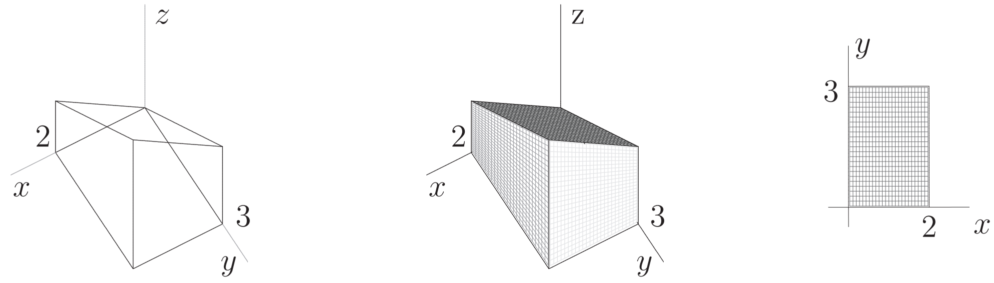
III.4. Fubiniova (Fubiniho) věta pro trojný integrál

• Vypočítejte trojné integrály na daných množinách $W \subset \mathbb{E}_3$:

***Příklad* 342**. $I = \iiint_W (x^2 + y^2)\,dx\,dy\,dz;$

$$W = \{[x, y, z] \in \mathbb{E}_3 : 0 \le z \le x + y,\ 0 \le y \le 3,\ 0 \le x \le 2\}$$

Řešení :



$$I = \int_0^2 \Big(\int_0^3 \Big(\int_0^{x+y} (x^2 + y^2)\,dz\Big)\,dy\Big)\,dx = \int_0^2 \Big(\int_0^3 (x^2 + y^2)\Big[z\Big]_0^{x+y} dy\Big)\,dx =$$

$$= \int_0^2 \Big(\int_0^3 (x^2 + y^2)(x + y)\,dy\Big)\,dx = \int_0^2 \Big(\int_0^3 (x^3 + x^2 y + x y^2 + y^3)\,dy\Big)\,dx =$$

$$= \int_0^2 \Big[x^3 y + \frac{x^2 y^2}{2} + \frac{x y^3}{3} + \frac{y^4}{4}\Big]_0^3 dx = \int_0^2 \Big(3x^3 + \frac{9}{2}x^2 + 9x + \frac{81}{4}\Big)\,dx =$$

$$= \Big[\frac{3}{4}x^4 + \frac{3}{2}x^3 + \frac{9}{2}x^2 + \frac{81}{4}x\Big]_0^2 = \frac{165}{2}.$$ ■

***Příklad* 343**. $I = \iiint_W \frac{x}{y}(z + 1)^2\,dx\,dy\,dz;$

$$W = \{[x, y, z] \in \mathbb{E}_3 : 0 \le x \le 1,\ 1 \le y \le \mathrm{e}^2,\ 0 \le z \le 2\}$$

Řešení :

$$I = \int_0^1 x\,dx \cdot \int_1^{\mathrm{e}^2} \frac{1}{y}\,dy \cdot \int_0^2 (z + 1)^2\,dx = \Big[\frac{x^2}{2}\Big]_0^1 \cdot \Big[\ln|y|\Big]_1^{\mathrm{e}^2} \cdot \Big[\frac{(z + 1)^3}{3}\Big]_0^2 =$$

$$= \frac{1}{2} \cdot 2 \cdot (9 - \frac{1}{3}) = \frac{26}{3}.$$ ■

POZNÁMKA: Je-li funkce typu $f(x, y, z) = g_1(x) \cdot g_2(y) \cdot g_3(z)$ a množina D je kvádr $D = \langle a,\, b\rangle \times \langle c,\, d\rangle \times \langle r,\, s\rangle$, pak

$$\iiint_D f(x, y, z)\,dx\,dy\,dz = \int_a^b g_1(x)\,dx \cdot \int_c^d g_2(y)\,dy \cdot \int_r^s g_3(z)\,dz.$$

