Mathematics II – Examples

III.4. Fubini's Theorem for triple integral

- **Compute the following triple integral on given region $W \subset \mathbb{E}_3$:**

Example 342: $I = \iiint_W (x^2 + y^2)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z;$

$$W = \{[x,y,z] \in \mathbb{E}_3:\ 0 \le z \le x + y,\ 0 \le y \le 3,\ 0 \le x \le 2\}.$$

Example 343: $I = \iiint_W \frac{x}{y}(z+1)^2\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z;$

$$W = \{[x,y,z] \in \mathbb{E}_3:\ 0 \le x \le 1,\ 1 \le y \le \mathrm{e}^2,\ 0 \le z \le 2\}.$$

Example 344*: $I = \iiint_W z^3 y \sin x\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z;$

$$W = \{[x,y,z] \in \mathbb{E}_3:\ 0 \le z \le \sin x,\ 0 \le y \le \sin^2 x,\ 0 \le x \le \tfrac{\pi}{2}\}.$$

Example 345: Compute $I = \iiint_W \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(1+3z)^3}$, where W is tetrahedron bounded by planes

$$x + 2y + 3z = 6,\ x = 0,\ y = 0,\ z = 0.$$

Example 346: Compute $I = \iiint_W y \cdot \cos(x+z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where the set W is bounded by surfaces

$$y = \sqrt{x},\ y = 0,\ z = 0,\ x + z = \tfrac{\pi}{2}.$$

- **Compute triple integral on region W bounded by given surfaces:**

Example 347: $I = \iiint_W (x + y + z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad W: x = 1,\ y = 0,\ y = x,\ z = 0,\ z = \sqrt{2}$

Example 348: $I = \iiint_W x\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad W: x = 0,\ y = 0,\ z = 0,\ z = xy,\ x + y = 1$

Example 349: $I = \iiint_W x^2 y z^3\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad W: z = xy,\ y = x,\ y = 1,\ z = 0$

Example 350: $I = \iiint_W (x + y)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad W: x = 0,\ y = 0,\ z = 0,\ x = a,\ y = a,\ z = a^2 - x^2 - y^2$

Example 351: $I = \iiint_W xz\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad W: x = 0,\ y = 0,\ z = 0,\ x + y = 1,\ z = 0,\ z = x^2 + y^2 + 1$

III.5. Substitution theorem for triple integral

- **Compute triple integral using transformation into cylindrical coordinates:**

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = w \end{cases} \qquad r > 0; \quad 0 \le \varphi < 2\pi, \qquad J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial w} \end{vmatrix} = r$$

Example 352: $I = \iiint_W (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 :\ x^2 + y^2 \le az,\ z \le a,\ (a > 0)\,\}.$$

Example 353: $I = \iiint_W \sqrt{x^2 + y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 :\ \sqrt{x^2 + y^2} \le z \le 6 - x^2 - y^2\,\}.$$

- **Compute triple integral using transformation into spherical coordinates:**

$$\begin{cases} x = r\cos\varphi\cos\vartheta \\ y = r\sin\varphi\cos\vartheta \\ z = r\sin\vartheta \end{cases} \qquad r > 0;\ 0 \le \varphi < 2\pi,\ -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}; \qquad J = r^2\cos\vartheta$$

Example 354: $I = \iiint_W \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 :\ x^2 + y^2 + z^2 \le 9,\ x \ge 0,\ y \le 0\,\}.$$

Example 355: $I = \iiint_W (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 :\ 1 \le x^2 + y^2 + z^2 \le 9,\ y \ge 0,\ z \le 0\,\}.$$

Example 356: $I = \iiint_W xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 :\ \tfrac{x^2}{4} + \tfrac{y^2}{9} + \tfrac{z^2}{4} \le 1,\ x \le 0,\ y \ge 0,\ z \ge 0\,\}.$$

Example 357: $I = \iiint_W z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 :\ x \le 0,\ y \ge 0,\ z \ge 0,\ \tfrac{x^2}{a^2} + \tfrac{y^2}{b^2} + \tfrac{z^2}{c^2} \le 1\,\}.$$

- **Compute triple integrals:**

Example 358: $I = \iiint_W \frac{1}{\sqrt{x^2+y^2+z^2}-4}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 : \; x^2+y^2+z^2 \le 1, \; y \ge 0, \; z \ge 0\}.$$

Example 359: $I = \iiint_W x^2 y \; \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 : \; z \le 4 - x^2 - y^2, \; y \ge 0, \; z \ge 0\}.$$

Example 360: $I = \iiint_W (x^2+z^2) \; \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 : \; 4x^2+y^2+\tfrac{z^2}{9} \le 1\}.$$

