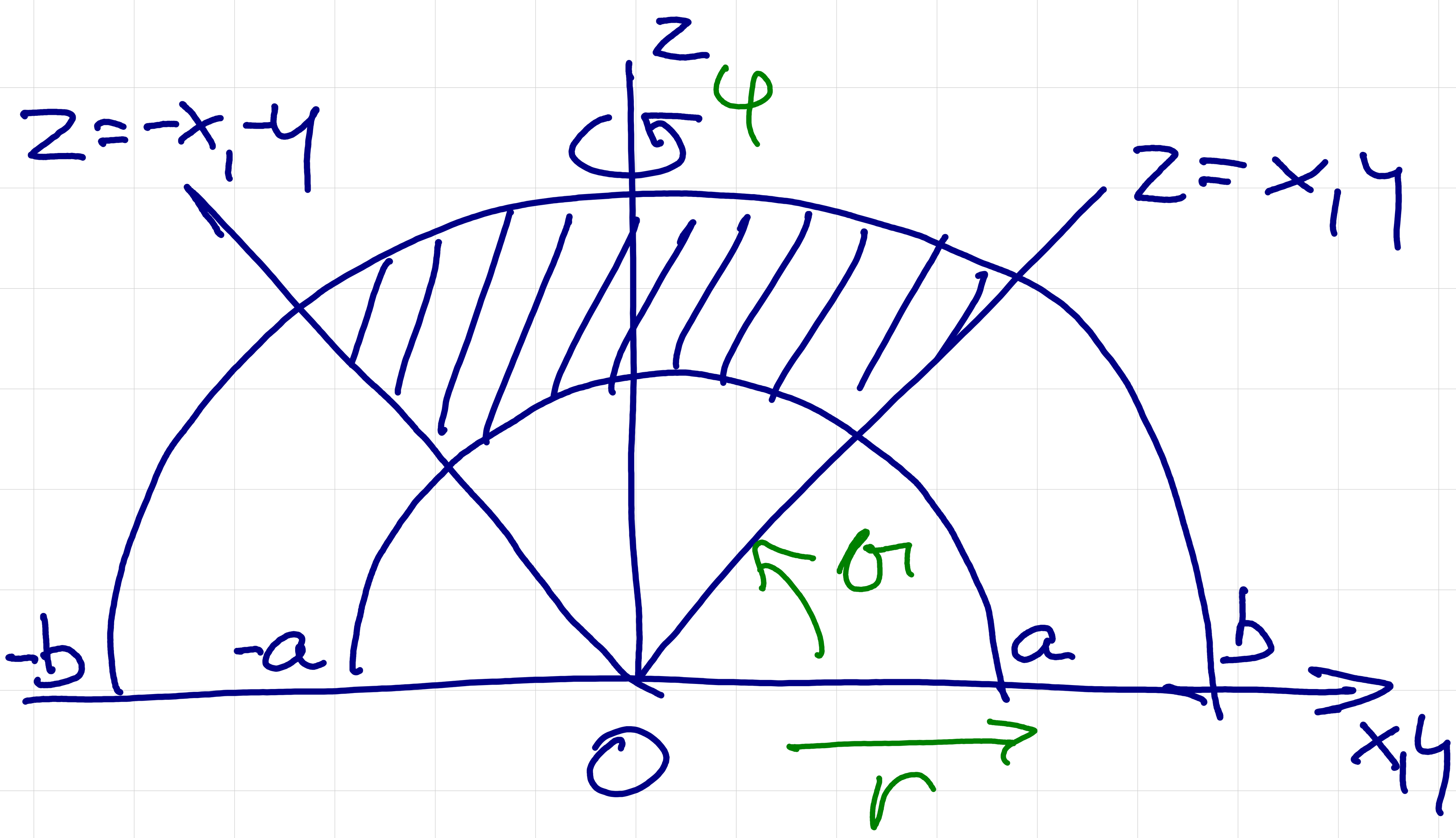
① $x^2+y^2+z^2 \le a^2 \qquad 0<a<b$

$x^2+y^2+z^2 \le b^2$

$x^2+y^2 = z^2 \quad (z \ge 0) \Rightarrow z = \pm\sqrt{x^2+y^2}$



SPHERICAL COORDINATES

$a \le r \le b$

$0 \le \varphi \le 2\pi$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$x = r\cos\theta\cos\varphi$