***Příklad* 344**.* $I = \iiint_W z^3 y \sin x\,dx\,dy\,dz;$

$$W = \Big\{[x, y, z] \in \mathbb{E}_3 : 0 \le z \le \sin x,\ 0 \le y \le \sin^2 x,\ 0 \le x \le \frac{\pi}{2}\Big\}$$

Řešení :

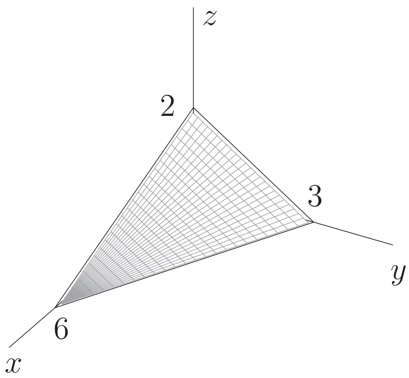
$$I = \int_0^{\pi/2} \Big(\int_0^{\sin^2 x} \Big(\int_0^{\sin x} y \sin x \cdot z^3\,dz\Big)\,dy\Big)\,dx = \int_0^{\pi/2} \Big(\int_0^{\sin^2 x} y \sin x \cdot \Big[\frac{z^4}{4}\Big]_0^{\sin x} dy\Big)\,dx =$$

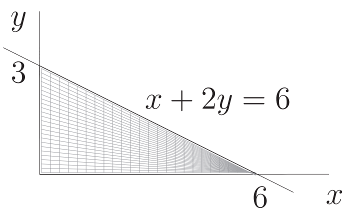
$$= \frac{1}{4}\int_0^{\pi/2}\Big(\int_0^{\sin^2 x} y\sin^5 x\,dy\Big)\,dx = \frac{1}{4}\int_0^{\pi/2}\sin^5 x\cdot\Big[\frac{y^2}{2}\Big]_0^{\sin^2 x}dx = \frac{1}{8}\int_0^{\pi/2}\sin^9 x\,dx =$$

(podle Wallisovy formule viz př. 21) $= \frac{1}{8}\cdot\frac{8\cdot 6\cdot 4\cdot 2}{9\cdot 7\cdot 5\cdot 3}\cdot 1 = \frac{16}{315}.$ ■

***Příklad* 345**. Vypočítejte $\iiint_W \frac{dx\,dy\,dz}{(1+3z)^3}$, kde W je čtyřstěn omezený rovinami $x+2y+3z=6,\ x=0,\ y=0,\ z=0.$

Řešení : Obecnou rovnici roviny napíšeme v kanonickém tvaru $\frac{x}{6}+\frac{y}{3}+\frac{z}{2}=1$





$$0\le z\le \frac{6-x-2y}{3}$$
$$0\le y\le \frac{6-x}{2}$$
$$0\le x\le 6$$

$$I = \int_0^6\Big(\int_0^{\frac{6-x}{2}}\Big(\int_0^{\frac{6-x-2y}{3}}\frac{1}{(1+3z)^3}\,dz\Big)\,dy\Big)\,dx = \frac{1}{3}\int_0^6\Big(\int_0^{\frac{6-x}{2}}\Big[\frac{-1}{2(1+3y)^2}\Big]_0^{\frac{6-x-2y}{3}}dy\Big)\,dx =$$

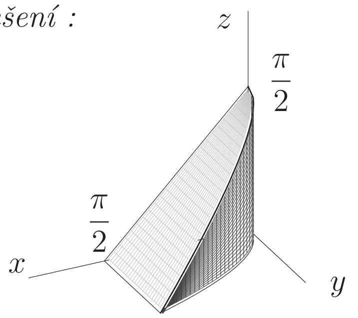
$$= \frac{1}{3}\int_0^6\Big(\int_0^{\frac{6-x}{2}}\Big(\frac{-1}{2(7-x-2y)^2}+\frac{1}{2}\Big)\,dy\Big)\,dx = \frac{1}{6}\int_0^6\Big(\int_0^{\frac{6-x}{2}}\Big(1-\frac{1}{(7-x-2y)^2}\Big)\,dy\Big)\,dx =$$