Example 361: $I = \iiint_W \frac{\sqrt{x^2+y^2+z^2+2}}{\sqrt{x^2+y^2+z^2}}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 : \; 1 \le x^2+y^2+z^2 \le 2, \; x \le 0, \; y \le 0, \; z \ge 0\}.$$

Example 362: $I = \iiint_W (x^2+y^2+z^2) \; \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 : \; x^2-2x+y^2 \le 0, \; -1 \le z \le 1\}.$$

Example 363: $I = \iiint_W xy \; \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$

$$W = \{[x,y,z] \in \mathbb{E}_3 : \; 0 \le x^2+z^2 \le 4, \; 0 \le y \le 4-x^2-z^2, \; x \ge 0\}.$$

(*Hint: use cylindrical coordinates* $x = t\cos\varphi, \; y = w, \; z = r\sin\varphi$.)

III.6. Applications of triple integrals

Example 364: Consider the region $T = \{[x,y,z] \in \mathbb{E}_3 : \; [x,y] \in B \subset \mathbb{E}_2, \; 0 \le z \le f(x,y)\}$. The region is located "amongst" xy-plane and graph of the function $z = f(x,y)$, which is continuous on measurable set B in $\mathbb{E}_2$. Show, that volume of the T can be computed using double integral $\iint_B f(x,y)\; \mathrm{d}x\, \mathrm{d}y$. (*Hint: use formula for solid volume* $V = \iiint_T 1 \; dx\, dy\, dz$.)

Remark: Due linearity of integral the above consecution can be generalized to the case, when a solid is bounded by graphs of two functions $f_1(x,y) \le z \le f_2(x,y)$. By direct application of Fubini's theorem we obtain $V = \iint_B (f_2(x,y) - f_1(x,y)) \; \mathrm{d}x\, \mathrm{d}y$.

- **Compute volume of solid W bounded by given surfaces:**

Example 365: $z = x^2 + y^2 + 4,\ x - y = 2,\ x = 0,\ y = 0,\ z = 0$

Example 366: $z = 2(x^2 + y^2),\ z^2 = 16(x^2 + y^2)$

Example 367: $z = 1 - x^2 - 4y^2,\ z = 0$

Example 368: $z = x^2 + y^2,\ z = x + y$

Example 369: Compute mass of a ball with mass density $\rho(x, y, z) = x^2 + y^2 + z^2$.

Example 370: Compute both mass and x-coordinate of center of mass of a solid bounded by planes $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$ having mass density $\rho(x, y, z) = 1$.

Example 371: Find center of mass of a solid bounded by surfaces $z = x^2 + y^2,\ z = 2$ with mass density $\rho(x, y, z) = k$.

Example 372: Find center of mass of a solid $W = \{[x, y, z] \in \mathbb{E}_3 :\ x^2 + y^2 + z^2 \leq a^2,\ z \geq 0\},\ \rho = 1$.

Example 373: Find center of mass of a cone with base $x^2 + y^2 \leq 16$ and apex at the point $[0, 0, 4]$, if the mass density is $\rho(x, y, z) = k$.

Example 374: Compute moment of inertia relative to point $[0, 0, 0]$ of the solid $W = \left\{[x, y, z] \in \mathbb{E}_3 :\ \dfrac{x^2 + y^2}{2} \leq z \leq \sqrt{3 - x^2 - y^2}\right\},\ \rho(x, y, z) = k$.

Example 375: Compute moment of inertia relative to z-axis of homogeneous solid $W = \{[x, y, z] \in \mathbb{E}_3 :\ \sqrt{x^2 + y^2} \leq z \leq 2\}$

Example 376: Compute static moment M_{yz} (relative to yz-plane) of the solid bounded by planes $x = 0,\ y = 0,\ z = 0,\ x + y = a,\ y = h,\ (a > 0,\ h > 0)$ with constant mass density $\rho(x, y, z)$.

Example 377: Compute moment of inertia J_{xy} (relative to xy-plane) of the solid $W = \{[x, y, z] \in \mathbb{E}_3 :\ x^2 - 2x + y^2 \leq 0,\ -1 \leq z \leq 1\},\ \rho = k$.

Example 378: Compute the integral and state its possible physical meaning: $\iiint_W \sqrt{x^2 + y^2 + z^2}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$, $W = \{[x, y, z] \in \mathbb{E}_3 :\ \sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^2 - y^2},\ y \geq 0\}$.