$y = r\cos\theta\sin\varphi$

$z = r\sin\theta$

$J = r^2\cos\theta$

$$V = \iiint_\Omega 1\,dx\,dy\,dz =$$

$$= \int_0^{2\pi}\left(\int_a^b\left(\int_{\pi/4}^{\pi/2} 1\cdot r^2\cos\theta\,d\theta\right)dr\right)d\varphi =$$

$$= 2\pi\cdot\int_a^b r^2\,dr\cdot\left[\sin\theta\right]_{\pi/4}^{\pi/2} = 2\pi\cdot\left[\frac{r^3}{3}\right]_a^b\cdot\left(1-\frac{\sqrt{2}}{2}\right) =$$

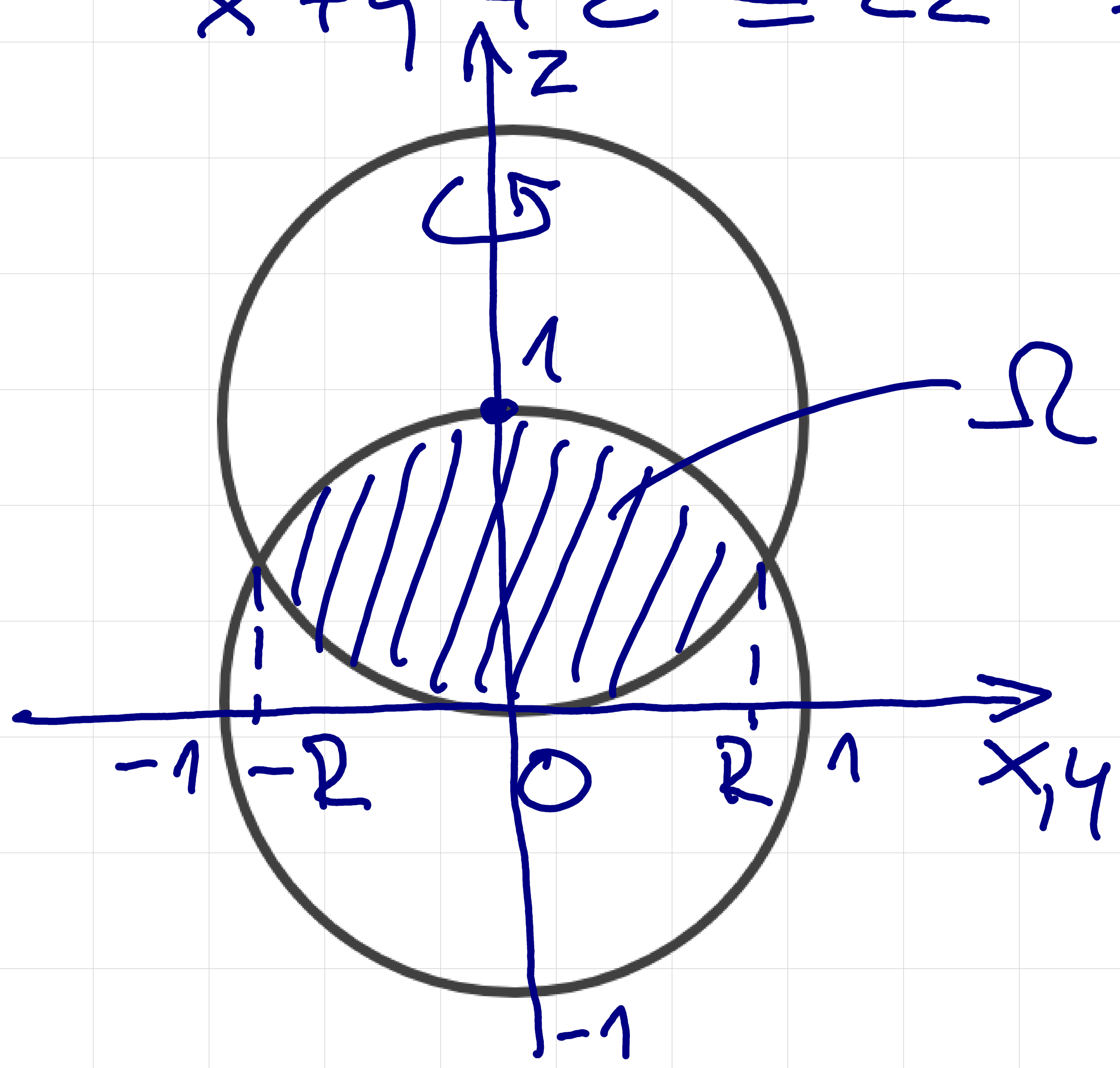
$$= (2-\sqrt{2})\pi\cdot\frac{1}{3}(b^3-a^3)$$