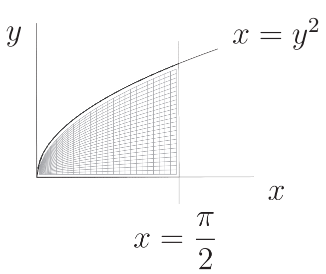
$$= \frac{1}{6}\int_0^6\Big[y-\frac{1}{2(7-x-2y)}\Big]_0^{\frac{6-x}{2}}dx = \frac{1}{6}\int_0^6\Big(\frac{6-x}{2}-\frac{1}{2}+\frac{1}{2(7-x)}\Big)\,dx =$$

$$= \frac{1}{12}\int_0^6\Big(5-x-\frac{1}{x-7}\Big)\,dx = \frac{1}{12}\Big[5x-\frac{x^2}{2}-\ln|x-7|\Big]_0^6 = \frac{12+\ln 7}{12} = 1+\frac{\ln 7}{12}.$$ ■

***Příklad* 346**. Vypočítejte $\iiint_W y\cdot\cos(x+z)\,dx\,dy\,dz$, kde množina W je omezená plochami $y=\sqrt{x},\ y=0,\ z=0,\ x+z=\frac{\pi}{2}.$

Řešení :





$$0\le z\le \frac{\pi}{2}-x$$
$$0\le y\le \sqrt{x}$$
$$0\le x\le \frac{\pi}{2}$$

$$I = \int_0^{\pi/2}\Big(\int_0^{\sqrt{x}}\Big(\int_0^{\frac{\pi}{2}-x} y\cdot\cos(x+z)\,dz\Big)\,dy\Big)\,dx = \int_0^{\pi/2}\Big(\int_0^{\sqrt{x}} y\Big[\sin(x+z)\Big]_0^{\frac{\pi}{2}-x}dy\Big)\,dx =$$

$$= \int_0^{\pi/2}\Big(\int_0^{\sqrt{x}} y\Big(\sin\frac{\pi}{2}-\sin x\Big)\,dy\Big)\,dx = \int_0^{\pi/2}\Big(\int_0^{\sqrt{x}} y(1-\sin x)\,dy\Big)\,dx =$$

$$=\int_0^{\pi/2}(1-\sin x)\cdot\left[\frac{y^2}{2}\right]_0^{\sqrt{x}}dx=\frac{1}{2}\int_0^{\pi/2}(1-\sin x)x\,dx=\frac{1}{2}\left[\frac{x^2}{2}\right]_0^{\pi/2}-\frac{1}{2}\int_0^{\pi/2}x\sin x\,dx=$$

$$=\left|\begin{array}{ll} u=x, & v'=\sin x \\ u'=1, & v=-\cos x\end{array}\right|=\frac{\pi^2}{16}-\frac{1}{2}\Big[-x\cos x+\sin x\Big]_0^{\pi/2}=\frac{\pi^2}{16}-\frac{1}{2}.$$

■

• Vypočítejte integrály na množinách W, které jsou omezeny danými plochami :

347. $\displaystyle\iiint_W (x+y+z)\,dx\,dy\,dz,\quad W: x=1,\ y=0,\ y=x,\ z=0,\ z=\sqrt{2}$ $\left[\frac{1+\sqrt{2}}{2}\right]$

348. $\displaystyle\iiint_W x\,dx\,dy\,dz,\quad W: x=0,\ y=0,\ z=0,\ z=xy,\ x+y=1$ $\left[\frac{1}{60}\right]$

349. $\displaystyle\iiint_W x^2yz^3\,dx\,dy\,dz,\quad W: z=xy,\ y=x,\ y=1,\ z=0$ $\left[\frac{1}{364}\right]$

350. $\displaystyle\iiint_W (x+y)\,dx\,dy\,dz,\quad W: x=0,\ y=0,\ z=0,\ x=a,\ y=a,\ z=a^2-x^2-y^2$ $\left[\frac{a^5}{6}\right]$

351. $\displaystyle\iiint_W xz\,dx\,dy\,dz,\quad W: x=0,\ y=0,\ z=0,\ x+y=1,\ z=x^2+y^2+1$ $\left[\frac{7}{120}\right]$

