Mathematics II – Examples

V. Surface integral

V.1. Parametrization of surfaces

- Suggest parametrization of a surface Q, which has orientation determined by normal vector $\vec{n}_Q$. Find if the surface Q is oriented in accordance with the suggested parametrization:

Example 591: Q is a parallelogram with vertices $A = [1,1,1],\ B = [1,4,4],\ C = [0,5,6],$ $D = [0,2,3],\ \vec{n}_Q \cdot \vec{k} > 0.$

Example 592: Q is a circle in the plane $x = 2$ with center at the point $[2,-1,3]$ and radius 4, $\vec{n}_Q = (-1,0,0)$

Example 593: $Q = \{[x,y,z] \in \mathbb{E}_3;\ z = x^2 + y^2,\ y \geq 0,\ z \leq 1\},\ \vec{n}_Q([0,0,0]) = (0,0,-1)$

Example 594*: Consider a half of sphere $Q = \{[x,y,z] \in \mathbb{E}_3;\ x^2 + y^2 + z^2 = a^2,\ z \geq 0,\ a > 0\}$, which is oriented in accordance with the normal vector $\vec{n} = (n_1, n_2, n_3)$, where $n_3 \geq 0$. Decide which of the following functions $P(u,v)$ is parametrization of surface Q:

a) $P(u,v) = \left[u, v, \sqrt{a^2 - u^2 - v^2}\right]$, where $[u,v] \in B = \{[u,v] \in \mathbb{E}_2;\ u^2 + v^2 \leq a^2\}$,

b) $P(u,v) = \left[\dfrac{2a^2u}{a^2+u^2+v^2}, \dfrac{2a^2v}{a^2+u^2+v^2}, \dfrac{2a^3}{a^2+u^2+v^2} - a\right]$, where $[u,v] \in B = \{[u,v] \in \mathbb{E}_2;\ u^2 + v^2 \leq a^2\}$,

c) $P(u,v) = [a\cos u \cos v, a \sin u \cos v, a \sin v]$, where $[u,v] \in B = \langle 0, 2\pi\rangle \times \langle 0, \frac{\pi}{2}\rangle$.

Example 595: Surface $Q = \{[x,y,z] \in \mathbb{E}_3;\ x^2 + y^2 = 4,\ x \geq 0,\ z \in \langle 1,4\rangle\}$, is oriented in accordance with normal vector $\vec{n}_Q$, for which $\vec{n}_Q \cdot \vec{i} \geq 0$ in any point.

a) Verify if $P(u,v) = (2\cos u, 2\sin u, v),\ [u,v] \in B = \left\langle -\dfrac{\pi}{2}, \dfrac{\pi}{2}\right\rangle \times \langle 1,4\rangle$ is a parametrization of surface Q and decide about orientation of this surface relative to this parametrization.

b) Justify that the parametrization $P(u,v) = \left[\sqrt{4-u^2}, u, v\right],\ [u,v] \in B = \langle -2,2\rangle \times \langle 1,4\rangle$ is not a parametrization of the surface Q.

Example 596: $Q = \{[x,y,z] \in \mathbb{E}_3;\ x^2 + y^2 = 4,\ x \geq 0,\ 0 \leq z \leq 4\},\ \vec{n}_Q([2,0,2]) = (1,0,0)$

Example 597: $Q = \{[x,y,z] \in \mathbb{E}_3;\ z = xy,\ x^2 + y^2 \leq a^2,\ a > 0\},\ \vec{n}_Q \cdot \vec{k} > 0$

Example 598: $Q = \{[x,y,z] \in \mathbb{E}_3;\ \dfrac{(y-1)^2}{4} + z^2 = 1,\ x \in \langle -1,3\rangle, z > 0\},\ \vec{n}_Q([0,0,1]) = (0,0,-1)$

Example 599: $Q = \{[x,y,z] \in \mathbb{E}_3;\ 2x + 3y + z = 6,\ x \geq 0,\ y \geq 0,\ z \geq 0\},\ \vec{n}_Q = (n_1, n_2, n_3),$ $n_1 > 0$