(2)

$$\iiint_{\Omega} z^2\,dx\,dy\,dz = ?$$

$\Omega:\ x^2+y^2+z^2 \le 1 \implies z = \pm\sqrt{1-(x^2+y^2)} = \pm\sqrt{1-r^2}$

$x^2+y^2+z^2 \le 2z \implies x^2+y^2+\underbrace{(z^2-2z+1)}_{(z-1)^2} \le 1 \implies z = 1\pm\sqrt{1-(x^2+y^2)} = 1\pm\sqrt{1-r^2}$



$1-z^2 = 2z-z^2$

$1 = 2z \implies z = 1/2\ ?$

$x^2+y^2 \le 1-z^2 = \frac{3}{4} \implies R = \sqrt{\frac{3}{4}}$

$x^2+y^2 \le 2z-z^2$

CYLINDRICAL COORDINATES

$x = r\cos\varphi$ $\qquad 0 \le r \le \sqrt{\frac{3}{4}}$

$y = r\sin\varphi$ $\qquad 0 \le \varphi \le 2\pi$

$z = w$ $\qquad 1-\sqrt{1-r^2} \le w \le \sqrt{1-r^2}$

$J = r$

$$\iiint_{\Omega} z^2\,dx\,dy\,dz = \int_0^{2\pi}\left(\int_0^{\sqrt{3/4}}\left(\int_{1-\sqrt{1-r^2}}^{\sqrt{1-r^2}} w^2\cdot r\,dw\right)dr\right)d\varphi =$$

$$= \frac{2\pi}{3}\int_0^{\sqrt{3/4}}\left[w^3\right]_{1-\sqrt{1-r^2}}^{\sqrt{1-r^2}}\cdot r\,dr = \frac{2\pi}{3}\int_0^{\sqrt{3/4}}\left(\left(\sqrt{1-r^2}\right)^3-\left(1-\sqrt{1-r^2}\right)^3\right)\cdot r\,dr =$$

$$= \left|\begin{array}{l} t = 1-r^2 \\ dt = -2r\,dr \\ r=0 \Rightarrow t=1,\ r=\sqrt{3/4} \Rightarrow t=\frac{1}{4} \end{array}\right| = -\frac{\pi}{3}\int_1^{1/4}\left(t^{3/2}-\left(1-t^{1/2}\right)^3\right)dt =$$

$$= -\frac{\pi}{3}\int_{1}^{1/4}\left(t^{3/2}-(1-t^{1/2})^{3}\right)dt = \frac{\pi}{3}\int_{1/4}^{1}\left(t^{3/2}-(1-3t^{1/2}+3t-t^{3/2})\right)dt =$$

$$= \frac{\pi}{3}\left[2\cdot\frac{2}{5}t^{5/2}+3\,\frac{2}{3}t^{3/2}-3\,\frac{1}{2}t^{2}-t\right]_{1/4}^{1} =$$