III.5. Substituční metoda pro trojný integrál

• Spočítejte integrály substitucí do cylindrických souřadnic :

$$\left.\begin{array}{l} x=r\cos\varphi \\ y=r\sin\varphi \\ z=w\end{array}\right\}\qquad J=\left|\begin{array}{ccc}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial w}\end{array}\right|=r$$

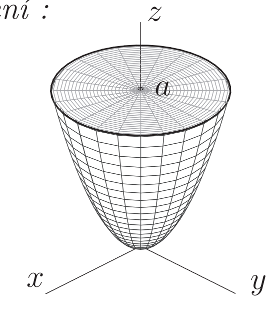
$r>0\ ;\quad 0\le\varphi<2\pi\ ;$

$x^2+y^2=r^2$

***Příklad* 352**. $\displaystyle\iiint_W (x^2+y^2)\,dx\,dy\,dz,$

$$W=\{[x,y,z]\in\mathbb{E}_3: x^2+y^2\le az,\ z\le a,\ (a>0)\}$$

Řešení :



W je část vnitřku rotačního paraboloidu :

$$x^2+y^2\le az\Longrightarrow r^2\le aw\Longrightarrow\frac{r^2}{a}\le w\le a,$$

$$z=a:\quad x^2+y^2=a^2\Longrightarrow r^2=a^2,$$

$$0\le r\le a,\quad 0\le\varphi\le 2\pi.$$

$$\iiint_W (x^2+y^2)\,dx\,dy\,dz=\int_0^{2\pi}\left(\int_0^a\left(\int_{\frac{r^2}{a}}^a r^2r\,dw\right)dr\right)d\varphi=\int_0^{2\pi}\left(\int_0^a r^3\Big[w\Big]_{\frac{r^2}{a}}^a dr\right)d\varphi=$$

$$= \int_0^{2\pi} 1\, d\varphi \cdot \int_0^a r^3\Big(a - \frac{r^2}{a}\Big)\, dr = 2\pi\Big[a\frac{r^4}{4} - \frac{r^6}{6a}\Big]_0^a = 2\pi\Big(\frac{a^5}{4} - \frac{a^5}{6}\Big) = \frac{2\pi\cdot a^5}{12} = \frac{\pi a^5}{6}.$$

V tomto příkladě jsme mohli postupovat i bez použití cylindrických souřadnic. Mohli jsme vyjádřit z přímo : $\dfrac{x^2+y^2}{a} \le z \le a$ a potom vzít v úvahu průmět tělesa do roviny (xy), což je řez tělesa rovinou $z = a$ tj. $x^2 + y^2 \le a^2$ a použít polární souřadnice pro dvojný integrál. Tedy

$$\iiint_W (x^2+y^2)\, dx\, dy\, dz = \iint\limits_{x^2+y^2\le a} \Big(\int_{\frac{x^2+y^2}{a}}^{a} (x^2+y^2)\, dz\Big)\, dx\, dy =$$

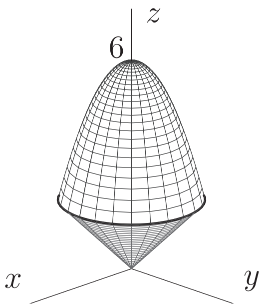
$$= \iint\limits_{x^2+y^2\le a} (x^2+y^2)\Big[z\Big]_{\frac{x^2+y^2}{a}}^{a} dx\, dy = \iint\limits_{x^2+y^2\le a} (x^2+y^2)\Big(a - \frac{x^2+y^2}{a}\Big)\, dx\, dy =$$

$$= \left| \begin{array}{lcl} x = r\cos\varphi & & 0\le r\le a \\ y = r\sin\varphi & | & 0 \le \varphi \le 2\pi \\ J = r & & \end{array} \right| = \int_0^{2\pi}\int_0^a r^2\Big(a - \frac{r^2}{a}\Big)\, r\, dr\, d\varphi = \frac{\pi a^5}{6}.$$ ■

***Příklad* 353**. $\displaystyle\iiint_W \sqrt{x^2+y^2}\, dx\, dy\, dz,$

$$W = \{[x,y,z] \in \mathbb{E}_3 : \sqrt{x^2+y^2} \le z \le 6 - x^2 - y^2\}$$

Řešení :



$z = \sqrt{x^2+y^2}$ je rovnice rotační kuželové plochy s vrcholem v počátku;

$z = 6 - x^2 - y^2$ je rovnice rotačního paraboloidu.