V.2. Surface integral of a scalar function

- Decide about existence of the given surface integral and if exists compute it:

Example 600: $\iint_Q \frac{xy \ln |x|}{z} \, \mathrm{d}p$, where $Q = \{[x, y, z] \in \mathbb{E}_3;\ (x-2)^2 + y^2 + z^2 = 1,\ z \geq 0\}$

Example 601*: $\iint_Q \frac{\mathrm{d}p}{x^2 + y^2 + z^2 - 1}$, $\quad Q = \{[x, y, z] \in \mathbb{E}_3;\ x^2 + y^2 + (z-3)^2 = a^2,\ a > 0\}$

- Compute given integral on surface $Q \subset \mathbb{E}_3$:

Example 602: $\iint_Q xz \, \mathrm{d}p$, where Q is a triangle $\triangle ABC$ with $A = [1, 0, 0],\ B = [0, 1, 0],$ $C = [0, 0, 1]$.

Example 603: $\iint_Q \sqrt{x^2 + y^2 + 1} \, \mathrm{d}p$, $\quad Q = \{[x, y, z] \in \mathbb{E}_3;\ 2z + x^2 + y^2 = 4,\ z \geq 0\}$

Example 604: $\iint_Q xy \, \mathrm{d}p$, $\quad Q = \{[x, y, z] \in \mathbb{E}_3;\ x^2 + y^2 = 4,\ 0 \leq z \leq 1\}$

Example 605*: $\iint_Q xyz \, \mathrm{d}p$, $\quad Q = \{[x, y, z] \in \mathbb{E}_3;\ y^2 + 9z^2 = 9,\ 1 \leq x \leq 3,\ y \geq 0,\ z \geq 0\}$

Example 606*: $\iint_Q (xy + yz + xz) \, \mathrm{d}p$, $\quad Q:\ y = \sqrt{x^2 + z^2}$, inside the surface $x^2 + z^2 = 2x$.

Example 607: $\iint_Q (x + y + z) \, \mathrm{d}p$, $\quad Q = \{[x, y, z] \in \mathbb{E}_3;\ x^2 + y^2 + z^2 = a^2,\ z \geq 0,\ a > 0\}$

Example 608: $\iint_Q (x^2 + y^2) \, \mathrm{d}p$, $\quad Q = \{[x, y, z] \in \mathbb{E}_3;\ z = \sqrt{x^2 + y^2},\ 0 \leq z \geq 1\}$

Example 609: $\iint_Q x \, \mathrm{d}p$, $\quad Q = \{[x, y, z] \in \mathbb{E}_3;\ z = \sqrt{a^2 - x^2 - y^2}\}$

Example 610: $\iint_Q z \, \mathrm{d}p$, $\quad Q = \{[x, y, z] \in \mathbb{E}_3;\ 2z = x^2 + y^2,\ 0 \leq z \geq 1\}$

V.3. Application of surface integral of a scalar function

- Compute area of a surface $Q \subset \mathbb{E}_3$ using a surface integral.

Example 611: Q is a part of the plane $12x + 3y + 4z = 12$ in the first octant.

Example 612: Q is a part of the sphere $x^2+y^2+z^2 = 16$ inside the cylindrical surface $x^2+y^2 = 9$.

Example 613: Q is a part of the surface $z = 2xy$ in the first octant and inside the cylindrical surface $x^2 + y^2 = a^2$.

Example 614: Q is a part of the conical surface $z = \sqrt{x^2 + y^2}$ inside the cylindrical surface $x^2 + y^2 = 2x$.

Example 615: Compute mass of the surface $Q = \{[x, y, z] \in \mathbb{E}_3;\ x^2 + y^2 + z^2 = 9,\ x \leq 0\}$, if the density is $\rho(x, y, z) = \dfrac{1}{z^2 + 9}$.