Example 379: Compute mass of a rotational cone around z-axis, with apex at origin, height $h = 4$ and radius of base $a = 1$. Mass of density is varying linearly from $\rho(x, y, z) = 5$ on base to $\rho(x, y, z) = 1$ in apex, depending on distance of a point from base.

- **There is given a set $D \subset \mathbb{E}_3$ and a function $f(x, y, z)$.**
  a) Sketch both the set D and its projection D_{xy} into plane $z = 0$.
  b) Verify assumptions for use of Fubini's theorem and compute volume integral $\iiint_D f(x, y, z)\, \mathbf{d}x\, \mathbf{d}y\, \mathbf{d}z$.
  c) Give some examples of possible physical meaning of this integral. State, if this integral can be interpreted as mass (under which density), statical moment or moment of inertia (under which density of mass, relative to which point or plane).

Example 380: $D = \{[x, y, z] \in \mathbb{E}_3 :\ 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 2 - x - y\},\quad f(x, y, z) = x^2.$

Example 381: $D = \{[x, y, z] \in \mathbb{E}_3 :\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 4 - x^2 - y^2\},\quad f(x, y, z) = xy.$

Example 382: $D = \{[x, y, z] \in \mathbb{E}_3 :\ 0 \le z \le 4 - x^2 - y^2,\ y \ge 0\},\quad f(x, y, z) = x^2 y.$

Example 383: $D = \{[x, y, z] \in \mathbb{E}_3 :\ 0 \le x \le 3,\ x \le y \le 3,\ 0 \le z \le xy\},\quad f(x, y, z) = (x^2 + y^2)z.$

Example 384: $D = \{[x, y, z] \in \mathbb{E}_3 :\ x^2 + y^2 \le 9,\ 0 \le z \le 2\},\quad f(x, y, z) = \sqrt{x^2 + y^2}.$

Example 385: $D = \{[x, y, z] \in \mathbb{E}_3 :\ x^2 + y^2 + z^2 \le R^2,\ z \le 0\},\quad f(x, y, z) = z,$ R is a positive constant.

- **Consider a solid $D \subset \mathbb{E}_3$, which is bounded by given surfaces.**
  a) Sketch both the solid D and its projection D_{xy} into plane $z = 0$.
  b) Express the set D as an elementary region in coordinate system, which is appropriated for computing a volume of D.
  c) Compute volume of D.

Example 386: $D:\quad 3x + 2y = 12,\ x = 0,\ y = 0,\ z = 0,\ z = x^2$

Example 387: $D:\quad x = 2,\ y = 2,\ xy = 1,\ z = 0,\ z = x^2 + y$

Example 388: $D:\quad x^2 + y^2 = 4,\ z = 0,\ z = 4 - x$

Example 389: $D:\quad x^2 + z^2 = 9,\ x = 0,\ x = 2$

Example 390: $D:\quad z = 0,\ z = a^2 - x^2,\ x^2 + y^2 = a^2$

Example 391: $D:\quad z = 0,\ z = 36 - 4x^2 - y^2$

- **Compute volume V of the solid $W \subset \mathbb{E}_3$:**

Example 392: $W:\quad z = 0,\ y + z = 1,\ y = \ln x,\ y = \ln^2 x$

Example 393: $W:\quad z = x^2 + y^2\ z = 18 - x^2 - y^2$

Example 394: $W:\quad 2z \ge x^2 + y^2,\ x^2 + y^2 + z^2 = 3$

Example 395: $W: \;\; x^2+y^2+z^2=a^2,\; x^2+y^2\le b^2,\; 0\ln b\le a$

Example 396: $W: \;\; (x^2+y^2)^2+2z=1,\; x\ge 0,\; y\ge 0,\; z\ge 0$

Example 397: $W: \;\; 1+x^2+y^2=z^2,\; z=5-x^2-y^2,\; z\ge 0$

- **Consider a solid $D\subset\mathbb{E}_3$, which is bounded by given surfaces.**
  a) Sketch both the solid D and its projection D_{xy} into plane $z=0$.
  b) Express the set D as an elementary region in appropriate coordinate system.
  c) Compute mass of D for a given mass density $\rho(x,y,z)$.

Example 398: $D: \;\; \dfrac{x^2}{4}+\dfrac{y^2}{16}=1,\; x=0,\; z=1,\; z=4,\; (x\le 0), \;\; \rho(x,y,z)=z$

Example 399: $D: \;\; z=\sqrt{x^2+y^2},\; z=4, \;\; \rho(x,y,z)=z$

Example 400: $D: \;\; z=x^2+y^2,\; z=4, \;\; \rho(x,y,z)=\sqrt{x^2+y^2}$

Example 401: $D: \;\; z=x^2+y^2+4,\; z=3-x^2-y^2,\; x^2+y^2=1, \;\; \rho(x,y,z)=k$

Example 402: $D=\{[x,y,z]\in\mathbb{E}_3:\; 1\le x^2+y^2+z^2\le 9,\; y\le 0\}, \;\; \rho(x,y,z)=\sqrt{x^2+y^2+z^2}$

- **Consider a solid $D\subset\mathbb{E}_3$.**
  a) Sketch both the solid D and its projection D_{xy} into plane $z=0$.
  b) Compute volume of D.
  c) Compute mass of D for a given mass density $\rho(x,y,z)$.

Example 403: $D=\{[x,y,z]\in\mathbb{E}_3:\; -1\le x\le 1,\; 0\le y\le 1,\; 0\le z\le 4-x^2-y^2\}, \;\; \rho(x,y,z)=x^2+y$

Example 404: $D: \;\; x=0,\; y=0,\; x+y=1,\; z=0,\; z=xy, \;\; \rho(x,y,z)=x$

Example 405: $D=\{[x,y,z]\in\mathbb{E}_3:\; x^2+y^2\le z\le 9\}, \;\; \rho(x,y,z)=x^2+y^2$

Example 406: $D=\{[x,y,z]\in\mathbb{E}_3:\; 0\le z\le 4-\sqrt{x^2+y^2}\}, \;\; \rho(x,y,z)=\sqrt{x^2+y^2}$

Example 407: $D: \;\; x^2+y^2=1,\; y=1,\; y=x^2+z^2+4, \;\; \rho(x,y,z)=y$

Example 408: $D: \;\; x=y^2,\; x=1,\; z=0,\; z=x, \;\; \rho(x,y,z)=z$

Example 409: $D=\{[x,y,z]\in\mathbb{E}_3:\; x^2+y^2+z^2\le 16,\; x^2+y^2\le 9\}, \;\; \rho(x,y,z)=k$

Example 410: $D: \;\; \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1, \;\; \rho(x,y,z)=3$

- **Compute mass m of the solid $W \subset \mathbb{E}_3$:**

Example 411: $W = \{[x,y,z] \in \mathbb{E}_3 :\ 0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c\},\ \rho = x+y+z$

Example 412: $W = \{[x,y,z] \in \mathbb{E}_3 :\ x \ge 0,\ y \ge 0,\ x+y \le 1,\ 0 \le z \le 4-x-2y\},\ \rho = x^2$

Example 413: $W = \{[x,y,z] \in \mathbb{E}_3 :\ 1 \le x^2+y^2+z^2 \le 9, z \ge 0\},\ \rho = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

Example 414: W is a ball with radius a, mass density equals to square distance from diameter. (*Hint: locate center of the ball at origin and diameter onto x-axis.*)

Example 415: W is bounded by surfaces $z=0,\ 2x+y+z=4,\ x=0,\ y=0$, and $\rho(x,y,z) = 4x$.

Example 416: W is bounded by surfaces $z = x^2+y^2+4,\ z = 3-x^2-y^2,\ x^2+y^2 = 1$, $\rho(x,y,z) = 2z(x^2-y^2)$.

- **Find center of mass T of the solid $W \subset \mathbb{E}_3$ bounded by surfaces:**

Example 417: $W:\quad z=0,\ z=x^2+y^2,\ x+y=5,\ x=0,\ y=0,\ \rho(x,y,z)=k$

Example 418: $W:\quad 2z = x^2+y^2,\ x^2+y^2+z^2 = 3,\ z \ge 0,\ \rho(x,y,z) = k$

Example 419: $W:\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1, zx = 0,\ y=0,\ z=0$ (in the first octant) $\rho(x,y,z)=k$

- **Compute moment of inertia of the solid $W \subset \mathbb{E}_3$:**

Example 420: $W:\quad x^2+y^2 = 2z,\ x+y = z$ relative to each ais, if $\rho(x,y,z) = 1$.

Example 421: $W:\quad x^2+y^2+z^2 = 2,\ x^2+y^2 = z^2\ (z \ge 0)$, relative to z-axis, if $\rho(x,y,z) = 1$.

Example 422: W is a rotational cylinder with base's radius a and height b relative to a prime p, which is parallel with cylinder axis and touch the cylinder surface. (*Hint: Consider the cylinder $(x-a)^2+y^2 \le a^2,\ 0 \le z \le b$, the prime p then coincides with z-axis.*)

Example 423: $W = \{[x,y,z] \in \mathbb{E}_3 :\ \sqrt{3x^2+3y^2} \le z \le 3\}$, relative to z-axis, if $\rho(x,y,z) = k$.