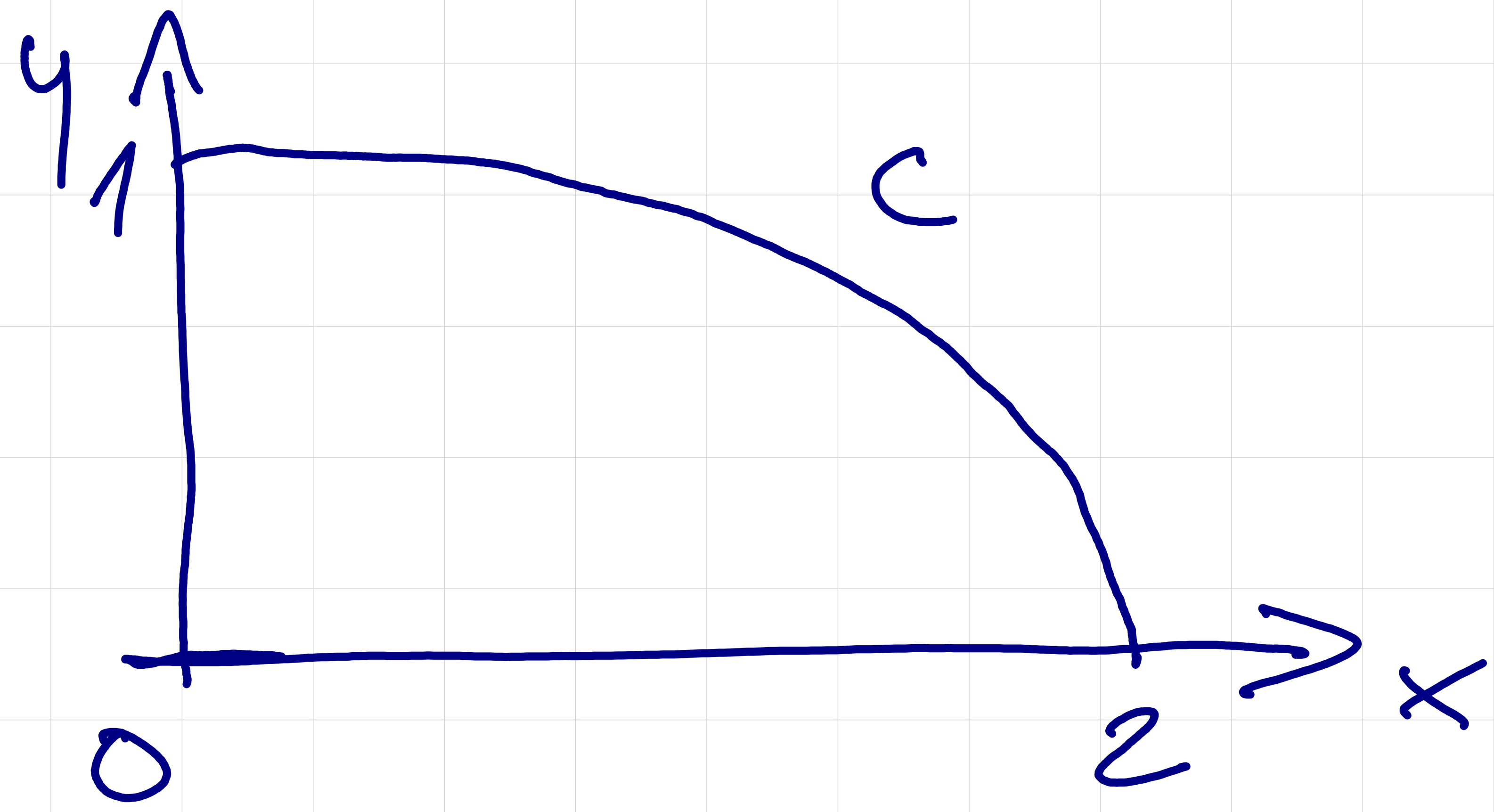
$$= \frac{\pi}{3}\left(\frac{4}{5}+\underbrace{2-\frac{3}{2}-1}_{-1/2}-\frac{4}{5}\cdot\frac{1}{32}-\cancel{2\cdot\frac{1}{8}}+\frac{3}{2}\cdot\frac{1}{16}+\cancel{\frac{1}{4}}\right) =$$

$$= \frac{\pi}{3}\left(\frac{4}{5}-\frac{1}{2}-\frac{1}{5\cdot 8}+\frac{3}{2\cdot 16}\right) = \frac{\pi}{3}\,\frac{128-80-4+15}{160} =$$

$$= \frac{59}{480}\pi$$

③ $m_x = ?\quad, \rho(x,y) = x$

$$C = \left\{[x;y] \in E_2 : \frac{x^2}{4} + y^2 = 1,\ x \geq 0,\ y \geq 0\right\}$$



$$x = 2\cos t$$
$$y = 1\sin t \qquad t \in \langle 0; \pi/2\rangle$$

$$P(t) = [2\cos t; \sin t] \qquad t \in \langle 0; \pi/2 \rangle$$

$$\dot{P}(t) = (-2\sin t; \cos t)$$

$$\|\dot{P}(t)\| = \sqrt{4\sin^2 t + \cos^2 t} = \sqrt{\sin^2 t + \cos^2 t + 3\sin^2 t} =$$
$$= \sqrt{1 + 3\sin^2 t}$$

$$m_x = \int_C y\rho\, ds = \int_C \underbrace{xy}_{f}\, ds = \int_0^{\pi/2} \underbrace{2\sin t\cos t}_{f(P(t))} \underbrace{\sqrt{1+3\sin^2 t}}_{\|\dot{P}(t)\|}\, dt =$$

$$= \left| \begin{array}{l} u = 1 + 3\sin^2 t \\ du = 3\cdot 2\sin t\cos t \\ t = 0 \Rightarrow u = 1,\ t = \pi/2 \Rightarrow u = 4 \end{array} \right| = \frac{1}{3}\int_1^4 \sqrt{u}\, du = \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_1^4 =$$

$$= \frac{1}{3}\cdot\frac{2}{3}\left(2^3 - 1^3\right) = \frac{14}{9}$$

(4) $\quad l = ?$

a) $P(t) = [3t; 3t^2; 2t^3] \quad t \in \langle 0; 1 \rangle$

$$\dot{P}(t) = (3; 6t; 6t^2)$$

$$\|\dot{P}(t)\| = \sqrt{9 + 36t^2 + 36t^4} = 3\sqrt{4t^4 + 4t^2 + 1} =$$

$$= 3\sqrt{(2t^2+1)(2t^2+1)} = 3(2t^2+1)$$

$$l = \int_C 1\, ds = \int_0^1 1 \cdot 3(2t^2+1)\, dt = 3\left[2\frac{t^3}{3} + t\right]_0^1 = 3 \cdot \frac{5}{3} = 5$$

b) $P(t) = [a\cos t; a\sin t; bt] \quad t \in \langle 0; 2\pi \rangle$

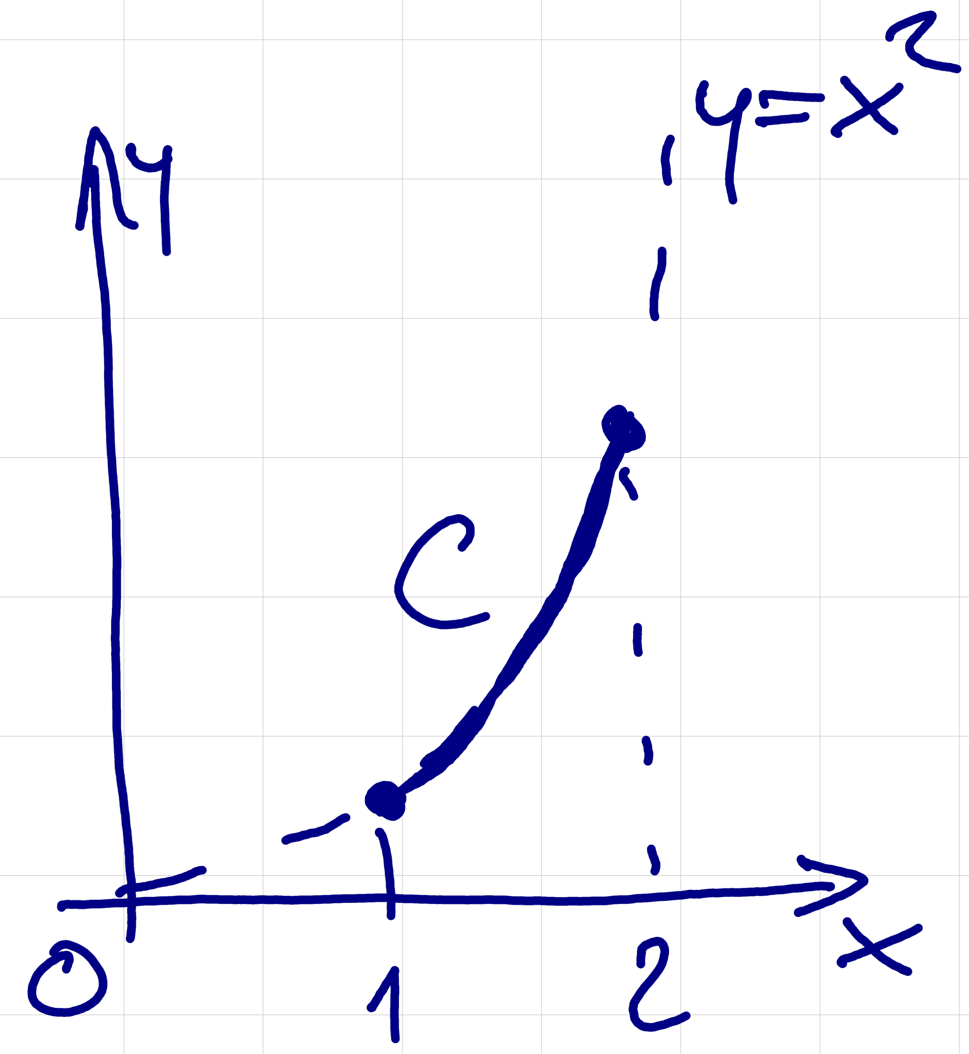
$$\dot{P}(t) = (-a\sin t; a\cos t; b)$$

$$\|\dot{P}(t)\| = \sqrt{\underbrace{a^2\sin^2 t + a^2\cos^2 t}_{a^2} + b^2} = \sqrt{a^2 + b^2}$$

$$l = \int_C 1\, ds = \int_0^{2\pi} 1\sqrt{a^2+b^2}\, dt = \sqrt{a^2+b^2}\int_0^{2\pi} 1\, dt = 2\pi\sqrt{a^2+b^2}$$

(5)

a) $C = \{[x;y] \in E_2 : y = x^2,\ x \in \langle 1;2 \rangle\}$



$P(t) = [t;\ t^2] \quad t \in \langle 1;2 \rangle$

$\dot{P}(t) = (1;\ 2t)$

$\|\dot{P}(t)\| = \sqrt{1+4t^2}$

$$\int_C x\,ds = \int_1^2 t\sqrt{1+4t^2}\,dt = \left| \begin{matrix} u = 1+4t^2 & t=1 \Rightarrow u=5 \\ du = 8t\,dt & t=2 \Rightarrow u=17 \end{matrix} \right| =$$

$$= \frac{1}{8}\int_5^{17} \sqrt{u}\,du = \frac{1}{8}\left[\frac{u^{3/2}}{3/2}\right]_5^{17} = \frac{1}{8}\,\frac{2}{3}\left(17^{3/2} - 5^{3/2}\right) =$$

$$= \frac{1}{12}\left(\sqrt{17^3} - \sqrt{5^3}\right)$$

$\int \rho\,ds \Rightarrow m: \rho = x \geq 0$

$\int_C x\,ds = \quad \int x\rho\,ds \Rightarrow m_y: \rho = 1 \geq 0$

$\int x^2\rho\,ds \Rightarrow J_y: \rho = \frac{1}{x} \geq 0$

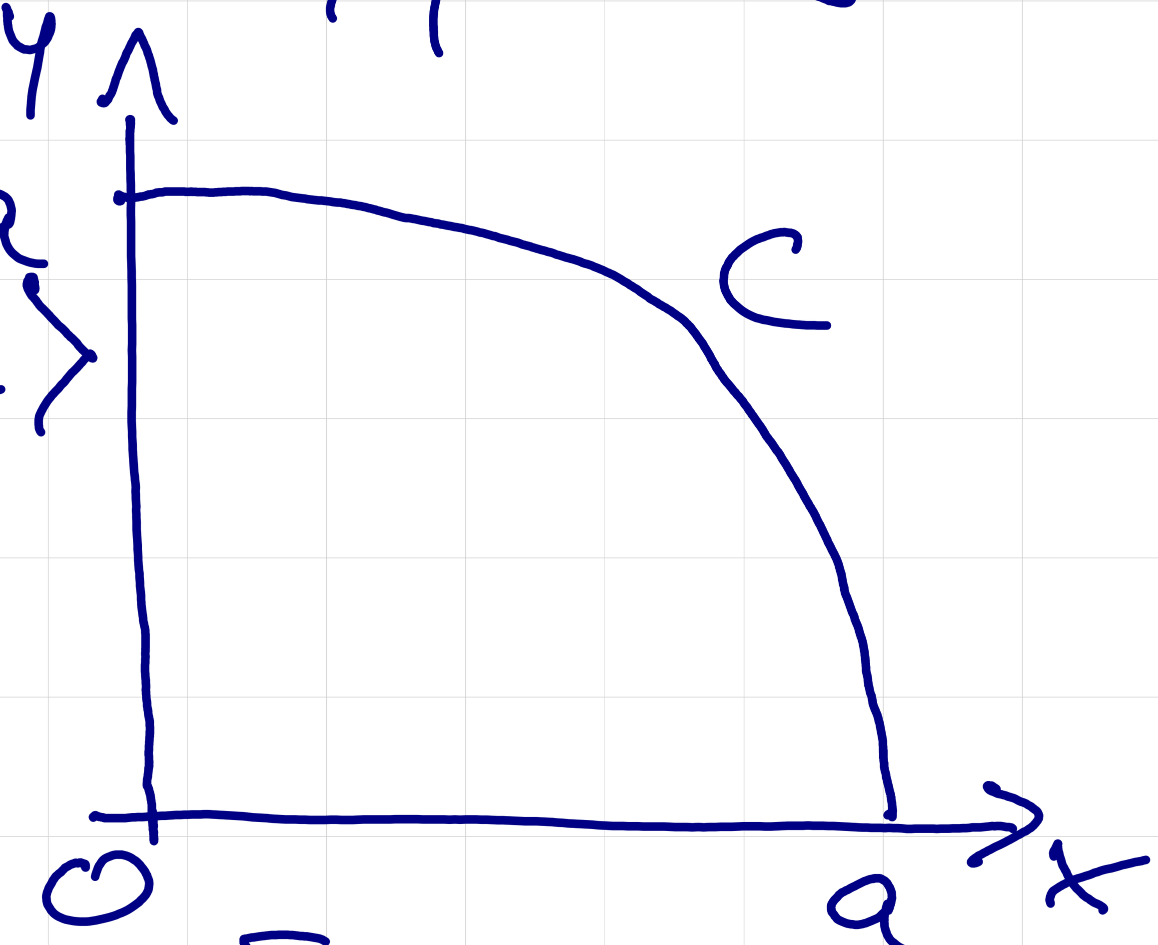
$\int y\rho\,ds \Rightarrow m_x: \rho = \frac{x}{y} \geq 0$

$\vdots$

ρ finite on C

b) $C = \{[x;y] \in E_2 : x^2+y^2=a^2,\ x \geq 0,\ y \geq 0\}$

$P(t) = [a\cos t;\ a\sin t]$ $\quad t \in \langle 0; \frac{\pi}{2} \rangle$



$\dot{P}(t) = (-a\sin t;\ a\cos t)$

$\|\dot{P}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

$$\int_C x^2 y \, ds = \int_0^{\pi/2} a^2\cos^2 t \, a\sin t \cdot a \, dt = a^4 \int_0^{\pi/2} \cos^2 t \sin t \, dt =$$

$$= \left| \begin{array}{l} u = \cos t \\ du = -\sin t \, dt \\ t=0 \Rightarrow u=1;\ t=\pi/2 \Rightarrow u=0 \end{array} \right| = a^4 \int_0^1 u^2 \, du = a^4 \left[\frac{u^3}{3} \right]_0^1 = \frac{a^4}{3}$$

OR $\quad x^2+y^2=a^2 \Rightarrow y^2 = a^2-x^2 \Rightarrow y = \pm\sqrt{a^2-x^2}$

$P(t) = [t;\ \sqrt{a^2-t^2}]$ $\quad t \in \langle 0; a \rangle$

$$\dot{P}(t) = \left(1;\ \frac{-\cancel{2}t}{\cancel{2}\sqrt{a^2-t^2}}\right)$$

$$\|\dot{P}(t)\| = \sqrt{1 + \frac{t^2}{a^2-t^2}} = \sqrt{\frac{a^2-t^2+t^2}{a^2-t^2}} = \frac{a}{\sqrt{a^2-t^2}}$$

$$\int_C \underbrace{x^2 y}_{f} \, ds = \int_0^a \underbrace{t^2 \cancel{\sqrt{a^2-t^2}}}_{f(P(t))} \cdot \underbrace{\frac{a}{\cancel{\sqrt{a^2-t^2}}}}_{\|\dot{P}(t)\|} \, dt = a\int_0^a t^2 \, dt = a\left[\frac{t^3}{3}\right]_0^a = \frac{a^4}{3}$$

$$\int_C \rho \, ds \Rightarrow m : \rho = x^2 y \geq 0$$

$$\int_C x^2 y \, ds = \int_C y \rho \, ds \Rightarrow m_x : \rho = x^2 \geq 0 \quad \text{on } C$$

$$\int_C x^2 \rho \, ds \Rightarrow J_y : \rho = y \geq 0$$

c) $C = \{[x;y;z] \in E_3 : x = a\cos t,\ y = a\sin t,\ z = bt,\ t \in \langle 0; 2\pi \rangle\}$

$P(t) = [a\cos t; a\sin t; bt] \quad t \in \langle 0; 2\pi \rangle$

$\dot{P}(t) = (-a\sin t; a\cos t; b)$

$\|\dot{P}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$

$$\int_C x^2(x^2+y^2)\,ds = \int_0^{2\pi} a^2\cos^2 t\,(a^2\cos^2 t + a^2\sin^2 t)\sqrt{a^2+b^2}\,dt =$$

$$= a^4\sqrt{a^2+b^2}\int_0^{2\pi}\cos^2 t\,dt = a^4\sqrt{a^2+b^2}\cdot\frac{1}{2}\int_0^{2\pi}(1+\cos 2t)\,dt =$$

$$= a^4\sqrt{a^2+b^2}\cdot\frac{1}{2}\Big(\underbrace{\int_0^{2\pi} 1\,dt}_{2\pi} + \underbrace{\int_0^{2\pi}\cos 2t\,dt}_{0}\Big) = \pi a^4\sqrt{a^2+b^2}$$

$\int_C \rho\,ds \Rightarrow m: \rho = x^2(x^2+y^2) \geq 0$

$\int_C x^2\rho\,ds \Rightarrow J_{yz}: \rho = (x^2+y^2) \geq 0$

$\int_C x^2(x^2+y^2)\,ds = \int_C (x^2+y^2)\rho\,ds \Rightarrow J_z: \rho = x \geq 0$ on C

$\int_C x\rho\,ds \Rightarrow m_y: \rho = x(x^2+y^2) \geq 0$