Example 616: Compute moment of inertia relative to z-axis of the surface $Q = \{[x, y, z] \in \mathbb{E}_3;\ z = \sqrt{x^2 + y^2},\ O \leq x \leq 2\}$ with constant density $\rho(x, y, z) = k$.

Example 617: Find center of mass of a part of the paraboloid $x^2 + y^2 = 2z$, bounded by the plane $z = 1$, if the density is $\rho(x, y, z) = 1$.

- Consider a surface $Q \subset \mathbb{E}_3$.

  a) Sketch the given surface along with its projection into xy-plane.

  b) Suggest an appropriate parametrization of surface Q and compute length of a normal vector to the surface Q under this parametrization.

  c) Compute area of the given surface.

Example 618: $Q = \{[x, y, z] \in \mathbb{E}_3;\ x + 2y + z = 6,\ 1 \leq x^2 + y^2 \leq 4\}$

Example 619: $Q = \{[x, y, z] \in \mathbb{E}_3;\ z = y^2 - x^2,\ 1 \leq x^2 + y^2 \leq 4\}$

Example 620: $Q = \{[x, y, z] \in \mathbb{E}_3;\ z = 4 - \sqrt{x^2 + y^2},\ 0 \leq z \leq 2\}$

Example 621: Q is a part of sphere $x^2 + y^2 + z^2 = 12$ inside the paraboloid $x^2 + y^2 = 4z$.

- Compute area of a surface $Q \subset \mathbb{E}_3$.

Example 622: $Q = \{[x, y, z] \in \mathbb{E}_3;\ z = x^2 + y^2,\ x^2 + y^2 \leq 1\}$

Example 623: $Q = \{[x,y,z] \in \mathbb{E}_3;\ 3x + 4y + z = 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$

Example 624: Q is a part of conical surface $z = 4\sqrt{x^2 + y^2}$ bounded by the plains $x = 0$, $y = 0, 2x + 3y = 6$.

Example 625: Q is a part of plane $2x + y - z = 0$ inside the elliptical cylindrical surface $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

Example 626: $Q = \{[x,y,z] \in \mathbb{E}_3;\ x = u + v,\ y = u - v,\ z = 4v, [u,v] \in \langle 0,1\rangle \times \langle 0,1\rangle\}$

- Consider a surface Q and areal mass density $\rho(x,y,z)$.

  a) Sketch the given surface along with its projection into plane $x = 0$.

  b) Suggest an appropriate parametrization of surface Q and compute length of a normal vector to the surface Q under this parametrization.

  c) Compute mass of the given surface.

Example 627: $Q = \{[x,y,z] \in \mathbb{E}_3;\ z = 4 - x^2 - y^2,\ z \geq 0\}$, $\rho(x,y,z) = \sqrt{1 + 4(x^2 + y^2)}$

Example 628: $Q = \{[x,y,z] \in \mathbb{E}_3;\ x^2 + y^2 = 4,\ x \geq 0,\ y \geq 0,\ z \in \langle 0,3\rangle\}$, $\rho(x,y,z) = xyz$

Example 629: $Q = \{[x,y,z] \in \mathbb{E}_3;\ z^2 = x^2 + y^2,\ x \geq 0,\ z \in \langle 0,1\rangle\}$, $\rho(x,y,z) = \sqrt{x^2 + y^2}$

- Consider a surface Q and a scalar function f.

  a) Sketch the given surface along with its projection into plane $x = 0$.

  b) Compute $\iint_Q f \,\mathrm{d}p$.

  c) Write a possible physical interpretation of the integral in b).