Obě plochy mají společnou osu rotace z, a proto se protínají v kružnici, jejíž poloměr dostaneme ze soustavy :

$\left\{\begin{array}{l} z^2 = x^2 + y^2, \\ z = 6 - x^2 - y^2. \end{array}\right.$ Tedy $z = 6 - z^2,\ z^2 + z - 6 = 0,$

$(z-2)(z+3) = 0$. Úloze vyhovuje řešení $z = 2 \Longrightarrow x^2+y^2 = 4$.

Použijeme cylindrické souřadnice a určíme příslušné meze : $\left\{\begin{array}{l} r \le w \le 6 - r^2 \\ 0 \le r \le 2 \\ 0 \le \varphi \le 2\pi \end{array}\right.$

$$\iiint_W \sqrt{x^2+y^2}\, dx\, dy\, dz = \int_0^{2\pi}\Big(\int_0^2\Big(\int_r^{6-r^2} r\cdot r\, dw\Big)\, dr\Big)\, d\varphi =$$

$$= \int_0^{2\pi}\Big(\int_0^2 r^2(6 - r^2 - r)\, dr\Big)\, d\varphi = 2\pi\cdot\Big[\frac{6r^3}{3} - \frac{r^5}{5} - \frac{r^4}{4}\Big]_0^2 = 2\pi\Big(16 - \frac{32}{5} - 4\Big) = \frac{56\pi}{5}.$$ ■

• Spočítejte integrály substitucí do sférických souřadnic :

$$\left.\begin{array}{l} x = r\cos\varphi\cos\vartheta \\ y = r\sin\varphi\cos\vartheta \\ z = r\sin\vartheta \end{array}\right\} \qquad J = \left| \begin{array}{ccc} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \vartheta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \vartheta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \vartheta} \end{array} \right| = r^2\cos\vartheta$$

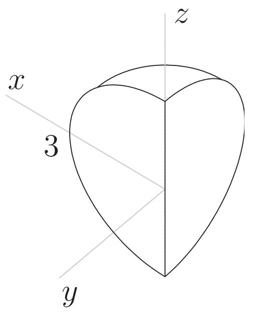
$r > 0$; $0 \le \varphi < 2\pi$; $-\dfrac{\pi}{2} < \vartheta < \dfrac{\pi}{2}$;

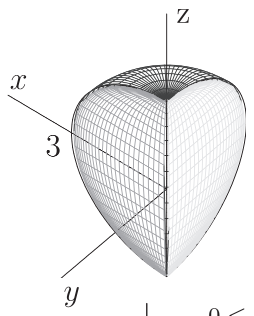
$x^2 + y^2 + z^2 = r^2$;

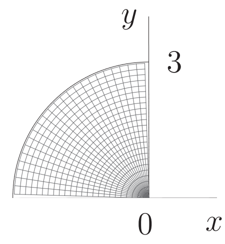
***Příklad* 354.** $\displaystyle\iiint_W \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz,$

$$W=\{[x,y,z]\in\mathbb{E}_3 : x^2+y^2+z^2\le 9,\ x\ge 0,\ y\le 0\}$$

Řešení :







$$\iiint_W \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz=\left|\begin{array}{c}0\le r\le 3\\ \frac{3\pi}{2}\le\varphi\le 2\pi\\ -\frac{\pi}{2}\le\vartheta\le\frac{\pi}{2}\end{array}\right|=$$

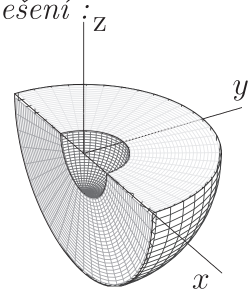
$$=\int_{-\pi/2}^{\pi/2}\Big(\int_{3\pi/2}^{2\pi}\Big(\int_0^3 r\cdot r^2\cos\vartheta\,dr\Big)\,d\varphi\Big)\,d\vartheta=\int_{-\pi/2}^{\pi/2}\cos\vartheta\,d\vartheta\cdot\int_{3\pi/2}^{2\pi}1\,d\varphi\cdot\int_0^3 r^3\,dr=$$

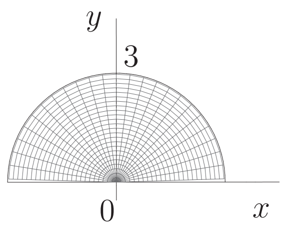
$$=2\cdot\frac{\pi}{2}\cdot\frac{81}{4}=\frac{81}{4}\pi.$$ ■

***Příklad* 355.** $\displaystyle\iiint_W (x^2+y^2)\,dx\,dy\,dz,$

$$W=\{[x,y,z]\in\mathbb{E}_3 : 1\le x^2+y^2+z^2\le 9,\ y\ge 0,\ z\le 0\}$$

Řešení :





$$\iiint_W (x^2+y^2)\,dx\,dy\,dz=$$

$$=\left|\begin{array}{ccc}1\le r\le 3 & & x^2+y^2=r^2\cos^2\vartheta\\ 0\le\varphi\le\pi & | & dx\,dy\,dz=J\,dr\,d\varphi\,d\vartheta\\ -\frac{\pi}{2}\le\vartheta\le 0 & & J=r^2\cos\vartheta\end{array}\right|=$$

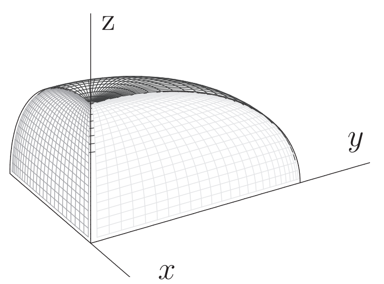
$$=\int_{-\pi/2}^{0}\Big(\int_0^{\pi}\Big(\int_1^3 r^2\cos^2\vartheta\cdot r^2\cos\vartheta\,dr\Big)\,d\varphi\Big)\,d\vartheta=\int_{-\pi/2}^{0}\cos^3\vartheta\,d\vartheta\cdot\int_0^{\pi}1d\varphi\cdot\int_1^3 r^4\,dr=$$

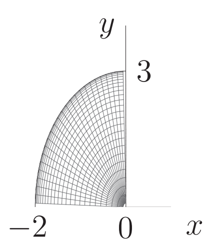
$$=\int_0^{\pi/2}\cos^3\vartheta\,d\vartheta\cdot\pi\cdot\left[\frac{r^5}{5}\right]_1^3=\frac{2}{3}\cdot\pi\cdot\frac{242}{5}=\frac{484}{15}\pi.$$ ■

***Příklad* 356.** $\displaystyle\iiint_W xy\,dx\,dy\,dz,$

$$W=\left\{[x,y,z]\in\mathbb{E}_3 : \tfrac{x^2}{4}+\tfrac{y^2}{9}+\tfrac{z^2}{4}\le 1,\ x\le 0,\ y\ge 0,\ z\ge 0\right\}$$

Řešení : Použijeme zobecněné sférické souřadnice.





$$\iiint_W xy\,dx\,dy\,dz=$$

$$=\left|\begin{array}{ccc}x=2r\cos\varphi\cos\vartheta & & 0\le r\le 1\\ y=3r\sin\varphi\cos\vartheta & | & \frac{\pi}{2}\le\varphi\le\pi\\ z=2r\sin\vartheta & & \\ J=12r^2\cos\vartheta & & 0\le\vartheta\le\frac{\pi}{2}\end{array}\right|=$$

$$= \int_0^{\frac{\pi}{2}} \Big(\int_{\frac{\pi}{2}}^{\pi} \Big(\int_0^1 6r^2 \sin\varphi \cos\varphi \cos^2\vartheta \cdot 12r^2 \cos\vartheta \, dr \Big) \, d\varphi \Big) \, d\vartheta =$$

$$= 72 \cdot \Big[\frac{r^5}{5} \Big]_0^1 \cdot \Big[\frac{\sin^2\varphi}{2} \Big]_{\frac{\pi}{2}}^{\pi} \cdot \Big[\frac{2}{3} \cdot 1 \Big] = -\frac{24}{5}. \qquad \blacksquare$$

***Příklad* 357**. Vypočítejte integrál $\iiint_W z^3 \, dx \, dy \, dz$,

$$W = \{[x, y, z] \in \mathbb{E}_3 : x \le 0,\ y \ge 0,\ z \ge 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$$

Řešení : Použijeme zobecněné sférické souřadnice :

$$\left.\begin{array}{l} x = ar\cos\varphi\cos\vartheta \\ y = br\sin\varphi\cos\vartheta \\ z = cr\sin\vartheta \end{array}\right\} \qquad J = abcr^2\cos\vartheta, \qquad \begin{array}{c} 0 \le r \le 1 \\ \frac{\pi}{2} \le \varphi \le \pi \\ 0 \le \vartheta \le \frac{\pi}{2} \end{array}$$

$$\iiint_W z^3 \, dx \, dy \, dz = \int_0^{\frac{\pi}{2}} \Big(\int_{\frac{\pi}{2}}^{\pi} \Big(\int_0^1 c^3 r^3 \sin^3\vartheta \cdot abcr^2 \cos\vartheta \, dr \Big) d\,\varphi \Big) \, d\,\vartheta =$$

$$= \int_0^{\frac{\pi}{2}} \sin^3\vartheta \cos\vartheta \, d\vartheta \cdot \int_{\frac{\pi}{2}}^{\pi} d\varphi \cdot \int_0^1 r^5 \, dr = abc^4 \Big[\frac{\sin^4\vartheta}{4} \Big]_0^{\frac{\pi}{2}} \cdot \frac{\pi}{2} \cdot \Big[\frac{r^6}{6} \Big]_0^1 = \frac{abc^4\pi}{48}. \qquad \blacksquare$$

• Vypočítejte integrály :

358. $\iiint_W \frac{1}{\sqrt{x^2+y^2+z^2} - 4} \, dx \, dy \, dz$,
$W = \{[x, y, z] \in \mathbb{E}_3 : x^2 + y^2 + z^2 \le 1,\ y \ge 0,\ z \le 0\}$ $\Big[\pi\Big(\frac{9}{2} + 16\ln 3\Big)\Big]$

359. $\iiint_W x^2 y \, dx \, dy \, dz$, $W = \{[x, y, z] \in \mathbb{E}_3 : z \le 4 - x^2 - y^2,\ y \ge 0,\ z \ge 0\}$ $\Big[\frac{512}{105}\Big]$

360. $\iiint_W (y^2 + z^2) \, dx \, dy \, dz$, $W = \{[x, y, z] \in \mathbb{E}_3 : 4x^2 + y^2 + \frac{z^2}{9} \le 1\}$ $[4\pi]$

361. $\iiint_W \frac{\sqrt{x^2+y^2+z^2+2}}{\sqrt{x^2+y^2+z^2}} \, dx \, dy \, dz$,
$W = \{[x, y, z] \in \mathbb{E}_3 : 1 \le x^2 + y^2 + z^2 \le 2,\ x \le 0,\ y \le 0,\ z \ge 0\}$ $\Big[\frac{\pi(8 - 3\sqrt{3})}{6}\Big]$

362. $\iiint_W (x^2 + y^2 + z^2) \, dx \, dy \, dz$,
$W = \{[x, y, z] \in \mathbb{E}_3 : x^2 - 2x + y^2 \le 0,\ -1 \le z \le 1\}$ $\Big[\frac{11}{3}\pi\Big]$

363. $\iiint_W xy \, dx \, dy \, dz$,
$W = \{[x, y, z] \in \mathbb{E}_3 : 0 \le x^2 + z^2 \le 4,\ 0 \le y \le 4 - x^2 - z^2,\ x \ge 0\}$
(Návod : cylindrické souř. $x = r\cos\varphi, y = w, z = r\sin\varphi$) $\Big[\frac{1024}{105}\Big]$