Example 630: $Q = \{[x,y,z] \in \mathbb{E}_3;\ x^2 + y^2 = 4,\ x \geq 0,\ z \in \langle 0,3\rangle\}$, $f(x,y,z) = xy^2$

Example 631: $Q = \{[x,y,z] \in \mathbb{E}_3;\ z = \frac{1}{2}(x^2 + y^2),\ z \in \langle 0,1\rangle\}$, $f(x,y,z) = kz,\ k > 0$

Example 632: $Q = \{[x,y,z] \in \mathbb{E}_3;\ z = \sqrt{9 - x^2 - y^2}\}$, $f(x,y,z) = x^2 + y^2$

Example 633: There is given the surface $Q = \{[x,y,z] \in \mathbb{E}_3;\ z = xy,\ x^2 + y^2 \leq 3\}$.

a) In standalone pictures sketch the three curves, which are given as intersections of graph of the function $z = xy$ with planes $z = 1,\ x - y = 0$ and $x + y = 0$.

b) Suggest an appropriate parametrization of Q. Write a normal vector to the surface Q and compute its length under this parametrization.

c) Compute area of Q.

Example 634: Compute a mass of the surface $Q = \{[x, y, z] \in \mathbb{E}_3;\ x^2+y^2+z^2 = a^2,\ z \geq 0,\ a > 0\}$, if the areal mass density is $\rho(x, y, z) = z$.

Example 635: Compute mass of the surface $Q = \{[x, y, z] \in \mathbb{E}_3;\ x + y + z = 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\ \}$, if the areal mass density is $\rho(x, y, z) = \frac{1}{(1+x+y)^2}$.

Example 636: Compute a center of mass of part of the conic surface $z = \sqrt{x^2 + y^2}$ inside the cylindrical surface $x^2 + y^2 = ax, (a > 0)$, if the mass density is constant $\rho(x, y, z) = k$.

Example 637: Compute a center of mass of the surface $Q = \{[x, y, z] \in \mathbb{E}_3;\ x = \sqrt{y^2 + z^2},\ y \geq 0,\ 0 \leq x \leq 2\}$ if the mass density is $\rho(x, y, z) = x$.

Example 638: Compute coordinate y_T of center of mass of the surface $Q = \{[x, y, z] \in \mathbb{E}_3;\ x^2 + z^2 = 4,\ x \geq 0,\ z \geq 0,\ y \in \langle 0, 3\rangle\ \}$ if the mass density is $\rho(x, y, z) = xyz$.

Example 639: Compute a static moment relative to axis of rotation of the homogeneous half-sphere with radius R, $\rho(x, y, z) = k$.

Example 640: Compute a moment of inertia relative to z-axis of the homogeneous triangle with vertices $[a, 0, 0]$, $[0, a, 0]$, $[0, 0, a]$, $(a > 0,\ \rho(x, y, z) = k)$.

Example 641: Compute a moment of inertia relative to z-axis of the homogeneous surface $Q = Q_1 \cup Q_2$, where $Q_1 = \{[x, y, z] \in \mathbb{E}_3;\ x^2 + y^2 \leq 16,\ z = 0\}$, $Q_2 = \{[x, y, z] \in \mathbb{E}_3;\ z = 4 - \sqrt{x^2 + y^2},\ z \geq 0\}, \rho(x, y, z) = k)$.

Example 642: Compute a moment of inertia relative to z-axis of the homogeneous surface $Q = \{[x, y, z] \in \mathbb{E}_3;\ z^2 = \frac{h^2}{a^2}(x^2 + y^2),\ 0 \leq z \leq h\},\ \rho(x, y, z) = k)$.

V.4. Surface integral of a vector function

- Compute given surface integral $\iint_Q \vec{f} \cdot \mathrm{d}\vec{p}$ on a given surface $Q \subset \mathbb{E}_3$, which is oriented in accordance with a given normal vector:

Example 643:

- Compute a flow of vector field $\vec{f}$ through the surface $Q \subset \mathbb{E}_3$, which is oriented in accordance with a given normal vector:

Example 650:

- Consider a vector function $\vec{f}$ and a surface Q.

  a) Sketch the given surface. Suggest its parametrization and write an orthogonal vector to surface Q.

  b) Compute a flow of a vector field given by $\vec{f}$ through the surface Q oriented with the given normal vector $\vec{n}$.

Example 656:

- Compute a flow of vector field $\vec{f}$ through the surface $Q \subset \mathbb{E}_3$, which is oriented in accordance with a given normal vector:

Example 662: